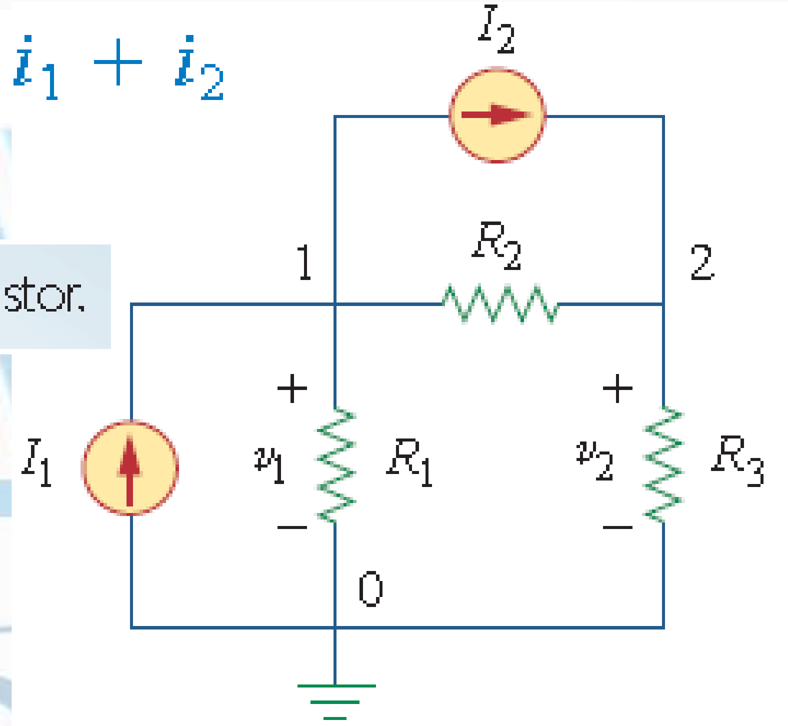
Nodal Analysis

Steps to Determine Node Voltages:

1. Select a node as the reference node. Assign voltages v_1, $v_2, \ldots, v_{n-1}$ to the remaining $n - 1$ nodes. The voltages are referenced with respect to the reference node.
2. Apply KCL to each of the $n - 1$ nonreference nodes. Use Ohm's law to express the branch currents in terms of node voltages.
3. Solve the resulting simultaneous equations to obtain the unknown node voltages.

At node 1, applying KCL gives $I_1 = I_2 + i_1 + i_2$

At node 2, $I_2 + i_2 = i_3$



Current flows from **a higher** potential to **a lower** potential in a resistor.

We can express this principle as

$$i = \frac{v_{higher} - v_{lower}}{R}$$

$$i_1 = \frac{v_1 - 0}{R_1} \quad \text{or} \quad i_1 = G_1 v_1$$

$$i_2 = \frac{v_1 - v_2}{R_2} \quad \text{or} \quad i_2 = G_2(v_1 - v_2)$$

$$i_3 = \frac{v_2 - 0}{R_3} \quad \text{or} \quad i_3 = G_3 v_2$$

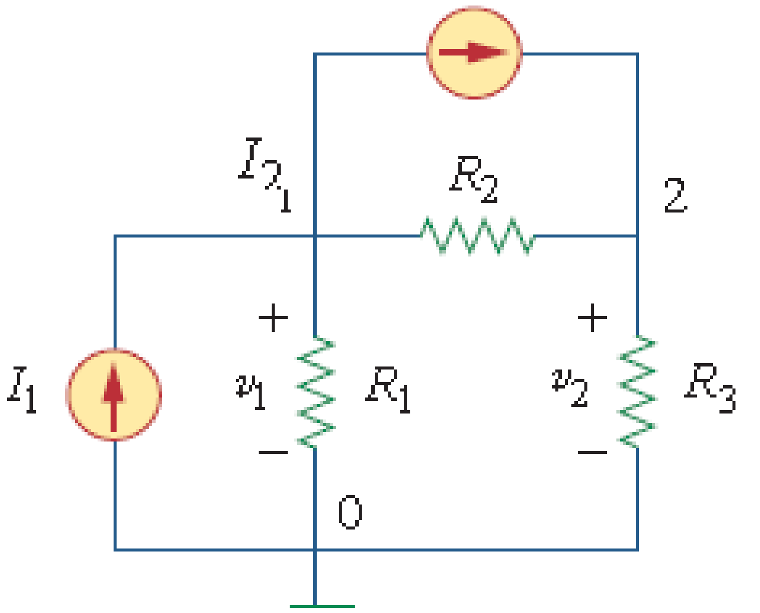
$$I_1 = I_2 + \frac{v_1}{R_1} + \frac{v_1 - v_2}{R_2}$$

$$I_2 + \frac{v_1 - v_2}{R_2} = \frac{v_2}{R_3}$$

$$I_1 = I_2 + G_1 v_1 + G_2(v_1 - v_2)$$

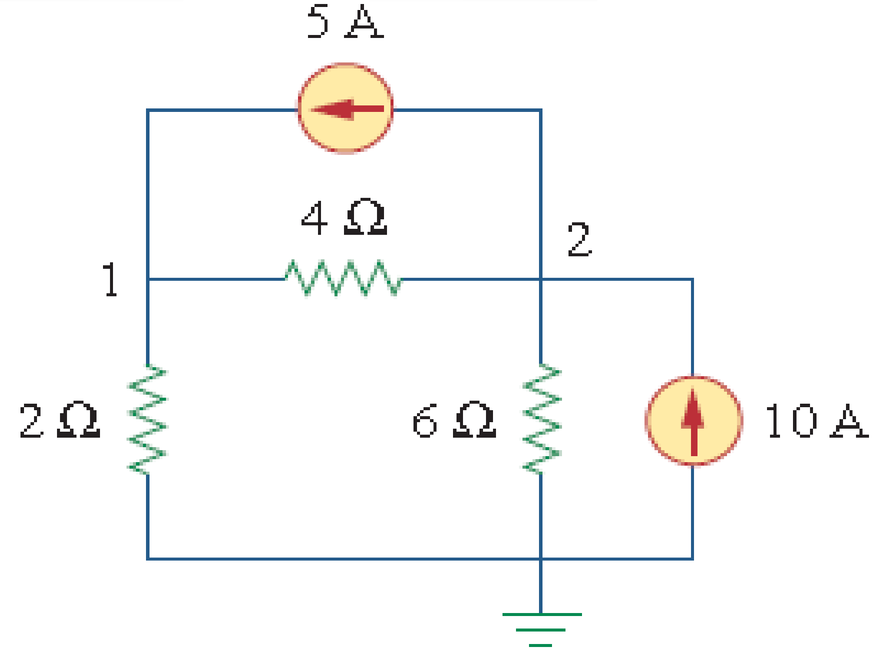
$$I_2 + G_2(v_1 - v_2) = G_3 v_2$$

$$\begin{bmatrix} G_1 + G_2 & -G_2 \\ -G_2 & G_2 + G_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} I_1 - I_2 \\ I_2 \end{bmatrix}$$

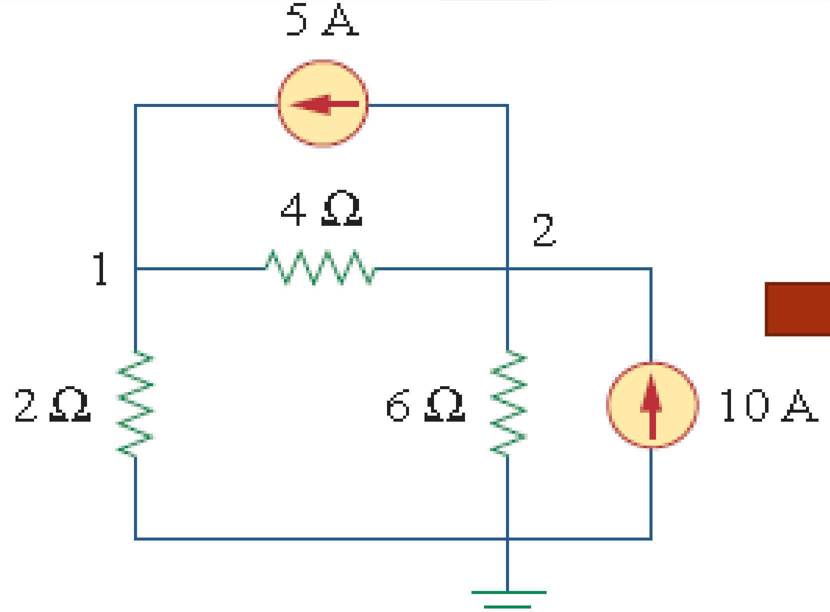


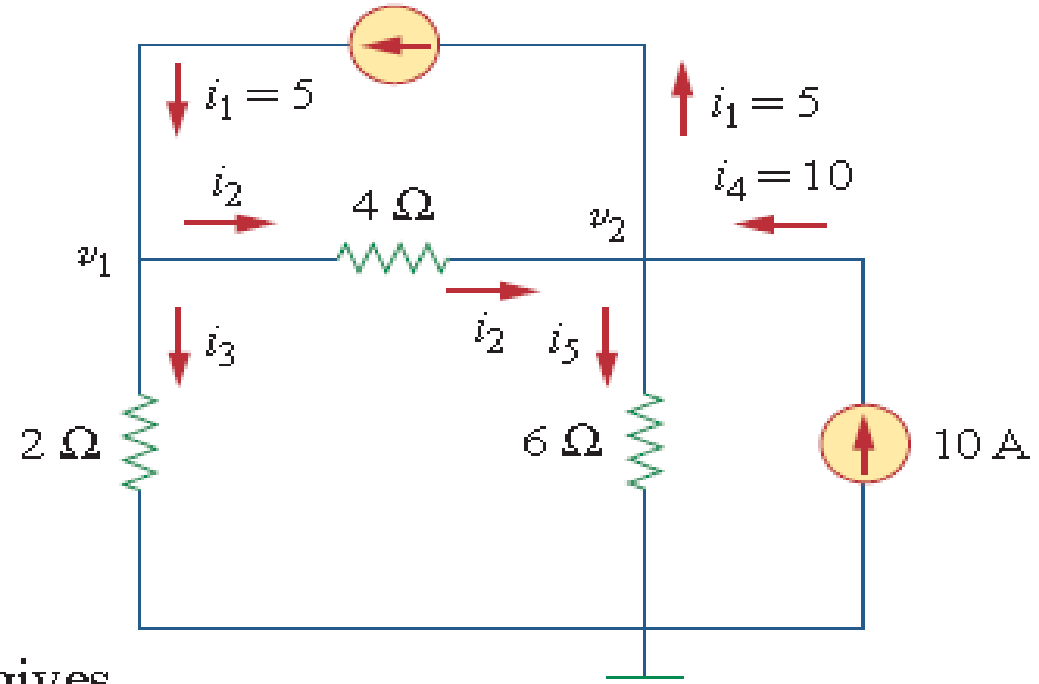
Example

Calculate the node voltages in the circuit shown in Fig



ELECTRICAL CIRCUIT (CEDC206):





At node 1, applying KCL and Ohm's law gives

$$i_1 = i_2 + i_3 \quad \Rightarrow \quad 5 = \frac{v_1 - v_2}{4} + \frac{v_1 - 0}{2}$$

$$20 = v_1 - v_2 + 2v_1 \qquad 3v_1 - v_2 = 20 \quad (1)$$

$$i_2 + i_4 = i_1 + i_5 \quad \Rightarrow \quad \frac{v_1 - v_2}{4} + 10 = 5 + \frac{v_2 - 0}{6}$$

$$3v_1 - 3v_2 + 120 = 60 + 2v_2 \qquad -3v_1 + 5v_2 = 60 \quad (2)$$

$$v_2 = 20\text{ V}$$

$$v_1 = \frac{40}{3} = 13.333\text{ V}$$

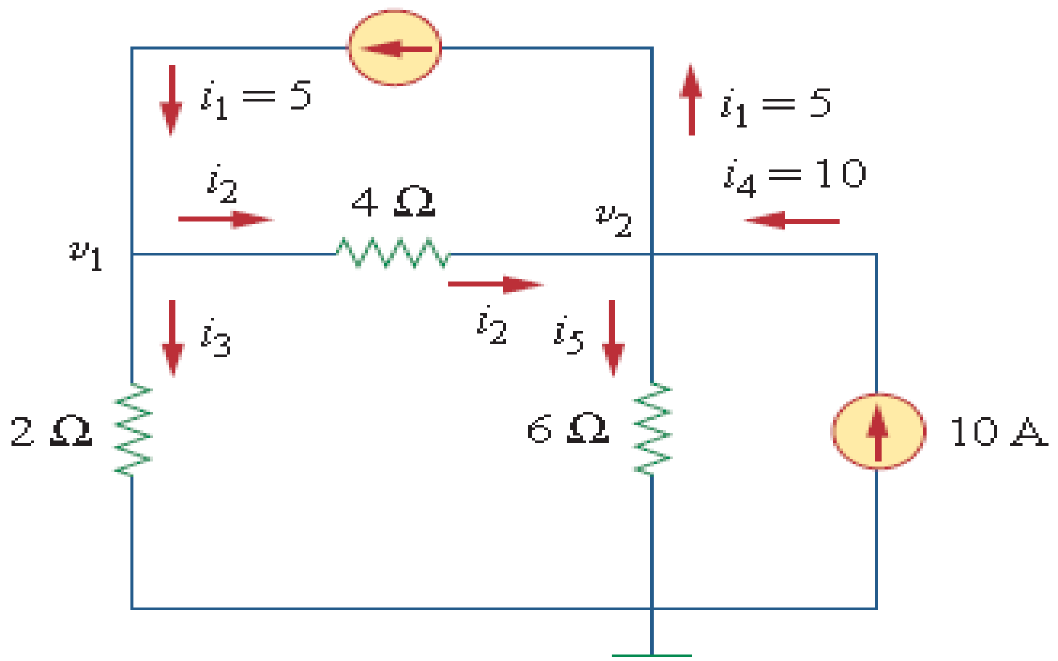
Robotics

METHOD 2

$$\begin{bmatrix} G_1 + G_2 & -G_2 \\ -G_2 & G_2 + G_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} =$$

$$\begin{bmatrix} 3 & -1 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 60 \end{bmatrix}$$

$$\Delta = \begin{vmatrix} 3 & -1 \\ -3 & 5 \end{vmatrix} = 15 - 3 = 12$$



$$v_1 = \frac{\Delta_1}{\Delta} = \frac{\begin{vmatrix} 20 & -1 \\ 60 & 5 \end{vmatrix}}{\Delta} = \frac{100 + 60}{12} = 13.333\text{ V}$$

$$v_2 = \frac{\Delta_2}{\Delta} = \frac{\begin{vmatrix} 3 & 20 \\ -3 & 60 \end{vmatrix}}{\Delta} = \frac{180 + 60}{12} = 20\text{ V}$$

If we need the currents, we can easily calculate them from the values of the nodal voltages.

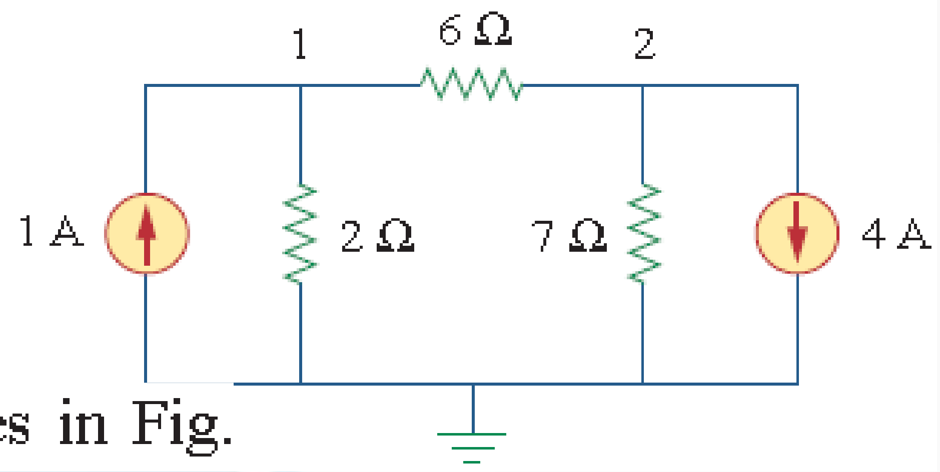
$$i_1 = 5\text{ A}, \qquad i_2 = \frac{v_1 - v_2}{4} = -1.6668\text{ A}, \qquad i_3 = \frac{v_1}{2} = 6.666\text{ A}$$

$$i_4 = 10\text{ A}, \qquad i_5 = \frac{v_2}{6} = 3.333\text{ A}$$

The fact that i_2 is negative shows that the current flows in the direction opposite to the one assumed.

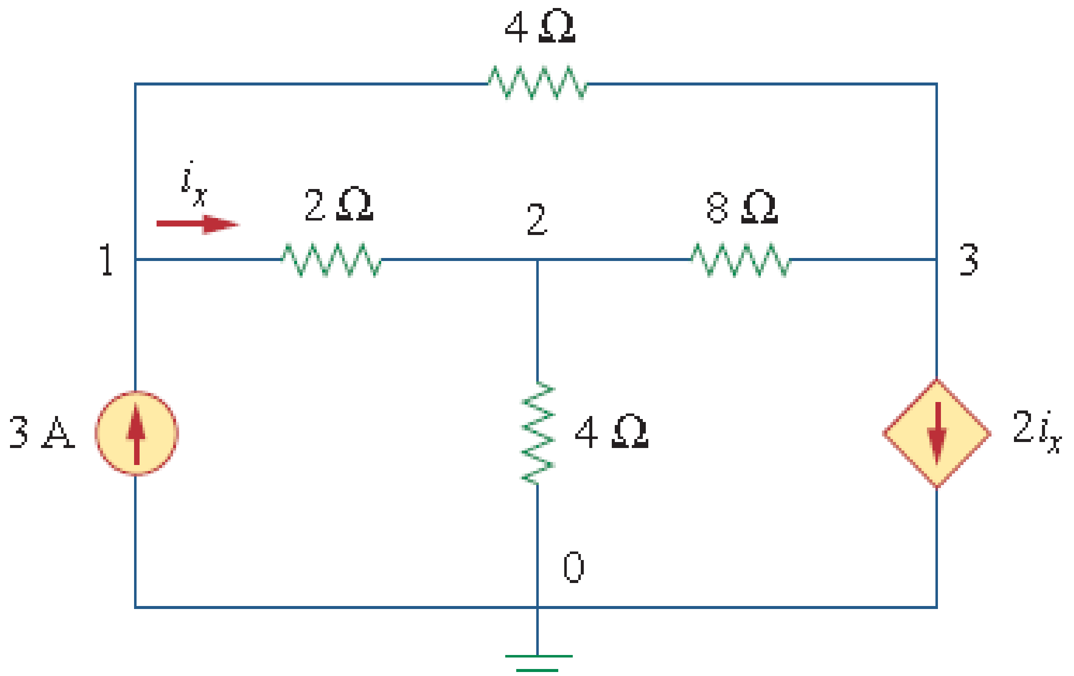
Practice Problem

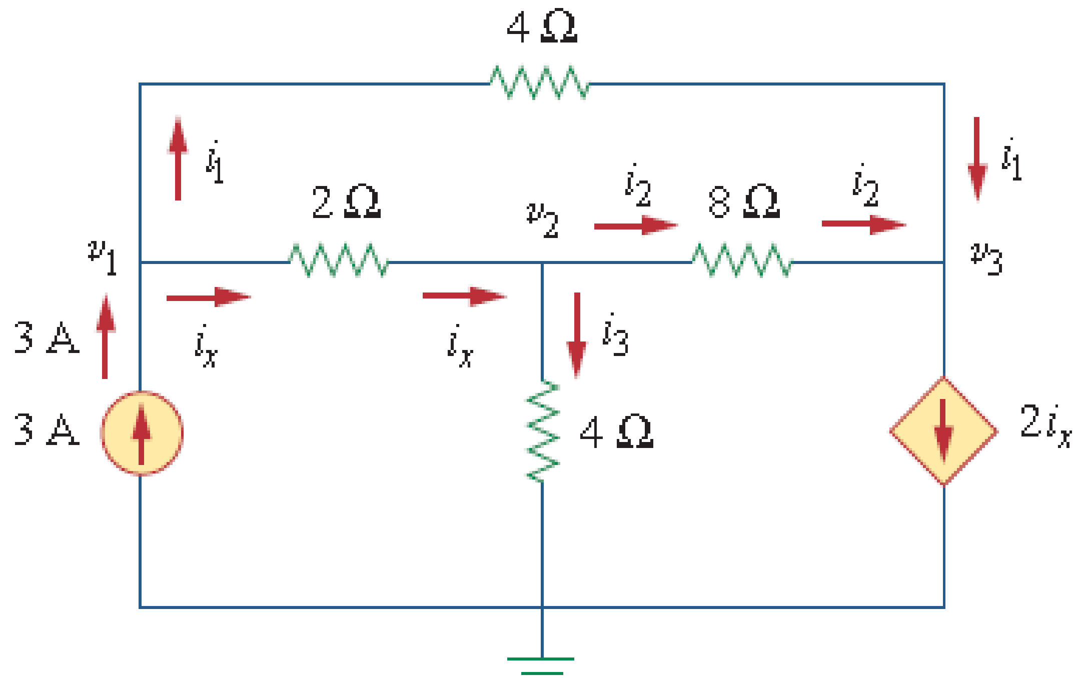
Obtain the node voltages in the circuit of Fig.



Answer: $v_1 = -2$ V, $v_2 = -14$ V.

Example Determine the voltages at the nodes in Fig.





At node 1,

$$3 = i_1 + i_x \quad \Rightarrow \quad 3 = \frac{v_1 - v_3}{4} + \frac{v_1 - v_2}{2}$$

$$3v_1 - 2v_2 - v_3 = 12$$

At node 2,

$$i_x = i_2 + i_3 \quad \Rightarrow \quad \frac{v_1 - v_2}{2} = \frac{v_2 - v_3}{8} + \frac{v_2 - 0}{4}$$

$$-4v_1 + 7v_2 - v_3 = 0$$

At node 3,

$$i_1 + i_2 = 2i_x \quad \Rightarrow \quad \frac{v_1 - v_3}{4} + \frac{v_2 - v_3}{8} = \frac{2(v_1 - v_2)}{2}$$

$$2v_1 - 3v_2 + v_3 = 0$$

$$\begin{bmatrix} 3 & -2 & -1 \\ -4 & 7 & -1 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix}$$

From this, we obtain

$$v_1 = \frac{\Delta_1}{\Delta}, \qquad v_2 = \frac{\Delta_2}{\Delta}, \qquad v_3 = \frac{\Delta_3}{\Delta}$$

$$\Delta = \begin{vmatrix} 3 & -2 & -1 \\ -4 & 7 & -1 \\ 2 & -3 & 1 \end{vmatrix} = \begin{array}{c|ccc|c} & 3 & -2 & -1 & \\ & -4 & 7 & -1 & \\ & 2 & -3 & 1 & \\ - & 3 & -2 & -1 & + \\ - & -4 & 7 & -1 & + \\ - & & & & + \end{array}$$

$$= 21 - 12 + 4 + 14 - 9 - 8 = 10$$

$$\Delta_1 = \begin{array}{c|ccc|c} & 12 & -2 & -1 & \\ & 0 & 7 & -1 & \\ & 0 & -3 & 1 & \\ - & 12 & -2 & -1 & + \\ - & 0 & 7 & -1 & + \\ - & & & & + \end{array}$$

$$= 84 + 0 + 0 - 0 - 36 - 0 = 48$$

$$\Delta_2 = \begin{array}{c|ccc|c} & 3 & 12 & -1 & \\ & -4 & 0 & -1 & \\ & 2 & 0 & 1 & \\ - & 3 & 12 & -1 & + \\ - & -4 & 0 & -1 & + \\ - & & & & + \end{array}$$

$$= 0 + 0 - 24 - 0 - 0 + 48 = 24$$

$$\Delta_3 = \begin{array}{c|ccc|c} & 3 & -2 & 12 & \\ & -4 & 7 & 0 & \\ & 2 & -3 & 0 & \\ - & 3 & -2 & 12 & + \\ - & -4 & 7 & 0 & + \\ - & & & & + \end{array}$$

$$= 0 + 144 + 0 - 168 - 0 - 0 = -24$$

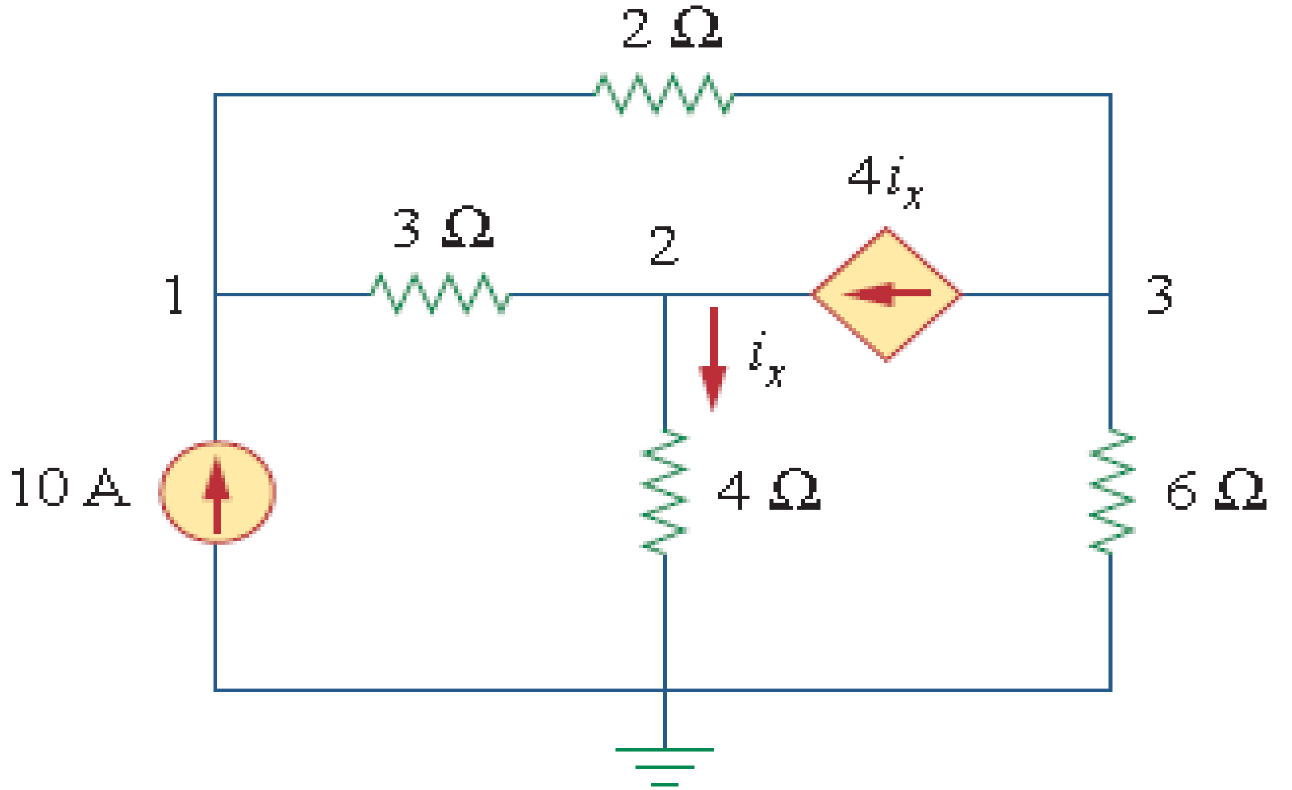
$$v_1 = \frac{\Delta_1}{\Delta} = \frac{48}{10} = 4.8 \text{ V},$$

$$v_2 = \frac{\Delta_2}{\Delta} = \frac{24}{10} = 2.4 \text{ V}$$

$$v_3 = \frac{\Delta_3}{\Delta} = \frac{-24}{10} = -2.4 \text{ V}$$

Practice Problem

Find the voltages at the three nonreference nodes in the circuit



Answer: $v_1 = 80$ V, $v_2 = -64$ V, $v_3 = 156$ V.

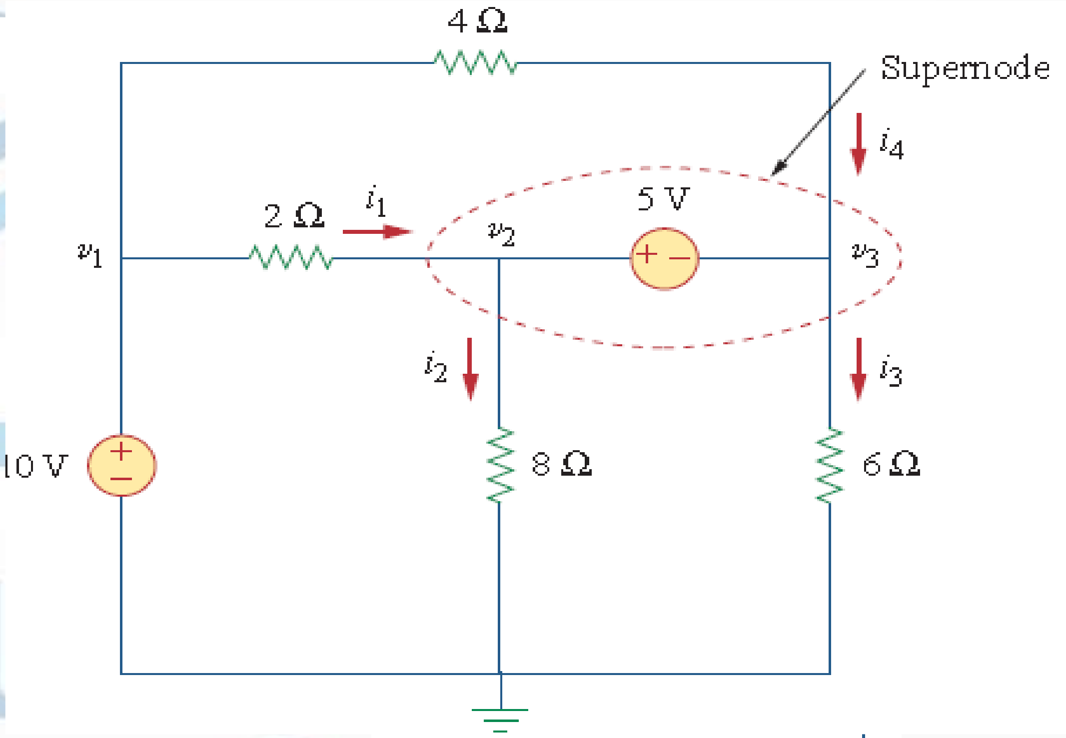
Robotics

Nodal Analysis with Voltage Source

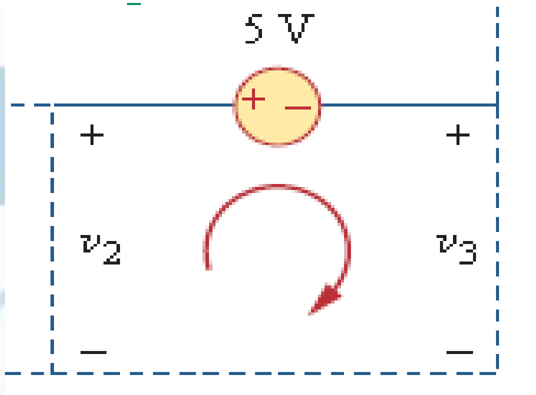
$$v_1 = 10\ \text{V}$$

$$i_1 + i_4 = i_2 + i_3$$

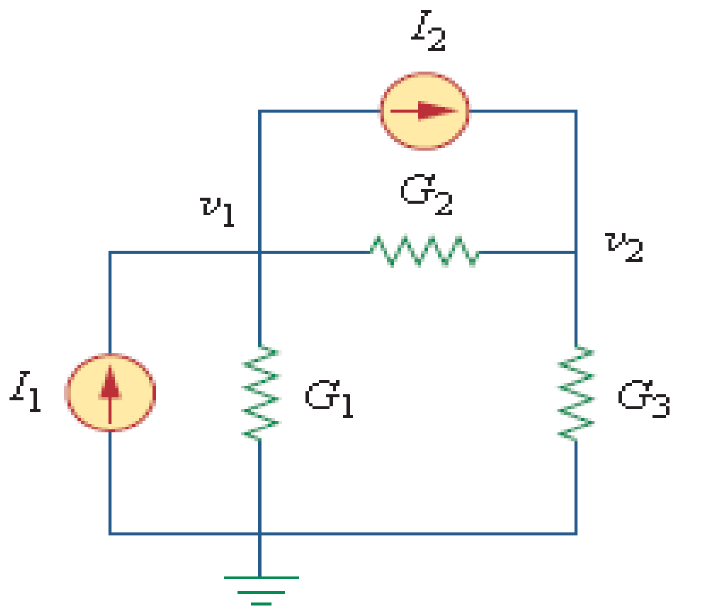
$$\frac{v_1 - v_2}{2} + \frac{v_1 - v_3}{4} = \frac{v_2 - 0}{8} + \frac{v_3 - 0}{6}$$



$$-v_2 + 5 + v_3 = 0 \quad \Rightarrow \quad v_2 - v_3 = 5$$



Nodal and Mesh Analyses by Inspection



$$\begin{bmatrix} G_1 + G_2 & -G_2 \\ -G_2 & G_2 + G_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} I_1 - I_2 \\ I_2 \end{bmatrix}$$

$$\begin{bmatrix} G_{11} & G_{12} & \dots & G_{1N} \\ G_{21} & G_{22} & \dots & G_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ G_{N1} & G_{N2} & \dots & G_{NN} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix} = \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_N \end{bmatrix}$$

where

G_{kk} = Sum of the conductances connected to node k

$G_{kj} = G_{jk}$ = Negative of the sum of the conductances directly connecting nodes k and j, $k \neq j$

v_k = Unknown voltage at node k

i_k = Sum of all independent current sources directly connected to node k, with currents entering the node treated as positive

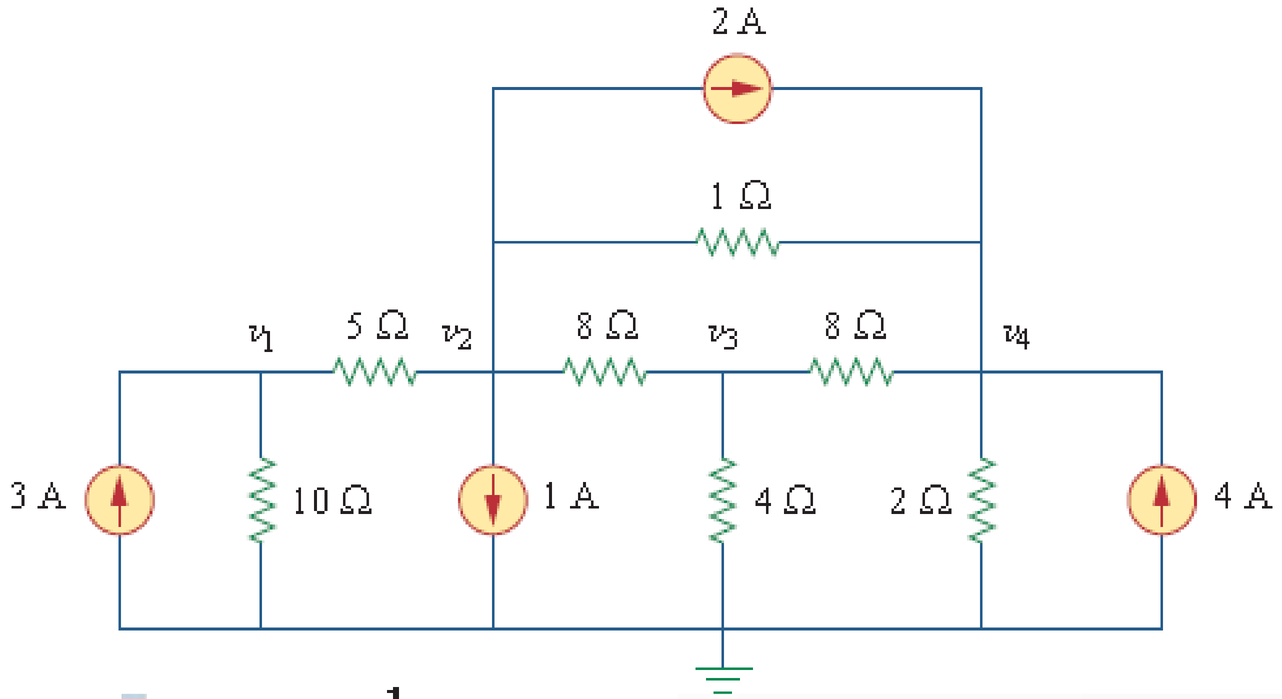
Robotics

$$G_{11} = \frac{1}{5} + \frac{1}{10} = 0.3,$$

$$G_{33} = \frac{1}{8} + \frac{1}{8} + \frac{1}{4} = 0.5,$$

$$G_{22} = \frac{1}{5} + \frac{1}{8} + \frac{1}{1} = 1.325$$

$$G_{44} = \frac{1}{8} + \frac{1}{2} + \frac{1}{1} = 1.625$$



$$G_{12} = -\frac{1}{5} = -0.2, \qquad G_{13} = G_{14} = 0 \qquad G_{34} = -\frac{1}{8} = -0.125$$

$$G_{23} = -\frac{1}{8} = -0.125, \qquad G_{24} = -\frac{1}{1} = -1 \qquad G_{41} = 0,$$

The input current vector **i** has the following terms, in amperes:

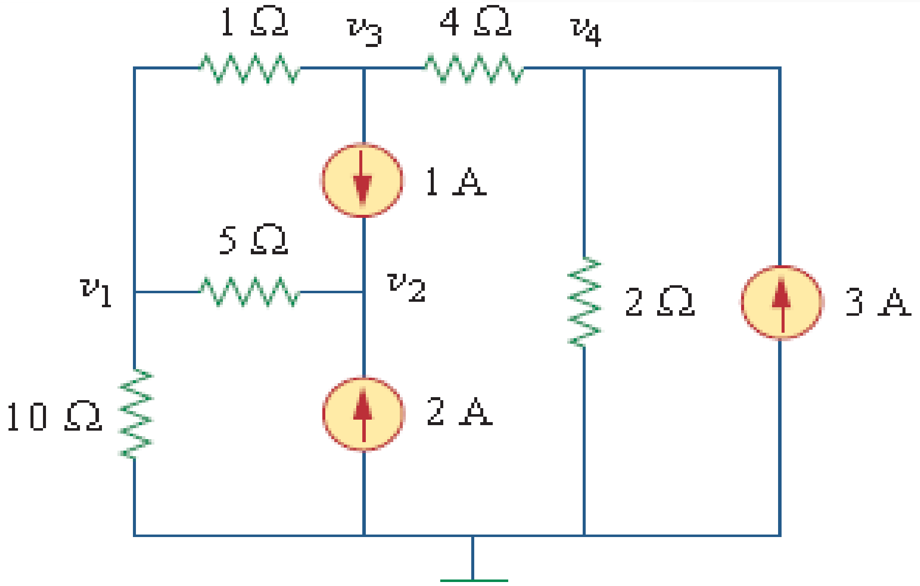
$$i_1 = 3, \qquad i_2 = -1 - 2 = -3, \qquad i_3 = 0, \qquad i_4 = 2 + 4 = 6$$

Thus the node-voltage equations are

$$\begin{bmatrix} 0.3 & -0.2 & 0 & 0 \\ -0.2 & 1.325 & -0.125 & -1 \\ 0 & -0.125 & 0.5 & -0.125 \\ 0 & -1 & -0.125 & 1.625 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 0 \\ 6 \end{bmatrix}$$

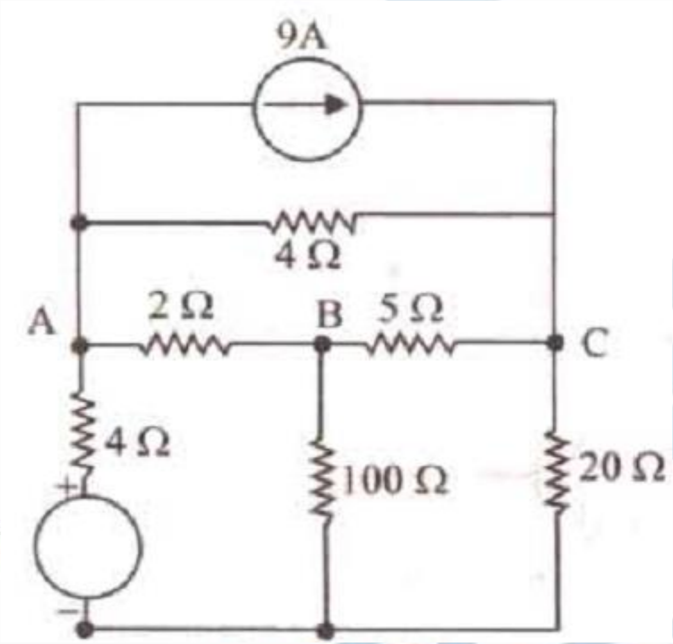
Practice Problem

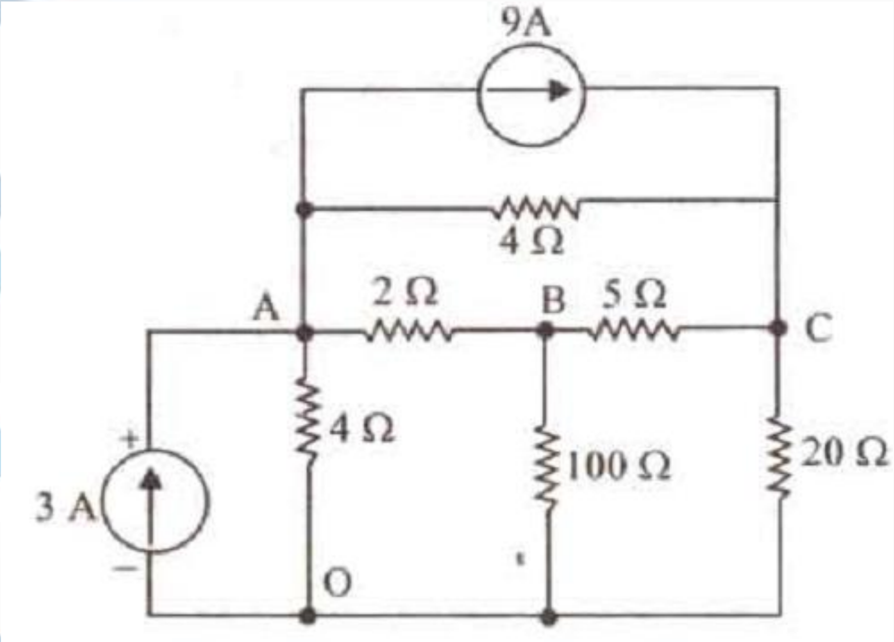
By inspection, obtain the node-voltage equations for the circuit



Answer:

$$\begin{bmatrix} 1.3 & -0.2 & -1 & 0 \\ -0.2 & 0.2 & 0 & 0 \\ -1 & 0 & 1.25 & -0.25 \\ 0 & 0 & -0.25 & 0.75 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -1 \\ 3 \end{bmatrix}$$





The potentials of three nodes to be found are : V_A, V_B, V_C

$$\begin{bmatrix} 1 & -0.5 & -0.25 \\ -0.5 & 0.71 & -0.20 \\ -0.25 & -0.20 & 0.5 \end{bmatrix} \begin{bmatrix} V_A \\ V_B \\ V_C \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \\ 9 \end{bmatrix}$$

$$\Delta = \begin{vmatrix} 1 & -0.5 & -0.25 \\ -0.5 & 0.71 & -0.20 \\ -0.25 & -0.20 & 0.5 \end{vmatrix}$$

$$\Delta = 1\times(0.710.5 - 0.04) + 0.5\,(-0.25 - 0.05) - 0.25\,(0.1 + 0.71\times 0.25) = 0.315 - 0.15 - 0.069375 = 0.095625$$

$$\Delta_a = \begin{vmatrix} -6 & -0.5 & -0.25 \\ -0.5 & 0.71 & -0.20 \\ 9 & -0.20 & +0.5 \end{vmatrix} = +0.6075$$

$$\Delta_b = \begin{vmatrix} 1 & -6 & -0.25 \\ -0.5 & 0 & -0.20 \\ -0.25 & 9 & 0.50 \end{vmatrix} = 1.125$$

$$\Delta_c = \begin{vmatrix} 1 & -0.5 & -6 \\ -0.5 & 0.71 & 0 \\ -0.25 & -0.20 & 9 \end{vmatrix} = 2.2475$$

$V_A = \Delta_a/\Delta = +0/6075/0.095625 = 6.353$ volts

$V_B = \Delta_b/\Delta = 1.125/0.095625 = 11.765$ volts

$V_C = \Delta_c/\Delta = 2.475/0.95625 = 25.882$ volts