Civil Engineering

CEVC201

Calculus 1

Lecture 7

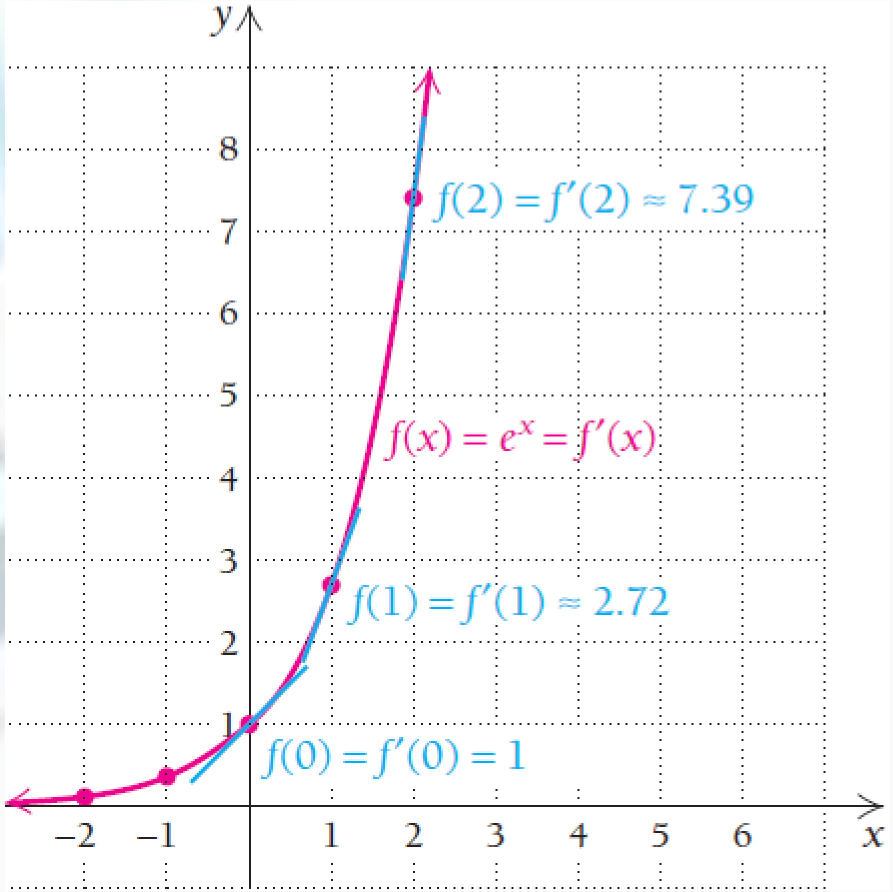
Some Special Functions

Chapter 5
Some Special Functions

5.1 Exponential functions

Definition: The number e, sometimes called the natural number, or Euler's number is defined by $e = \lim_{h \to 0} (1+h)^{1/h} \approx 2.718281828$

Definition: The exponential function e^x is defined for all $x \in R$ by $f(x) = e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$



Note:

$$(e^x)' = 0 + 1 + \frac{2x}{2!} + \frac{3x}{3!} + \cdots = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x$$

Theorem: The derivative of the function f given by $f(x) = e^x$ is itself $f'(x) = f(x)$ or $(e^x)' = e^x$

Note: $e^1 = e = 1 + 1 + \dfrac{1}{2!} + \dfrac{1}{3!} + \cdots$

$$\frac{d}{dx}(3e^x) = 3\frac{d}{dx}e^x = 3e^x$$

$$\frac{d}{dx}(x^2e^x) = 2xe^x + x^2e^x = xe^x(x+2)$$

Theorem: $\frac{d}{dx}e^u = e^u\frac{du}{dx}$

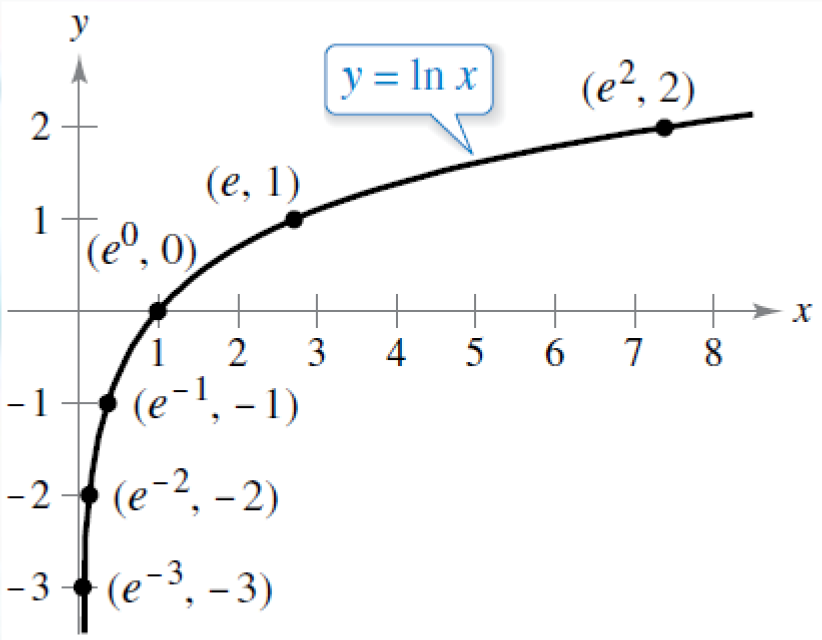
$$\frac{d}{dx}(e^{x^2-5x}) = e^{x^2-5x}(x^2-5x)' = e^{x^2-5x}(2x-5)$$

$$\frac{d}{dx}e^{\sqrt{x^2-3}} = e^{\sqrt{x^2-3}}(\sqrt{x^2-3})' = e^{\sqrt{x^2-3}}\tfrac{1}{2}(x^2-3)^{-1/2}\cdot 2x = \frac{xe^{\sqrt{x^2-3}}}{\sqrt{x^2-3}}$$

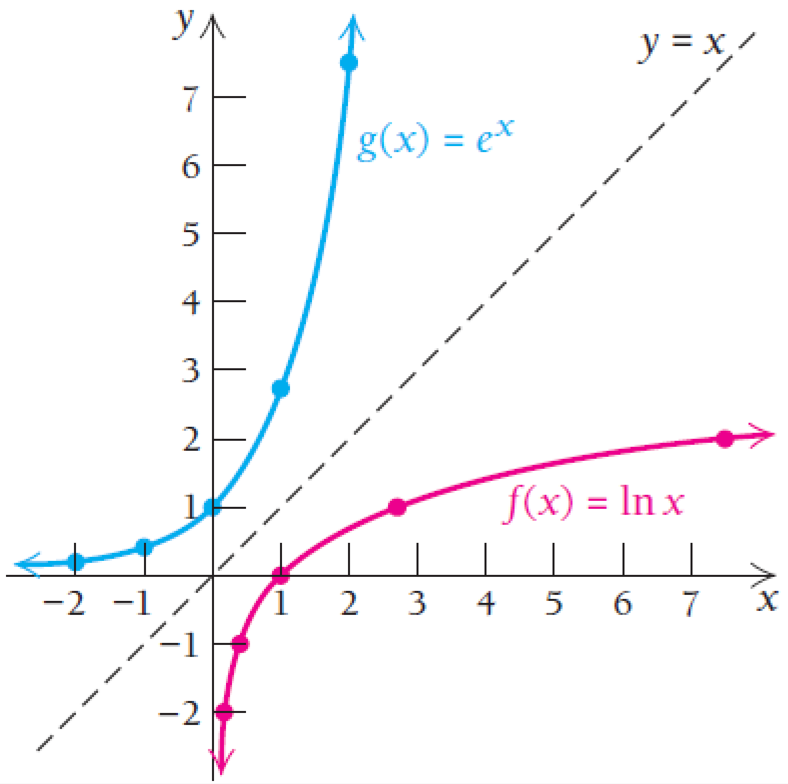
5.2 Logarithmic functions

Definition: The Natural Logarithmic function ln$x$, is defined as follows

$$\ln x = y \Leftrightarrow e^y = x, \quad x > 0$$



Note: each function of e^x and ln x is the inverse of the other



ln$e^x = x$, for all $x \in R$

$e^{\ln x} = x$, for all $x > 0$

Theorem:

ln$x$ exists only for positive numbers x. The domain is (0, ∞)

ln$x$ < 0 for 0 < x < 1

ln$x$ = 0 when x = 1

ln$x$ > 0 for x > 1

The function given by $f(x)$ = ln$x$ is always increasing. The range is the entire real line, (–∞, ∞), or the set of real numbers, R.

Theorem: For any positive number x, $\frac{d}{dx}\ln x = \frac{1}{x}$

$$\frac{d}{dx}(x^2\ln x + 5x) = \frac{d}{dx}(x^2\ln x) + \frac{d}{dx}(5x) = 2x\cdot\ln x + x^2\frac{1}{x} + 5 = 2x\cdot\ln x + x + 5$$

Theorem: $\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$

$$\frac{d}{dx}\ln(x^2-5x)=\frac{2x-5}{x^2-5x}$$

$$\ln(AB)=\ln A+\ln B \quad A, B>0$$

$$\ln\frac{A}{B}=\ln A-\ln B \quad A, B>0$$

$$\frac{d}{dx}\ln\frac{x^2-5}{x}=\frac{d}{dx}\left[\ln(x^2-5)-\ln x\right]=\frac{d}{dx}\ln(x^2-5)-\frac{d}{dx}\ln x=\frac{2x}{x^2-5}-\frac{1}{x}$$

5.3 Inverse Trigonometric Functions

Definitions: The Inverse Trigonometric Functions

The **inverse sine** function:

$$y = \arcsin x = \operatorname{asin} x = \sin^{-1} x \Leftrightarrow x = \sin y,\ -1 \le x \le 1 \text{ and } -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

The **inverse cosine** function:

$$y = \arccos x = \operatorname{acos} x = \cos^{-1} x \Leftrightarrow x = \cos y,\ -1 \le x \le 1 \text{ and } 0 \le y \le \pi$$

The **inverse tangent** function:

$$y = \arctan x = \operatorname{atan} x = \tan^{-1} x \Leftrightarrow x = \tan y,\ -\infty \le x \le \infty \text{ and } -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

$$\cos^{-1}\frac{1}{2} = \frac{\pi}{3} \qquad \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3} \qquad \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6} \qquad \sin^{-1}-\frac{1}{2} = -\frac{\pi}{6}$$

$$\sin^{-1} 0.3 \approx 0.305$$

$$y = \operatorname{arccot} x = \operatorname{acot} x = \cot^{-1} x \Leftrightarrow x = \cot y, \ -\infty \le x \le \infty \text{ and } 0 \le y \le \pi$$

$$y = \operatorname{arcsec} x = \operatorname{asec} x = \sec^{-1} x \Leftrightarrow x = \sec y, \ |x| \ge 1 \text{ and } 0 \le y \le \pi, \quad y \ne \frac{\pi}{2}$$

$$y = \operatorname{arccsc} x = \operatorname{acsc} x = \csc^{-1} x \Leftrightarrow x = \csc y, \ |x| \ge 1 \text{ and } -\frac{\pi}{2} \le y \le \frac{\pi}{2}, \quad y \ne 0$$

Properties of Inverse Trigonometric Functions

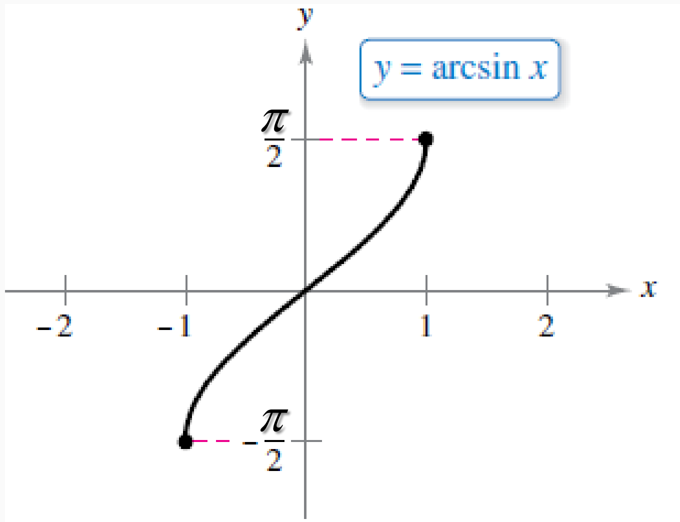
$$\text{if } -1 \le x \le 1 \text{ and } -\frac{\pi}{2} \le y \le \frac{\pi}{2}, \text{ then } \quad \sin(\arcsin x) = x \quad \text{and} \quad \arcsin(\sin y) = y$$

$$\text{if } -\frac{\pi}{2} \le y \le \frac{\pi}{2}, \text{ then } \quad \tan(\arctan x) = x \quad \text{and} \quad \arctan(\tan y) = y$$

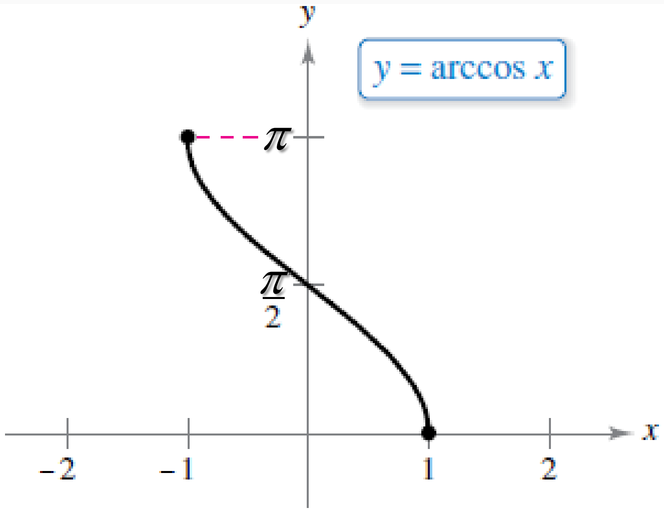
$$\text{if } |x| \ge 1 \text{ and } 0 \le y \le \frac{\pi}{2} \text{ or } \frac{\pi}{2} \le y \le \pi, \text{ then } \quad \sec(\operatorname{arcsec} x) = x \quad \text{and} \quad \operatorname{arcsec}(\sec y) = y$$

$$\arctan(2x-3) = \frac{\pi}{4} \Leftrightarrow \tan[\arctan(2x-3)] = \tan\frac{\pi}{4} \Leftrightarrow 2x - 3 = 1 \Rightarrow x = 2$$

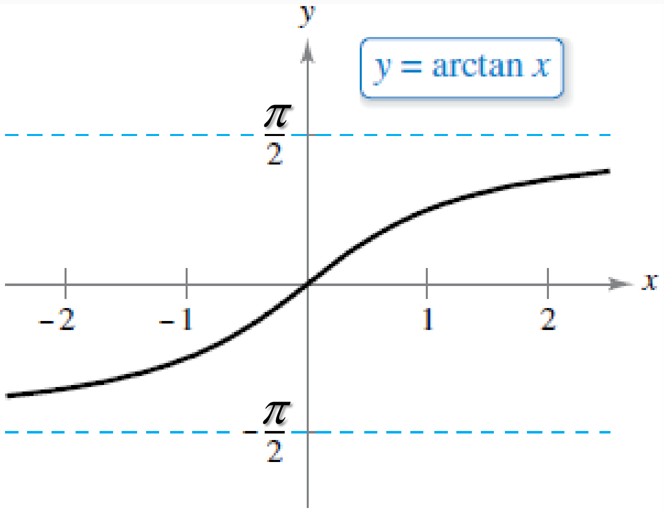
Calculus 1 (CEVC201): Some Special Functions



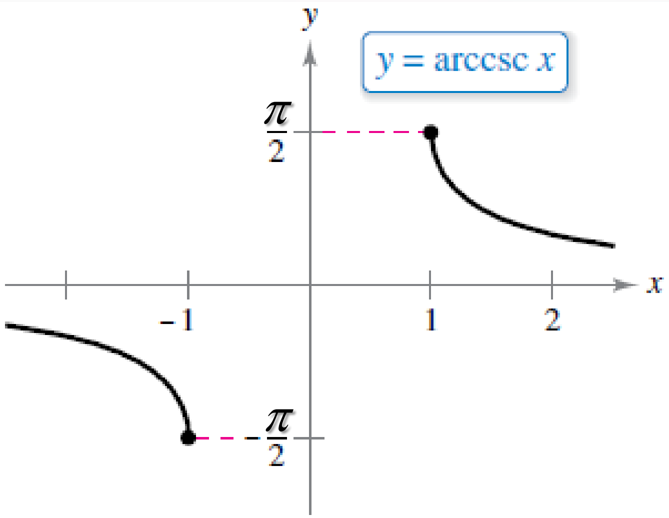
Domain: $[-1, 1]$
Range: $[-\pi/2, \pi/2]$



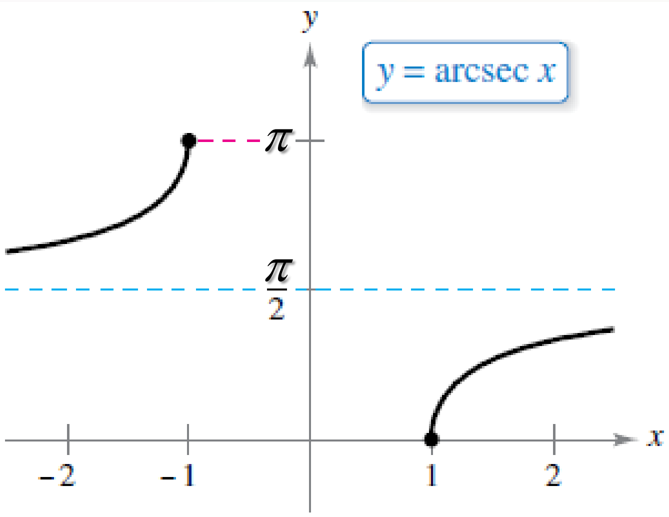
Domain: $[-1, 1]$
Range: $[0, \pi]$



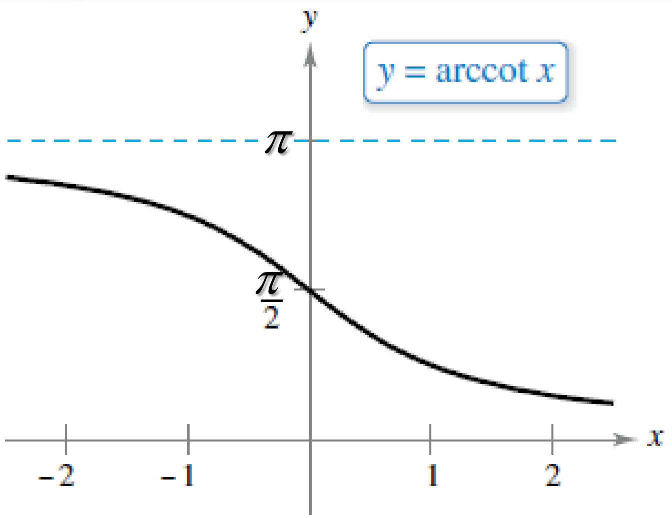
Domain: $(-\infty, \infty)$
Range: $(-\pi/2, \pi/2)$



Domain: $(-\infty, -1] \cup [1, \infty)$
Range: $[-\pi/2, 0) \cup (0, \pi/2]$



Domain: $(-\infty, -1] \cup [1, \infty)$
Range: $[0, \pi/2) \cup (\pi/2, \pi]$



Domain: $(-\infty, \infty)$
Range: $(0, \pi)$

Derivatives of Inverse Trigonometric Functions

Let u be a differentiable function of x

$$\frac{d}{dx}\arcsin u=\frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}\arccos u=\frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}\arctan u=\frac{u'}{1+u^2} \qquad \frac{d}{dx}\operatorname{arccot} u=\frac{-u'}{1+u^2}$$

$$\frac{d}{dx}\operatorname{arcsec} u=\frac{u'}{|u|\sqrt{u^2-1}} \qquad \frac{d}{dx}\operatorname{arccsc} u=\frac{-u'}{|u|\sqrt{u^2-1}}$$

$$\frac{d}{dx}\arcsin 2x=\frac{2}{\sqrt{1-4x^2}} \qquad \frac{d}{dx}\arctan x^2=\frac{2x}{1+x^4}$$

$$\frac{d}{dx}\operatorname{arcsec} e^{2x}=\frac{2e^{2x}}{e^{2x}\sqrt{e^{4x}-1}}=\frac{2}{\sqrt{e^{4x}-1}}$$

5.4 Hyperbolic Functions

Definitions: Hyperbolic Functions

$$\text{sh}x = \sinh x = \frac{e^x - e^{-x}}{2}$$

$$\text{ch}x = \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\text{th}x = \tanh x = \frac{\sinh x}{\cosh x}$$

$$\text{csch}x = \frac{1}{\sinh x}, \quad x > 0$$

$$\text{sech}x = \frac{1}{\cosh x}$$

$$\coth x = \frac{1}{\tanh x}, \quad x > 0$$

$$\cosh^2 x - \sinh x = 1$$

$$\cos^2 x + \sin x = 1$$

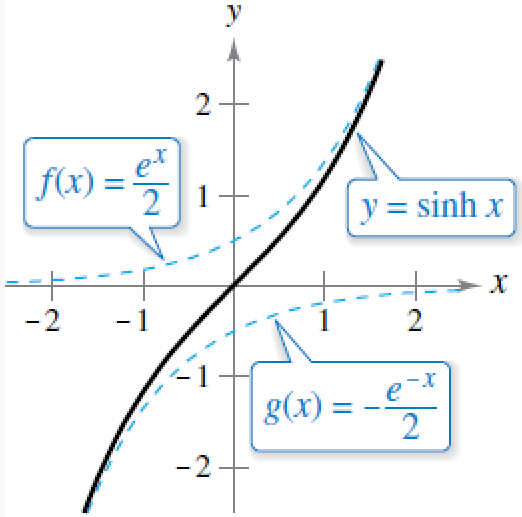
$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

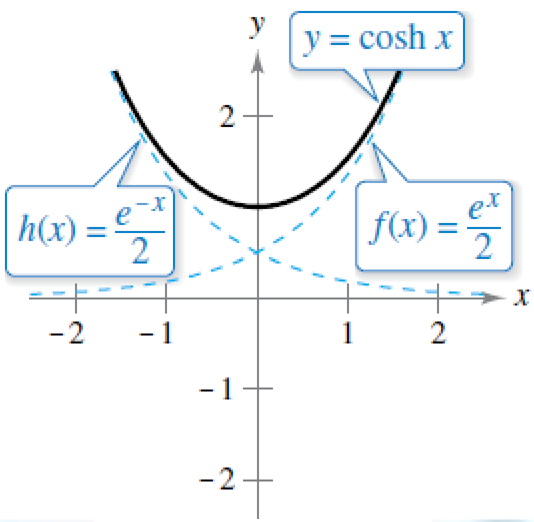
$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

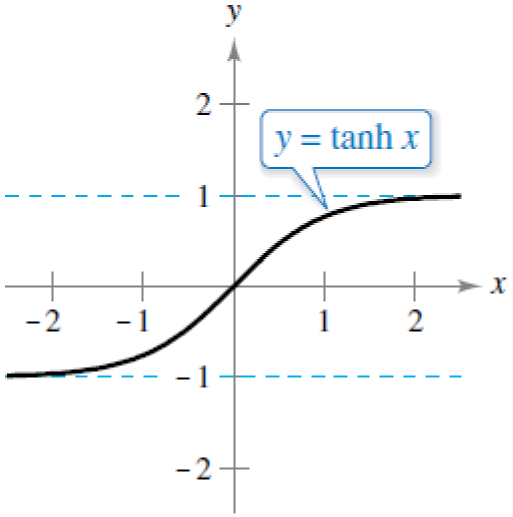
Calculus 1 (CEVC201): Some Special Functions



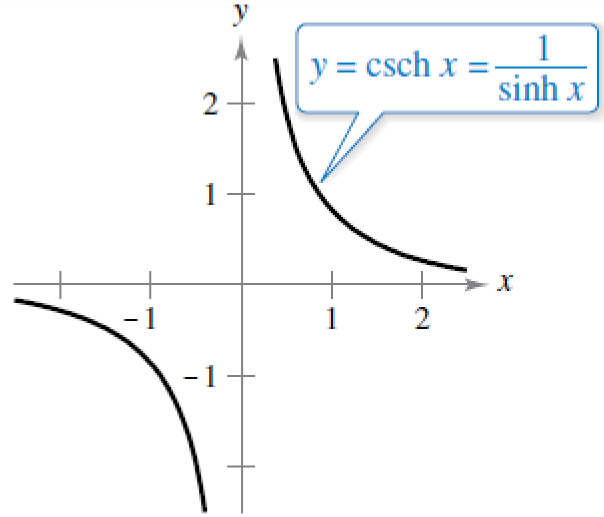
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$



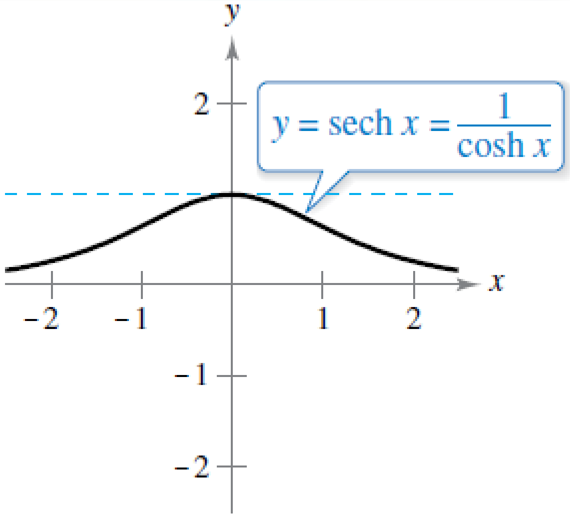
Domain: $(-\infty, \infty)$
Range: $[1, \infty)$



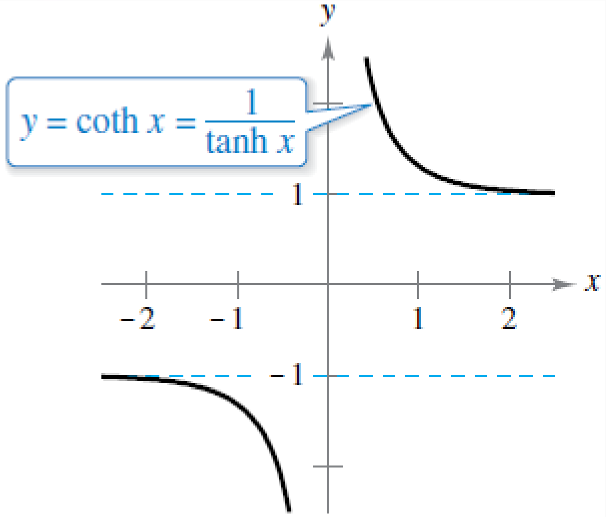
Domain: $(-\infty, \infty)$
Range: $(-1, 1)$



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$



Domain: $(-\infty, \infty)$
Range: $(0, 1]$



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, -1) \cup (1, \infty)$

Differentiation of Hyperbolic Functions

Let u be a differentiable function of x

$$\frac{d}{dx}\sinh u = \cosh u \cdot u'$$

$$\frac{d}{dx}\coth u = -\text{csch}^2 u \cdot u'$$

$$\frac{d}{dx}\cosh u = \sinh u \cdot u'$$

$$\frac{d}{dx}\text{sech}\, u = -\text{sech}\, u \cdot \tanh u \cdot u'$$

$$\frac{d}{dx}\tanh u = \text{sech}^2 u \cdot u'$$

$$\frac{d}{dx}\text{csch}\, u = -\text{csch}\, u \cdot \coth u \cdot u'$$

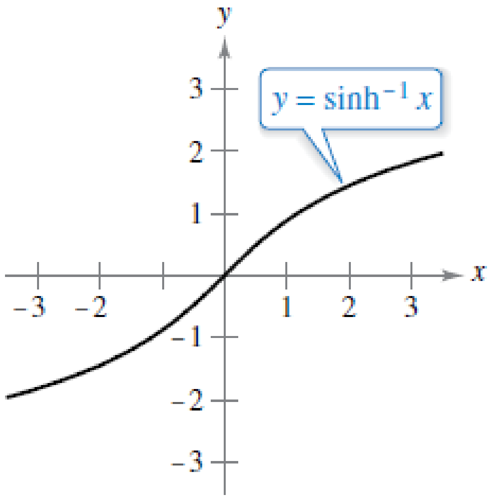
$$\frac{d}{dx}\sinh(x^2-3) = 2x\cosh(x^2-3)$$

$$\frac{d}{dx}\ln(\cosh x) = \frac{\sinh x}{\cosh x} = \tanh x$$

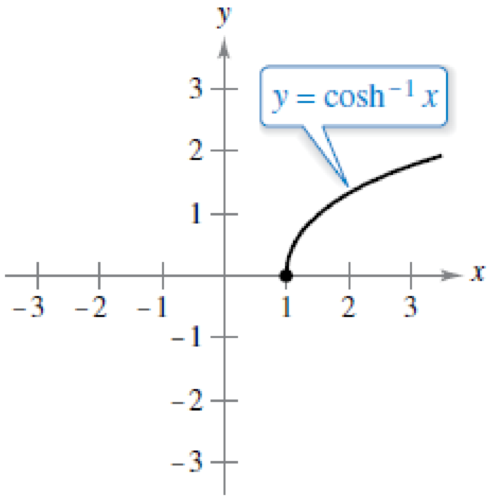
$$\frac{d}{dx}\left[x\sinh x - \cosh x\right] = x\cosh x + \sinh x - \sinh x = x\cosh x$$

5.5 Inverse Hyperbolic Functions

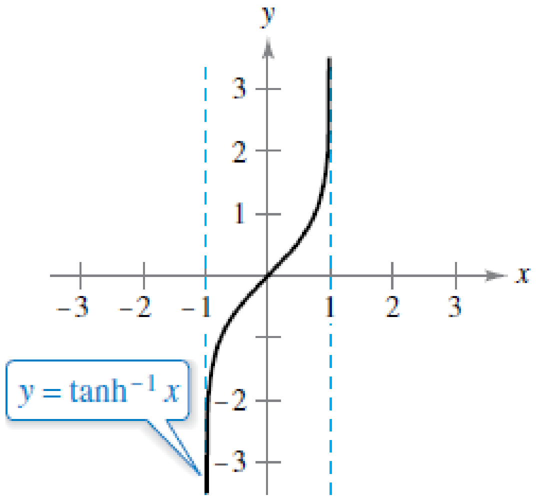
$\sinh^{-1}x = \ln(x+\sqrt{x^2+1})$	$(-\infty,\infty)$
$\cosh^{-1}x = \ln(x+\sqrt{x^2-1})$	$[1,\infty)$
$\tanh^{-1}x = \dfrac{1}{2}\ln\dfrac{1+x}{1-x}$	$(-1,1)$
$\coth^{-1}x = \dfrac{1}{2}\ln\dfrac{x+1}{x-1}$	$(-\infty,-1)\cup(1,\infty)$
$\operatorname{sech}^{-1}x = \ln\dfrac{1+\sqrt{1-x^2}}{x}$	$(0,1]$
$\operatorname{csch}^{-1}x = \ln\left(\dfrac{1}{x}+\dfrac{\sqrt{1+x^2}}{\lvert x\rvert}\right)$	$(-\infty,0)\cup(0,\infty)$



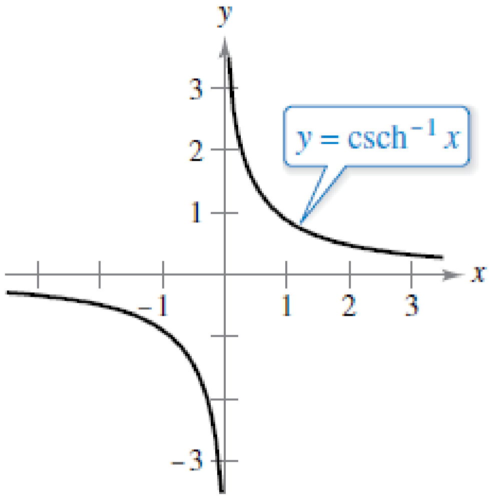
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$



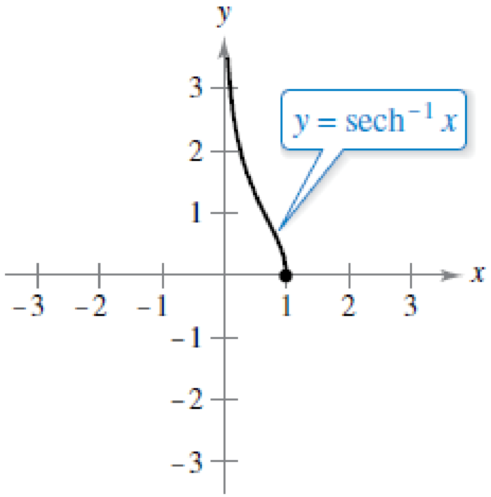
Domain: $[1, \infty)$
Range: $[0, \infty)$



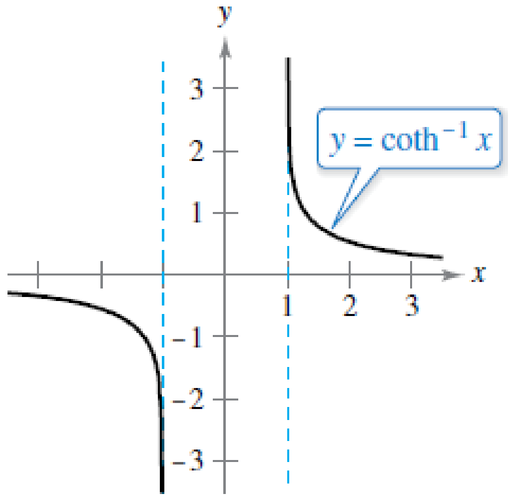
Domain: $(-1, 1)$
Range: $(-\infty, \infty)$



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$



Domain: $(0, 1]$
Range: $[0, \infty)$



Domain: $(-\infty, -1) \cup (1, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$

Derivatives of Inverse Hyperbolic Functions

Let u be a differentiable function of x

$$\frac{d}{dx}\sinh^{-1}u = \frac{u'}{\sqrt{u^2+1}} \qquad \frac{d}{dx}\coth^{-1}u = \frac{u'}{\sqrt{u^2-1}}$$

$$\frac{d}{dx}\cosh^{-1}u = \frac{u'}{\sqrt{u^2-1}} \qquad \frac{d}{dx}\operatorname{sech}^{-1}u = \frac{-u'}{u\sqrt{1-u^2}}$$

$$\frac{d}{dx}\tanh^{-1}u = \frac{u'}{1-u^2} \qquad \frac{d}{dx}\operatorname{csch}^{-1}u = \frac{-u'}{|u|\sqrt{1+u^2}}$$

$$\frac{d}{dx}\sinh^{-1}2x = \frac{2}{\sqrt{4x^2+1}}$$

$$\frac{d}{dx}\tanh^{-1}x^3 = \frac{3x^2}{1-x^2}$$