CALCULUS 2

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The Ratio Test:

absolute and conditional convergence

- $\sum_{n=0}^{\infty} |a_n|$ converges $\Longrightarrow \sum_{n=0}^{\infty} a_n$ converges absolutely
- $\sum_{n=0}^{\infty} |a_n|$ diverges and $\sum_{n=0}^{\infty} a_n$ converges $\Longrightarrow \sum_{n=0}^{\infty} a_n$ converges conditionally

  Ex. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges conditionally

Ratio Test

Let $\Sigma\, a_n$ be a series with nonzero terms.

1. The series $\Sigma\, a_n$ converges absolutely when $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$.
2. The series $\Sigma\, a_n$ diverges when $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$ or $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$.
3. The Ratio Test is inconclusive when $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$.

Exapmles:

Determine the convergence or divergence of

$$\sum_{n=0}^{\infty} \frac{2^n}{n!}.$$

Solution

$$a_n = \frac{2^n}{n!}$$

you can write the following.

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left[\frac{2^{n+1}}{(n+1)!} \div \frac{2^n}{n!} \right]$$

$$= \lim_{n \to \infty} \left[\frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \right]$$

$$= \lim_{n \to \infty} \frac{2}{n+1}$$

$$= 0 < 1$$

Determine whether each series converges or diverges.

a. $\displaystyle\sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n}$ **b.** $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

Solution

a. This series converges because the limit of $|a_{n+1}/a_n|$ is less than 1.

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left[(n+1)^2 \left(\frac{2^{n+2}}{3^{n+1}} \right) \left(\frac{3^n}{n^2 2^{n+1}} \right) \right]$$

$$= \lim_{n \to \infty} \frac{2(n+1)^2}{3n^2}$$

$$= \frac{2}{3} < 1$$

b. This series diverges because the limit of $|a_{n+1}/a_n|$ is greater than 1.

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left[\frac{(n+1)^{n+1}}{(n+1)!} \left(\frac{n!}{n^n} \right) \right]$$

$$= \lim_{n \to \infty} \left[\frac{(n+1)^{n+1}}{(n+1)} \left(\frac{1}{n^n} \right) \right]$$

$$= \lim_{n \to \infty} \frac{(n+1)^n}{n^n}$$

$$= \lim_{n \to \infty} \left(1 + \frac{1}{n} \right)^n$$

$$= e > 1$$

Root Test

1. The series Σa_n converges absolutely when $\lim_{n \to \infty} \sqrt[n]{|a_n|} < 1$.
2. The series Σa_n diverges when $\lim_{n \to \infty} \sqrt[n]{|a_n|} > 1$ or $\lim_{n \to \infty} \sqrt[n]{|a_n|} = \infty$.
3. The Root Test is inconclusive when $\lim_{n \to \infty} \sqrt[n]{|a_n|} = 1$.

Example

Determine the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}.$$

Solution You can apply the Root Test as follows.

$$\begin{aligned}\lim_{n \to \infty} \sqrt[n]{|a_n|} &= \lim_{n \to \infty} \sqrt[n]{\frac{e^{2n}}{n^n}} \\ &= \lim_{n \to \infty} \frac{e^{2n/n}}{n^{n/n}} \\ &= \lim_{n \to \infty} \frac{e^2}{n} \\ &= 0 < 1\end{aligned}$$

Because this limit is less than 1, you can conclude that the series converges absolutely

Summary of tests for series

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence
nth-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n \to \infty} a_n \neq 0$
Geometric Series ($r \neq 0$)	$\sum_{n=0}^{\infty} ar^n$	$\lvert r \rvert < 1$	$\lvert r \rvert \geq 1$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+1})$	$\lim_{n \to \infty} b_n = L$	
p-Series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$0 < p \leq 1$
Alternating Series ($a_n > 0$)	$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$	$a_{n+1} \leq a_n$ and $\lim_{n \to \infty} a_n = 0$	
Integral (f is continuous, positive, and decreasing)	$\sum_{n=1}^{\infty} a_n$, $a_n = f(n) \geq 0$	$\int_1^{\infty} f(x)\, dx$ converges	$\int_1^{\infty} f(x)\, dx$ diverges

Summary of tests for series:

Root	$\sum_{n=1}^{\infty} a_n$	$\lim_{n \to \infty} \sqrt[n]{\lvert a_n \rvert} < 1$	$\lim_{n \to \infty} \sqrt[n]{\lvert a_n \rvert} > 1$ or $= \infty$
Ratio	$\sum_{n=1}^{\infty} a_n$	$\lim_{n \to \infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert < 1$	$\lim_{n \to \infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert > 1$ or $= \infty$
Direct Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$0 < a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges	$0 < b_n \leq a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges
Limit Comparison ($a_n, b_n > 0$)	$\sum_{n=1}^{\infty} a_n$	$\lim_{n \to \infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ converges	$\lim_{n \to \infty} \frac{a_n}{b_n} = L > 0$ and $\sum_{n=1}^{\infty} b_n$ diverges

Definition of Taylor and Maclaurin Series

If a function f has derivatives of all orders at $x = c$, then the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

is called the **Taylor series for f at c**. Moreover, if $c = 0$, then the series is the **Maclaurin series for f**.

Example

Use the function $f(x) = \sin x$ to form the Maclaurin series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \acute{f}(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \ldots..$$

and determine the interval of convergence.

Solution Taking successive derivatives of f yields

$$
\begin{aligned}
f(x) &= \sin x & f(0) &= \sin 0 = 0\\
f'(x) &= \cos x & f'(0) &= \cos 0 = 1\\
f''(x) &= -\sin x & f''(0) &= -\sin 0 = 0\\
f'''(x) &= -\cos x & f'''(0) &= -\cos 0 = -1\\
f^{(4)}(x) &= \sin x & f^{(4)}(0) &= \sin 0 = 0\\
f^{(5)}(x) &= \cos x & f^{(5)}(0) &= \cos 0 = 1
\end{aligned}
$$

and so on. The pattern repeats after the third derivative. So, the power series is as follows.

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots\\
&= 0 + (1)x + \frac{0}{2!}x^2 + \frac{(-1)}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \frac{0}{6!}x^6 + \frac{(-1)}{7!}x^7 + \cdots\\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}
\end{aligned}
$$

By the Ratio Test, you can conclude that this series converges for all x.

Example

Use the function $f(x) = \ln x$ to form the Taylor series at $c = 1$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!} x^n = f(1) + \acute{f}(1)(x-1) + \frac{f^{''}(1)}{2!}(x-1)^2 + \frac{f^{'''}(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4 + ...$$

Solution: Taking successive derivatives of f yields

$f(x) = \ln x$	$f(1) = \ln 1 = 0$
$\acute{f}(x) = \frac{1}{x}$	$\acute{f}(1) = \frac{1}{1} = 1$
$f^{''}(1) = \frac{-1}{x^2}$	$f^{''}(1) = -1$
$f^{'''}(x) = \frac{2}{x^3}$	$f^{'''}(1) = 2$
$f^{(4)}(x) = \frac{-6}{x^4}$	$f^{(4)}(1) = -6$
$f^{(5)}(x) = \frac{+24}{x^4}$	$f^{(4)}(1) = 24$

The series power as follows,

$$\ln x = 0 + (x-1) + \frac{-1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 + \frac{-6}{4!}(x-1)^4 + \frac{24}{5!}(x-1)^5 .....$$

$$\ln x = 0 + (x-1) + \frac{-1!}{2!}(x-1)^2 + \frac{2!}{3!}(x-1)^3 + \frac{-3!}{4!}(x-1)^4 + \frac{4!}{5!}(x-1)^5 .....$$

$$\ln x = (x-1) + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \frac{-(x-1)^4}{4} + \frac{(x-1)^5}{5} ..... + (-1)^{n-1}\frac{(x-1)^n}{n} + ...$$

$$f(x) = f(c) + \acute{f}(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots\ldots\ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \ldots\ldots$$

$$f(x) = f(c) + \acute{f}(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots\ldots\ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x)$$

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}; \qquad z \in]c, x[$$

Convergence of Taylor Series

If $\lim_{n \to \infty} R_n = 0$ for all x in the interval I, then the Taylor series for f converges and equals $f(x)$,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

Deriving Taylor Series from a Basic List

POWER SERIES FOR ELEMENTARY FUNCTIONS

Function	Interval of Convergence
$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - \cdots + (-1)^n(x-1)^n + \cdots$	$0 < x < 2$
$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots + (-1)^n x^n + \cdots$	$-1 < x < 1$
$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + \frac{(-1)^{n-1}(x-1)^n}{n} + \cdots$	$0 < x \leq 2$
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!} + \cdots$	$-\infty < x < \infty$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$	$-\infty < x < \infty$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$	$-\infty < x < \infty$
$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$	$-1 \leq x \leq 1$
$\arcsin x = x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots + \frac{(2n)!x^{2n+1}}{(2^n n!)^2(2n+1)} + \cdots$	$-1 \leq x \leq 1$
$(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \cdots + \frac{k(k-1)\cdots(k-n+1)x^n}{n!} + \cdots$	$-1 < x < 1$*

Example

Find the power series for

$$f(x) = \cos\sqrt{x}.$$

Solution Using the power series

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

you can replace x by

$$\sqrt{x}$$

to obtain the series

$$\cos\sqrt{x} = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \frac{x^4}{8!} - \cdots.$$

This series converges for all x in the domain of $\cos\sqrt{x}$—that is, for $x \geq 0$.

A Power Series for $\sin^2 x$

Example

Find the power series for

$$f(x) = \sin^2 x.$$

Solution Consider rewriting $\sin^2 x$ as

$$\sin^2 x = \frac{1 - \cos 2x}{2} = \frac{1}{2} - \frac{1}{2}\cos 2x.$$

Now, use the series for $\cos x$.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$\cos 2x = 1 - \frac{2^2}{2!}x^2 + \frac{2^4}{4!}x^4 - \frac{2^6}{6!}x^6 + \frac{2^8}{8!}x^8 - \cdots$$

$$-\frac{1}{2}\cos 2x = -\frac{1}{2} + \frac{2}{2!}x^2 - \frac{2^3}{4!}x^4 + \frac{2^5}{6!}x^6 - \frac{2^7}{8!}x^8 + \cdots$$

$$\frac{1}{2} - \frac{1}{2}\cos 2x = \frac{1}{2} - \frac{1}{2} + \frac{2}{2!}x^2 - \frac{2^3}{4!}x^4 + \frac{2^5}{6!}x^6 - \frac{2^7}{8!}x^8 + \cdots$$

So, the series for $f(x) = \sin^2 x$ is

$$\sin^2 x = \frac{2}{2!}x^2 - \frac{2^3}{4!}x^4 + \frac{2^5}{6!}x^6 - \frac{2^7}{8!}x^8 + \cdots.$$

This series converges for $-\infty < x < \infty$.

Example

Find the Maclaurin series for $f(x) = (1 + x)^k$ and determine its radius of convergence. Assume that k is not a positive integer and $k \neq 0$.

Solution By successive differentiation, you have

$$\begin{aligned}
f(x) &= (1 + x)^k & f(0) &= 1 \\
f'(x) &= k(1 + x)^{k-1} & f'(0) &= k \\
f''(x) &= k(k-1)(1 + x)^{k-2} & f''(0) &= k(k-1) \\
f'''(x) &= k(k-1)(k-2)(1 + x)^{k-3} & f'''(0) &= k(k-1)(k-2) \\
&\vdots & &\vdots \\
f^{(n)}(x) &= k \cdots (k-n+1)(1 + x)^{k-n} & f^{(n)}(0) &= k(k-1) \cdots (k-n+1)
\end{aligned}$$

which produces the series

$$1 + kx + \frac{k(k-1)x^2}{2} + \cdots + \frac{k(k-1)\cdots(k-n+1)x^n}{n!} + \cdots.$$

By the Ratio Test, you can conclude that the radius of convergence is $R = 1$. So, the series converges to some function in the interval $(-1, 1)$.

Finding a Binomial Series

Example

Find the power series for

$$f(x) = \sqrt[3]{1 + x}.$$

Solution Using the binomial series

$$(1 + x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \cdots$$

let $k = \frac{1}{3}$ and write

$$(1 + x)^{1/3} = 1 + \frac{x}{3} - \frac{2x^2}{3^2 2!} + \frac{2 \cdot 5x^3}{3^3 3!} - \frac{2 \cdot 5 \cdot 8x^4}{3^4 4!} + \cdots.$$

Thank you for your attention