CALCULUS 2

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Plane Curves and Parametric Equations:

Definition

If f and g are continuous functions of t on an interval I, then the equations

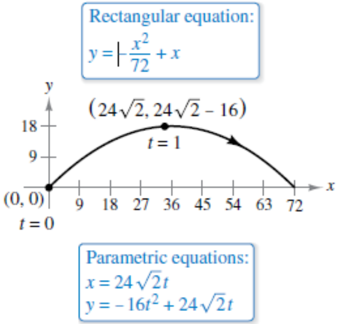
$$\begin{cases} x = f(t) & ; t \in I \\ y = g(t) & ; t \in I \end{cases}$$

are parametric equations and t is the parameter.

The set of points (x, y) obtained as t varies over the interval I is the graph of the parametric equations

the parametric equations and the graph are a plane curve, denoted by C.

Example:



Example

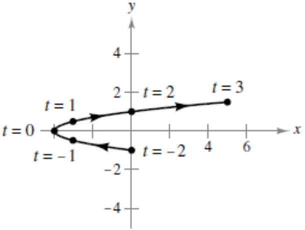
Sketch the curve described by the parametric equations

$$x = f(t) = t^2 - 4$$

$$y = g(t) = \frac{t}{2}; \quad -2 \le t \le 3$$

Soluttion: For values of t on the given interval, the parametric equations yield the points (x, y) shown in the table.

t	-2	-1	0	1	2	3
x	0	-3	-4	-3	0	5
y	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$



Parametric equations:
$x = t^2 - 4$ and $y = \frac{t}{2}$, $-2 \quad t \quad 3$

Eliminating the Parameter

Finding a rectangular equation that represents the graph of a set of parametric equations is called eliminating the parameter.

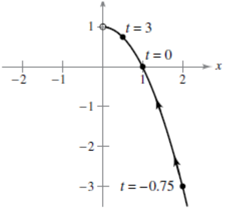
Parametric equations	⇨	Solve for t in one equation.	⇨	Substitute into second equation.	⇨	Rectangular equation
$x = t^2 - 4$ $y = t/2$		$t = 2y$		$x = (2y)^2 - 4$		$x = 4y^2 - 4$

Example

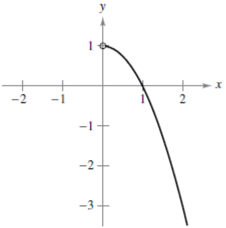
Sketch the curve represented by the equations

$$x = \frac{1}{\sqrt{t+1}}, \qquad y = \frac{t}{t+1}; \quad t > -1$$

by eliminating the parameter and adjusting the domain of the resulting rectangular equation.



Parametric equations:
$x = \frac{1}{\sqrt{t+1}}, y = \frac{t}{t+1}, t > -1$



Rectangular equation:
$y = 1 - x^2, x > 0$

$$x = \frac{1}{\sqrt{t+1}}$$ Parametric equation for x

$$x^2 = \frac{1}{t+1}$$ Square each side.

$$t + 1 = \frac{1}{x^2}$$

$$t = \frac{1}{x^2} - 1$$

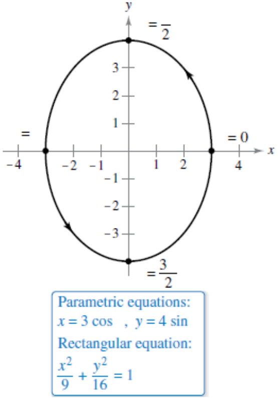
$$t = \frac{1 - x^2}{x^2}$$ Solve for t.

Now, substituting into the parametric equation for y produces

$$y = \frac{t}{t+1}$$ Parametric equation for y

$$y = \frac{(1 - x^2)/x^2}{[(1 - x^2)/x^2] + 1}$$ Substitute $(1 - x^2)/x^2$ for t.

$$y = 1 - x^2.$$ Simplify.



Sketch the curve represented by

$$x = 3\cos\theta \quad \text{and} \quad y = 4\sin\theta, \quad 0 \le \theta \le 2\pi$$

by eliminating the parameter and finding the corresponding rectangular equation.

Solution Begin by solving for $\cos\theta$ and $\sin\theta$ in the given equations.

$$\cos\theta = \frac{x}{3} \qquad \text{Solve for } \cos\theta.$$

and

$$\sin\theta = \frac{y}{4} \qquad \text{Solve for } \sin\theta.$$

Next, make use of the identity

$$\sin^2\theta + \cos^2\theta = 1$$

to form an equation involving only x and y.

$$\cos^2\theta + \sin^2\theta = 1 \qquad \text{Trigonometric identity}$$

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1 \qquad \text{Substitute.}$$

$$\frac{x^2}{9} + \frac{y^2}{16} = 1 \qquad \text{Rectangular equation}$$

Example:

Find a set of parametric equations that represents the graph of $y = 1 - x^2$, using each of the following parameters.

a. $x = t$, b. The slope $m = \frac{dy}{dx}$ at the point (x, y)

a. Letting $x = t$ produces the parametric equations

$$x = t \quad \text{and} \quad y = 1 - x^2 = 1 - t^2.$$

b. To write x and y in terms of the parameter m, you can proceed as follows.

$$m = \frac{dy}{dx}$$

$$m = -2x \qquad \text{Differentiate } y = 1 - x^2.$$

$$x = -\frac{m}{2} \qquad \text{Solve for } x.$$

This produces a parametric equation for x. To obtain a parametric equation for y substitute $-m/2$ for x in the original equation.

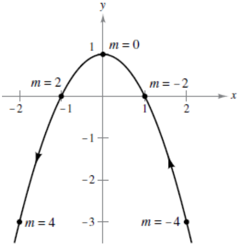
$$y = 1 - x^2 \qquad \text{Write original rectangular equation.}$$

$$y = 1 - \left(-\frac{m}{2}\right)^2 \qquad \text{Substitute } -m/2 \text{ for } x.$$

$$y = 1 - \frac{m^2}{4} \qquad \text{Simplify.}$$

So, the parametric equations are

$$x = -\frac{m}{2} \quad \text{and} \quad y = 1 - \frac{m^2}{4}.$$



Definition of a Smooth Curve:

- A curve C represented by $x = f(t)$ and $y = g(t)$ on an interval I is called smooth when
  - $\acute{f}$, $\acute{g}$ are continuous on I, and
  - Not simultaneously 0, i.e. $(\acute{f})^2 + (\acute{g})^2 \neq 0$

Parametric Form of the Derivative

If a smooth curve C is given by the equations

$$x = f(t) \quad \text{and} \quad y = g(t)$$

then the slope of C at (x, y) is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \quad \frac{dx}{dt} \neq 0.$$

Find dy/dx for the curve given by $x = \sin t$ and $y = \cos t$.

Solution

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\sin t}{\cos t} = -\tan t$$

Finding Slope and Concavity

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{dx/dt}$$ Second derivative

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left[\frac{d^2y}{dx^2}\right] = \frac{\frac{d}{dt}\left[\frac{d^2y}{dx^2}\right]}{dx/dt}.$$ Third derivative

For the curve given by

$$x = \sqrt{t} \quad \text{and} \quad y = \frac{1}{4}(t^2 - 4), \quad t \geq 0$$

find the slope and concavity at the point (2, 3).

Solution Because

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{(1/2)t}{(1/2)t^{-1/2}} = t^{3/2}$$

you can find the second derivative to be

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{dx/dt} = \frac{\frac{d}{dt}[t^{3/2}]}{dx/dt} = \frac{(3/2)t^{1/2}}{(1/2)t^{-1/2}} = 3t.$$

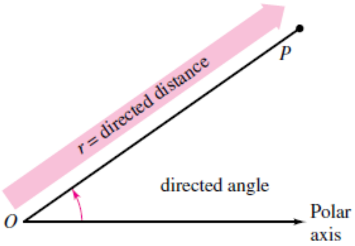
At $(x, y) = (2, 3)$, it follows that $t = 4$, and the slope is

$$\frac{dy}{dx} = (4)^{3/2} = 8.$$

Moreover, when $t = 4$, the second derivative is

$$\frac{d^2y}{dx^2} = 3(4) = 12 > 0$$

Polar Coordinates



r = *directed distance* from O to P

θ = *directed angle*, counterclockwise from polar axis to segment $\overline{OP}$

Coordinate Conversion

The polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) of the point as follows.

Polar-to-Rectangular	**Rectangular-to-Polar**
$x = r\cos\theta$	$\tan\theta = \dfrac{y}{x}$
$y = r\sin\theta$	$r^2 = x^2 + y^2$

Polar-to-Rectangular Conversion

a. For the point $(r, \theta) = (2, \pi)$,

$$x = r\cos\theta = 2\cos\pi = -2 \quad \text{and} \quad y = r\sin\theta = 2\sin\pi = 0.$$

So, the rectangular coordinates are $(x, y) = (-2, 0)$.

b. For the point $(r, \theta) = \left(\sqrt{3}, \pi/6\right)$,

$$x = \sqrt{3}\cos\frac{\pi}{6} = \frac{3}{2} \quad \text{and} \quad y = \sqrt{3}\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2}.$$

So, the rectangular coordinates are $(x, y) = \left(3/2, \sqrt{3}/2\right)$.

a. For the second-quadrant point $(x, y) = (-1, 1)$,

$$\tan\theta = \frac{y}{x} = -1 \quad \Longrightarrow \quad \theta = \frac{3\pi}{4}.$$

Because θ was chosen to be in the same quadrant as (x, y), use a positive value of r.

$$r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$$

This implies that *one* set of polar coordinates is $(r, \theta) = \left(\sqrt{2}, 3\pi/4\right)$.

b. Because the point $(x, y) = (0, 2)$ lies on the positive y-axis, choose $\theta = \pi/2$ and $r = 2$, and one set of polar coordinates is $(r, \theta) = (2, \pi/2)$.

Thank you for your attention