CALCULUS 2

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Definition of Partial Derivatives of a Function of Two Variables

If $z = f(x, y)$, then the **first partial derivatives** of f with respect to x and y are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$ Partial derivative with respect to x

and

$$f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$ Partial derivative with respect to y

provided the limits exist.

Finding Partial Derivatives

a. To find f_x for $f(x, y) = 3x - x^2y^2 + 2x^3y$, consider y to be constant and differentiate with respect to x.

$$f_x(x, y) = 3 - 2xy^2 + 6x^2y \qquad \text{Partial derivative with respect to } x$$

To find f_y, consider x to be constant and differentiate with respect to y.

$$f_y(x, y) = -2x^2y + 2x^3 \qquad \text{Partial derivative with respect to } y$$

b. To find f_x for $f(x, y) = (\ln x)(\sin x^2y)$, consider y to be constant and differentiate with respect to x.

$$f_x(x, y) = (\ln x)(\cos x^2y)(2xy) + \frac{\sin x^2y}{x} \qquad \text{Partial derivative with respect to } x$$

To find f_y, consider x to be constant and differentiate with respect to y.

$$f_y(x, y) = (\ln x)(\cos x^2y)(x^2) \qquad \text{Partial derivative with respect to } y$$

Notation for First Partial Derivatives

For $z = f(x, y)$, the partial derivatives f_x and f_y are denoted by

$$\frac{\partial}{\partial x} f(x, y) = f_x(x, y) = z_x = \frac{\partial z}{\partial x}$$ Partial derivative with respect to x

and

$$\frac{\partial}{\partial y} f(x, y) = f_y(x, y) = z_y = \frac{\partial z}{\partial y}.$$ Partial derivative with respect to y

The first partials evaluated at the point (a, b) are denoted by

$$\left.\frac{\partial z}{\partial x}\right|_{(a, b)} = f_x(a, b)$$

and

$$\left.\frac{\partial z}{\partial y}\right|_{(a, b)} = f_y(a, b).$$

For $f(x, y) = xe^{x^2y}$, find f_x and f_y, and evaluate each at the point $(1, \ln 2)$.

Solution Because

$$f_x(x, y) = xe^{x^2y}(2xy) + e^{x^2y} \qquad \text{Partial derivative with respect to } x$$

the partial derivative of f with respect to x at $(1, \ln 2)$ is

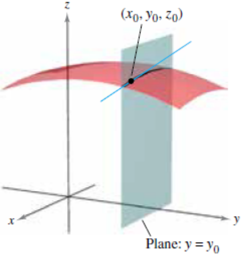
$$\begin{aligned} f_x(1, \ln 2) &= e^{\ln 2}(2 \ln 2) + e^{\ln 2} \\ &= 4 \ln 2 + 2. \end{aligned}$$

Because

$$\begin{aligned} f_y(x, y) &= xe^{x^2y}(x^2) \\ &= x^3e^{x^2y} \qquad \text{Partial derivative with respect to } y \end{aligned}$$

the partial derivative of f with respect to y at $(1, \ln 2)$ is

$$\begin{aligned} f_y(1, \ln 2) &= e^{\ln 2} \\ &= 2. \end{aligned}$$



$\frac{\partial f}{\partial x}$ = slope in x-direction

Partial Derivatives of a Function of Three or More Variables:

a. To find the partial derivative of $f(x, y, z) = xy + yz^2 + xz$ with respect to z, consider x and y to be constant and obtain

$$\frac{\partial}{\partial z}[xy + yz^2 + xz] = 2yz + x.$$

b. To find the partial derivative of $f(x, y, z) = z \sin(xy^2 + 2z)$ with respect to z, consider x and y to be constant. Then, using the Product Rule, you obtain

$$\begin{aligned}\frac{\partial}{\partial z}[z \sin(xy^2 + 2z)] &= (z)\frac{\partial}{\partial z}[\sin(xy^2 + 2z)] + \sin(xy^2 + 2z)\frac{\partial}{\partial z}[z] \\ &= (z)[\cos(xy^2 + 2z)](2) + \sin(xy^2 + 2z) \\ &= 2z \cos(xy^2 + 2z) + \sin(xy^2 + 2z).\end{aligned}$$

c. To find the partial derivative of

$$f(x, y, z, w) = \frac{x + y + z}{w}$$

with respect to w, consider x, y, and z to be constant and obtain

$$\frac{\partial}{\partial w}\left[\frac{x + y + z}{w}\right] = -\frac{x + y + z}{w^2}.$$

Higher-Order Partial Derivatives:

1. Differentiate twice with respect to x:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}.$$

2. Differentiate twice with respect to y:

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}.$$

3. Differentiate first with respect to x and then with respect to y:

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}.$$

4. Differentiate first with respect to y and then with respect to x:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}.$$

Finding Second Partial Derivatives:

Find the second partial derivatives of

$$f(x, y) = 3xy^2 - 2y + 5x^2y^2$$

and determine the value of $f_{xy}(-1, 2)$.

Solution Begin by finding the first partial derivatives with respect to x and y.

$$f_x(x, y) = 3y^2 + 10xy^2 \quad \text{and} \quad f_y(x, y) = 6xy - 2 + 10x^2y$$

Then, differentiate each of these with respect to x and y.

$$f_{xx}(x, y) = 10y^2 \qquad \text{and} \quad f_{yy}(x, y) = 6x + 10x^2$$
$$f_{xy}(x, y) = 6y + 20xy \quad \text{and} \quad f_{yx}(x, y) = 6y + 20xy$$

At $(-1, 2)$, the value of f_{xy} is

$$f_{xy}(-1, 2) = 12 - 40 = -28.$$

Equality of Mixed Partial Derivatives

If f is a function of x and y such that f_{xy} and f_{yx} are continuous on an open disk R, then, for every (x, y) in R,

$$f_{xy}(x, y) = f_{yx}(x, y).$$

Show that $f_{xz} = f_{zx}$ and $f_{xzz} = f_{zxz} = f_{zzx}$ for the function

$$f(x, y, z) = ye^x + x \ln z.$$

Solution

First partials:

$$f_x(x, y, z) = ye^x + \ln z, \quad f_z(x, y, z) = \frac{x}{z}$$

Second partials (note that the first two are equal):

$$f_{xz}(x, y, z) = \frac{1}{z}, \quad f_{zx}(x, y, z) = \frac{1}{z}, \quad f_{zz}(x, y, z) = -\frac{x}{z^2}$$

Third partials (note that all three are equal):

$$f_{xzz}(x, y, z) = -\frac{1}{z^2}, \quad f_{zxz}(x, y, z) = -\frac{1}{z^2}, \quad f_{zzx}(x, y, z) = -\frac{1}{z^2}$$

Definition of Total Differential

If $z = f(x, y)$ and Δx and Δy are increments of x and y, then the **differentials** of the independent variables x and y are

$$dx = \Delta x \quad \text{and} \quad dy = \Delta y$$

and the **total differential** of the dependent variable z is

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = f_x(x, y)\,dx + f_y(x, y)\,dy.$$

Find the total differential for each function.

a. $z = 2x \sin y - 3x^2y^2$ **b.** $w = x^2 + y^2 + z^2$

Solution

a. The total differential dz for $z = 2x \sin y - 3x^2y^2$ is

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \qquad \text{Total differential } dz$$

$$= (2 \sin y - 6xy^2)\,dx + (2x \cos y - 6x^2y)\,dy.$$

b. The total differential dw for $w = x^2 + y^2 + z^2$ is

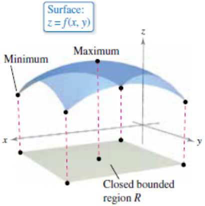
$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz \qquad \text{Total differential } dw$$

$$= 2x\,dx + 2y\,dy + 2z\,dz.$$

Extreme Value Theorem

Let f be a continuous function of two variables x and y defined on a closed bounded region R in the xy-plane.

1. There is at least one point in R at which f takes on a minimum value.
2. There is at least one point in R at which f takes on a maximum value.



Definition of Relative Extrema

Let f be a function defined on a region R containing (x_0, y_0).

1. The function f has a **relative minimum** at (x_0, y_0) if

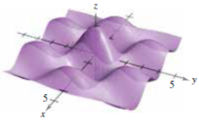
$$f(x, y) \geq f(x_0, y_0)$$

for all (x, y) in an *open* disk containing (x_0, y_0).

2. The function f has a **relative maximum** at (x_0, y_0) if

$$f(x, y) \leq f(x_0, y_0)$$

for all (x, y) in an *open* disk containing (x_0, y_0).



Relative extrema

Critical Point

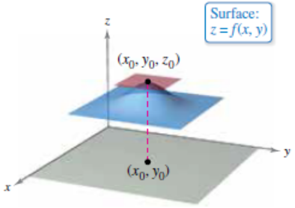
Definition of Critical Point

Let f be defined on an open region R containing (x_0, y_0). The point (x_0, y_0) is a **critical point** of f if one of the following is true.

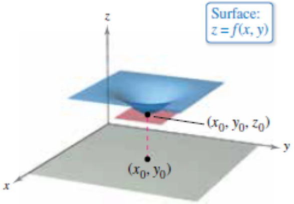
1. $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$
2. $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.

Relative Extrema Occur Only at Critical Points

If f has a relative extremum at (x_0, y_0) on an open region R, then (x_0, y_0) is a critical point of f.

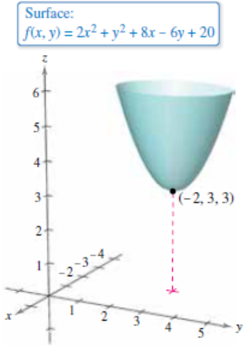


Relative maximum



Relative minimum

Finding a Relative Extremum



The function $z = f(x, y)$ has a relative minimum at $(-2, 3)$.

Determine the relative extrema of

$$f(x, y) = 2x^2 + y^2 + 8x - 6y + 20.$$

Solution Begin by finding the critical points of f. Because

$$f_x(x, y) = 4x + 8 \qquad \text{Partial with respect to } x$$

and

$$f_y(x, y) = 2y - 6 \qquad \text{Partial with respect to } y$$

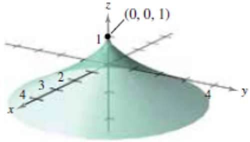
are defined for all x and y, the only critical points are those for which both first partial derivatives are 0. To locate these points, set $f_x(x, y)$ and $f_y(x, y)$ equal to 0, and solve the equations

$$4x + 8 = 0 \quad \text{and} \quad 2y - 6 = 0$$

to obtain the critical point $(-2, 3)$. By completing the square for f, you can see that for all $(x, y) \neq (-2, 3)$

$$f(x, y) = 2(x + 2)^2 + (y - 3)^2 + 3 > 3.$$

So, a relative *minimum* of f occurs at $(-2, 3)$. The value of the relative minimum is $f(-2, 3) = 3$,



$f_x(x, y)$ and $f_y(x, y)$ are undefined at (0, 0).

Determine the relative extrema of

$$f(x, y) = 1 - (x^2 + y^2)^{1/3}.$$

Solution Because

$$f_x(x, y) = -\frac{2x}{3(x^2 + y^2)^{2/3}} \qquad \text{Partial with respect to } x$$

and

$$f_y(x, y) = -\frac{2y}{3(x^2 + y^2)^{2/3}} \qquad \text{Partial with respect to } y$$

it follows that both partial derivatives exist for all points in the xy-plane except for (0, 0). Moreover, because the partial derivatives cannot both be 0 unless both x and y are 0, you can conclude that (0, 0) is the only critical point note that $f(0, 0)$ is 1. For all other (x, y), it is clear that

$$f(x, y) = 1 - (x^2 + y^2)^{1/3} < 1.$$

So, f has a relative *maximum* at (0, 0).

The Second Partials Test

Second Partials Test

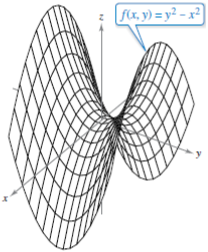
Let f have continuous second partial derivatives on an open region containing a point (a, b) for which

$$f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0.$$

To test for relative extrema of f, consider the quantity

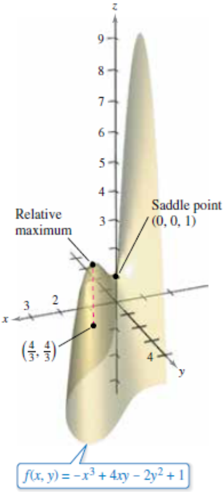
$$d = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2.$$

1. If $d > 0$ and $f_{xx}(a, b) > 0$, then f has a **relative minimum** at (a, b).
2. If $d > 0$ and $f_{xx}(a, b) < 0$, then f has a **relative maximum** at (a, b).
3. If $d < 0$, then $(a, b, f(a, b))$ is a **saddle point.**
4. The test is inconclusive if $d = 0$.



Saddle point at $(0, 0, 0)$:
$f_x(0, 0) = f_y(0, 0) = 0$

Using the Second Partials Test



Find the relative extrema of $f(x, y) = -x^3 + 4xy - 2y^2 + 1$.

Solution Begin by finding the critical points of f. Because

$$f_x(x, y) = -3x^2 + 4y \quad \text{and} \quad f_y(x, y) = 4x - 4y$$

exist for all x and y, the only critical points are those for which both first partial derivatives are 0. To locate these points, set $f_x(x, y)$ and $f_y(x, y)$ equal to 0 to obtain

$$-3x^2 + 4y = 0 \quad \text{and} \quad 4x - 4y = 0.$$

From the second equation, you know that $x = y$, and, by substitution into the first equation, you obtain two solutions: $y = x = 0$ and $y = x = \frac{4}{3}$. Because

$$f_{xx}(x, y) = -6x, \quad f_{yy}(x, y) = -4, \quad \text{and} \quad f_{xy}(x, y) = 4$$

it follows that, for the critical point $(0, 0)$,

$$d = f_{xx}(0, 0)f_{yy}(0, 0) - [f_{xy}(0, 0)]^2 = 0 - 16 < 0$$

and, by the Second Partials Test, you can conclude that $(0, 0, 1)$ is a saddle point of f. Furthermore, for the critical point $\left(\frac{4}{3}, \frac{4}{3}\right)$,

$$\begin{aligned} d &= f_{xx}\left(\frac{4}{3}, \frac{4}{3}\right)f_{yy}\left(\frac{4}{3}, \frac{4}{3}\right) - \left[f_{xy}\left(\frac{4}{3}, \frac{4}{3}\right)\right]^2 \\ &= -8(-4) - 16 \\ &= 16 \\ &> 0 \end{aligned}$$

and because $f_{xx}\left(\frac{4}{3}, \frac{4}{3}\right) = -8 < 0$, you can conclude that f has a relative maximum at $\left(\frac{4}{3}, \frac{4}{3}\right)$.

Thank you for your attention