CALCULUS 2

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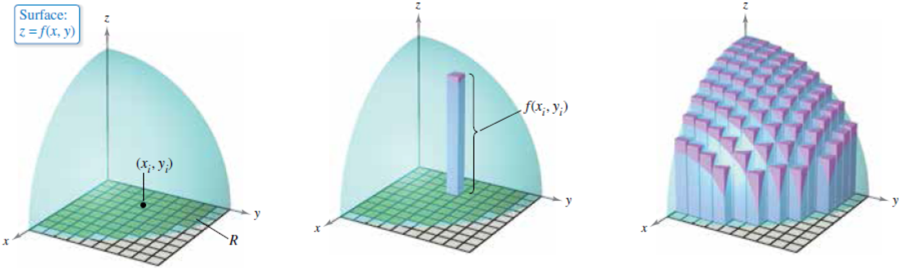
Double Integrals and Volume



Definition of Double Integral

If f is defined on a closed, bounded region R in the xy-plane, then the **double integral of f over R** is

$$\iint_R f(x, y)\, dA = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i)\, \Delta A_i$$

provided the limit exists. If the limit exists, then f is **integrable** over R.

Volume of a Solid Region

If f is integrable over a plane region R and $f(x, y) \geq 0$ for all (x, y) in R, then the volume of the solid region that lies above R and below the graph of f is

$$V = \int_R\int f(x, y)\, dA.$$

Properties of Double Integrals

Let f and g be continuous over a closed, bounded plane region R, and let c be a constant.

1. $\displaystyle\int_R\int cf(x, y)\, dA = c\int_R\int f(x, y)\, dA$
2. $\displaystyle\int_R\int [f(x, y) \pm g(x, y)]\, dA = \int_R\int f(x, y)\, dA \pm \int_R\int g(x, y)\, dA$
3. $\displaystyle\int_R\int f(x, y)\, dA \geq 0, \quad \text{if } f(x, y) \geq 0$
4. $\displaystyle\int_R\int f(x, y)\, dA \geq \int_R\int g(x, y)\, dA, \quad \text{if } f(x, y) \geq g(x, y)$
5. $\displaystyle\int_R\int f(x, y)\, dA = \int_{R_1}\int f(x, y)\, dA + \int_{R_2}\int f(x, y)\, dA$, where R is the union of two nonoverlapping subregions R_1 and R_2.

Finding Partial Derivatives:

Notation for First Partial Derivatives

For $z = f(x, y)$, the partial derivatives f_x and f_y are denoted by

$$\frac{\partial}{\partial x} f(x, y) = f_x(x, y) = z_x = \frac{\partial z}{\partial x}$$ Partial derivative with respect to x

and

$$\frac{\partial}{\partial y} f(x, y) = f_y(x, y) = z_y = \frac{\partial z}{\partial y}.$$ Partial derivative with respect to y

The first partials evaluated at the point (a, b) are denoted by

$$\left.\frac{\partial z}{\partial x}\right|_{(a, b)} = f_x(a, b)$$

and

$$\left.\frac{\partial z}{\partial y}\right|_{(a, b)} = f_y(a, b).$$

Fubini's Theorem

Let f be continuous on a plane region R.

1. If R is defined by $a \le x \le b$ and $g_1(x) \le y \le g_2(x)$, where g_1 and g_2 are continuous on $[a, b]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

2. If R is defined by $c \le y \le d$ and $h_1(y) \le x \le h_2(y)$, where h_1 and h_2 are continuous on $[c, d]$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

Double Integral:

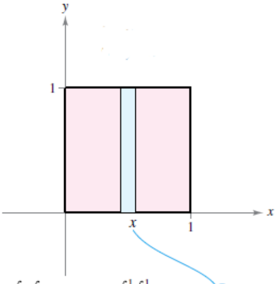
Evaluate

$$\int_R\int\left(1-\frac{1}{2}x^2-\frac{1}{2}y^2\right)dA$$

where R is the region given by

$$0 \le x \le 1, \quad 0 \le y \le 1.$$

Solution Because the region R is a square, it is both vertically and horizontally simple, and you can use either order of integration. Choose $dy\,dx$ by placing a vertical representative rectangle in the region (see the figure at the right). This produces the following.



$$\int_R\int f(x,y)\,dA = \int_0^1\int_0^1 f(x,y)\,dy\,dx$$

$$\begin{aligned}
\int_R\int\left(1-\frac{1}{2}x^2-\frac{1}{2}y^2\right)dA &= \int_0^1\int_0^1\left(1-\frac{1}{2}x^2-\frac{1}{2}y^2\right)dy\,dx \\
&= \int_0^1\left[\left(1-\frac{1}{2}x^2\right)y-\frac{y^3}{6}\right]_0^1 dx \\
&= \int_0^1\left(\frac{5}{6}-\frac{1}{2}x^2\right)dx \\
&= \left[\frac{5}{6}x-\frac{x^3}{6}\right]_0^1 \\
&= \frac{2}{3}
\end{aligned}$$

Double Integrals in Polar Coordinates

Change of Variables to Polar Form

Let R be a plane region consisting of all points $(x, y) = (r\cos\theta, r\sin\theta)$ satisfying the conditions $0 \le g_1(\theta) \le r \le g_2(\theta)$, $\alpha \le \theta \le \beta$, where $0 \le (\beta - \alpha) \le 2\pi$. If g_1 and g_2 are continuous on $[\alpha, \beta]$ and f is continuous on R, then

$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta) r\, dr\, d\theta.$$

Example

Let R be the annular region lying between the two circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 5$. Evaluate the integral

$$\int_R\int (x^2 + y)\, dA.$$

Solution The polar boundaries are $1 \le r \le \sqrt{5}$ and $0 \le \theta \le 2\pi$, . Furthermore, $x^2 = (r\cos\theta)^2$ and $y = r\sin\theta$. So, you have

$$\begin{aligned}
\int_R\int (x^2 + y)\, dA &= \int_0^{2\pi}\int_1^{\sqrt{5}} (r^2\cos^2\theta + r\sin\theta) r\, dr\, d\theta \\
&= \int_0^{2\pi}\int_1^{\sqrt{5}} (r^3\cos^2\theta + r^2\sin\theta)\, dr\, d\theta \\
&= \int_0^{2\pi} \left(\frac{r^4}{4}\cos^2\theta + \frac{r^3}{3}\sin\theta\right)\Bigg]_1^{\sqrt{5}} d\theta \\
&= \int_0^{2\pi} \left(6\cos^2\theta + \frac{5\sqrt{5}-1}{3}\sin\theta\right) d\theta \\
&= \int_0^{2\pi} \left(3 + 3\cos 2\theta + \frac{5\sqrt{5}-1}{3}\sin\theta\right) d\theta \\
&= \left(3\theta + \frac{3\sin 2\theta}{2} - \frac{5\sqrt{5}-1}{3}\cos\theta\right)\Bigg]_0^{2\pi} \\
&= 6\pi.
\end{aligned}$$

Use polar coordinates to find the volume of the solid region bounded above by the hemisphere

$$z = \sqrt{16 - x^2 - y^2}$$ Hemisphere forms upper surface.

and below by the circular region R given by

$$x^2 + y^2 \le 4$$ Circular region forms lower surface.

as shown in Figure 14.30.

Solution you can see that R has the bounds

$$-\sqrt{4 - y^2} \le x \le \sqrt{4 - y^2}, \quad -2 \le y \le 2$$

and that $0 \le z \le \sqrt{16 - x^2 - y^2}$. In polar coordinates, the bounds are

$$0 \le r \le 2 \quad \text{and} \quad 0 \le \theta \le 2\pi$$

with height $z = \sqrt{16 - x^2 - y^2} = \sqrt{16 - r^2}$. Consequently, the volume V is

$$V = \int_R\int f(x, y)\, dA$$ Formula for volume

$$= \int_0^{2\pi}\int_0^2 \sqrt{16 - r^2}\, r\, dr\, d\theta$$ Polar coordinates

$$= -\frac{1}{3}\int_0^{2\pi} (16 - r^2)^{3/2}\Big]_0^2\, d\theta$$ Integrate with respect to r.

$$= -\frac{1}{3}\int_0^{2\pi} \left(24\sqrt{3} - 64\right) d\theta$$

$$= -\frac{8}{3}\left(3\sqrt{3} - 8\right)\theta\Big]_0^{2\pi}$$ Integrate with respect to θ.

$$= \frac{16\pi}{3}\left(8 - 3\sqrt{3}\right)$$

$$\approx 46.979.$$

Thank you for your attention