Sequences and Summations

Epp, section 4.1

Definitions

- Sequence: an ordered list of elements
  - Like a set, but:
    - Elements can be duplicated
    - Elements are ordered

Sequences

- A sequence is a function from a subset of **Z** to a set S
  - Usually from the positive or non-negative ints
  - a_n is the image of n

- a_n is a term in the sequence
- $\{a_n\}$ means the entire sequence
  - The same notation as sets!

Sequence examples

- $a_n = 3n$
  - The terms in the sequence are a_1, a_2, a_3, …
  - The sequence $\{a_n\}$ is { 3, 6, 9, 12, … }

- $b_n = 2^n$
  - The terms in the sequence are b_1, b_2, b_3, …
  - The sequence $\{b_n\}$ is { 2, 4, 8, 16, 32, … }

- Note that sequences are indexed from 1
  - Not in all other textbooks, though!

Geometric vs. arithmetic sequences

- The difference is in how they grow

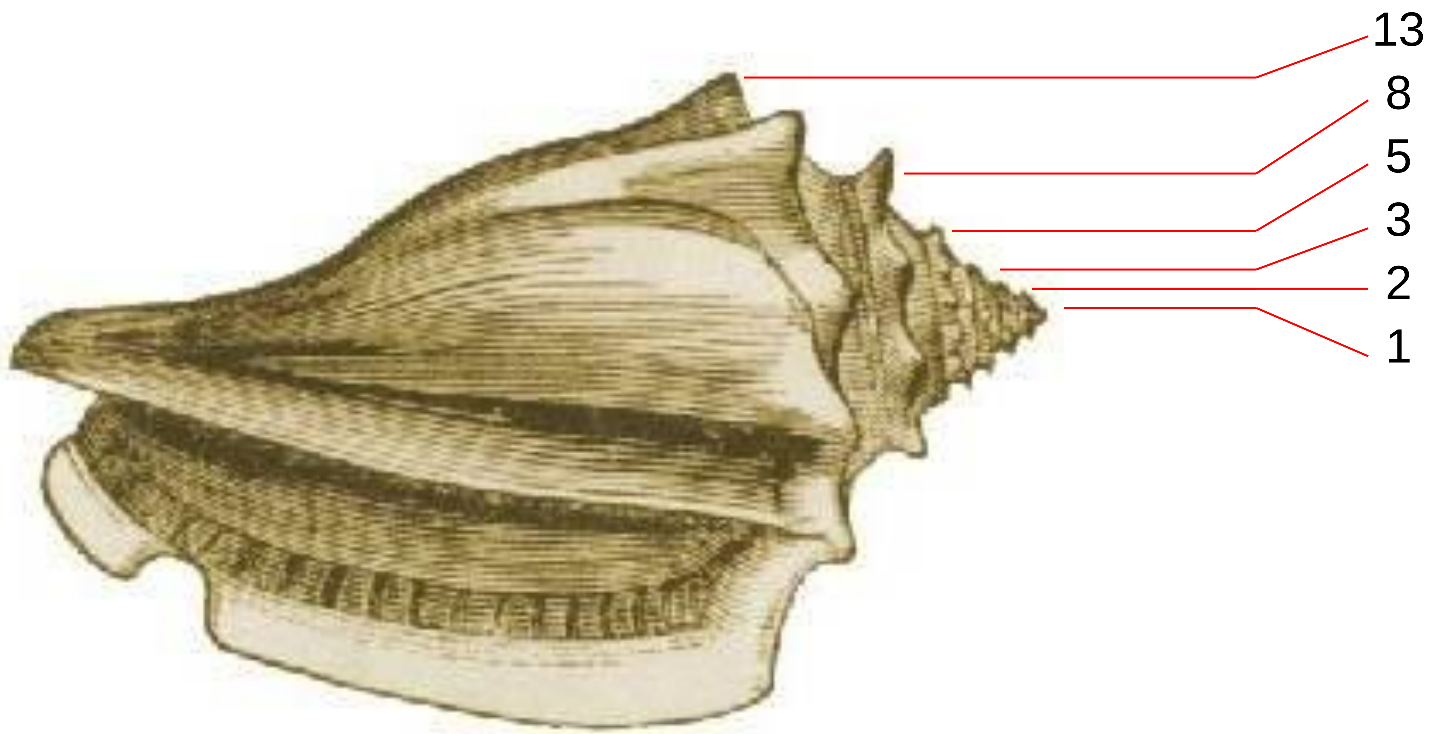
- Arithmetic sequences increase by a constant *amount*
  - $a_n = 3n$
  - The sequence $\{a_n\}$ is { 3, 6, 9, 12, … }
  - Each number is 3 more than the last
  - Of the form: $f(x) = dx + a$

- Geometric sequences increase by a constant *factor*
  - $b_n = 2^n$
  - The sequence $\{b_n\}$ is { 2, 4, 8, 16, 32, … }
  - Each number is twice the previous
  - Of the form: $f(x) = ar^x$

Fibonacci sequence

- Sequences can be neither geometric or arithmetic
  - $F_n = F_{n-1} + F_{n-2}$, where the first two terms are 1
    - Alternative, $F(n) = F(n-1) + F(n-2)$
  - Each term is the sum of the previous two terms
  - Sequence: { 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, … }
  - This is the Fibonacci sequence
  - Full formula: $$F(n) = \frac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{\sqrt{5}\cdot 2^n}$$

Fibonacci sequence in nature



Reproducing rabbits

- You have one pair of rabbits on an island
  - The rabbits repeat the following:
    - Get pregnant one month
    - Give birth (to another pair) the next month
  - This process repeats indefinitely (no deaths)
  - Rabbits get pregnant the month they are born

- How many rabbits are there after 10 months?

Reproducing rabbits

- First month: 1 pair
  - The original pair
- Second month: 1 pair
  - The original (and now pregnant) pair
- Third month: 2 pairs
  - The child pair (which is pregnant) and the parent pair (recovering)
- Fourth month: 3 pairs
  - “Grandchildren”: Children from the baby pair (now pregnant)
  - Child pair (recovering)
  - Parent pair (pregnant)
- Fifth month: 5 pairs
  - Both the grandchildren and the parents reproduced
  - 3 pairs are pregnant (child and the two new born rabbits)

Reproducing rabbits

- Sixth month: 8 pairs
  - All 3 new rabbit pairs are pregnant, as well as those not pregnant in the last month (2)
- Seventh month: 13 pairs
  - All 5 new rabbit pairs are pregnant, as well as those not pregnant in the last month (3)
- Eighth month: 21 pairs
  - All 8 new rabbit pairs are pregnant, as well as those not pregnant in the last month (5)
- Ninth month: 34 pairs
  - All 13 new rabbit pairs are pregnant, as well as those not pregnant in the last month (8)
- Tenth month: 55 pairs
  - All 21 new rabbit pairs are pregnant, as well as those not pregnant in the last month (13)

Reproducing rabbits

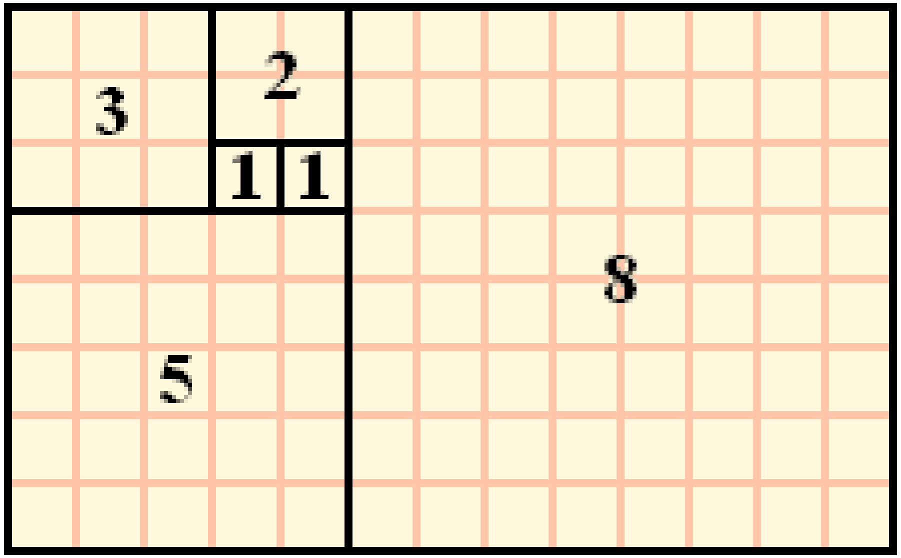
- Note the sequence:

  { 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, … }

- The Fibonacci sequence again

Fibonacci sequence

- Another application:



- Fibonacci references from http://en.wikipedia.org/wiki/Fibonacci_sequence

Fibonacci sequence

- As the terms increase, the ratio between successive terms approaches 1.618

$$\lim_{n\to\infty} \frac{F(n+1)}{F(n)} = \phi = \frac{\sqrt{5}+1}{2} = 1.618933989$$

- This is called the “golden ratio”
  - Ratio of human leg length to arm length
  - Ratio of successive layers in a conch shell

- Reference: http://en.wikipedia.org/wiki/Golden_ratio

Determining the sequence formula

- Given values in a sequence, how do you determine the formula?

- Steps to consider:
  - Is it an arithmetic progression (each term a constant amount from the last)?
  - Is it a geometric progression (each term a factor of the previous term)?
  - Does the sequence it repeat (or cycle)?
  - Does the sequence combine previous terms?
  - Are there runs of the same value?

Determining the sequence formula

a) 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1, …

- The sequence alternates 1's and 0's, increasing the number of 1's and 0's each time

b) 1, 2, 2, 3, 4, 4, 5, 6, 6, 7, 8, 8, …

- This sequence increases by one, but repeats all even numbers once

c) 1, 0, 2, 0, 4, 0, 8, 0, 16, 0, …

- The non-0 numbers are a geometric sequence (2^n) interspersed with zeros

d) 3, 6, 12, 24, 48, 96, 192, …

- Each term is twice the previous: geometric progression
- $a_n = 3*2^{n-1}$

Determining the sequence formula

e) 15, 8, 1, -6, -13, -20, -27, …

- Each term is 7 less than the previous term
- $a_n = 22 - 7n$

f) 3, 5, 8, 12, 17, 23, 30, 38, 47, …

- The difference between successive terms increases by one each time
- $a_1 = 3$, $a_n = a_{n-1} + n$
- $a_n = n(n+1)/2 + 2$

g) 2, 16, 54, 128, 250, 432, 686, …

- Each term is twice the cube of n
- $a_n = 2*n^3$

h) 2, 3, 7, 25, 121, 721, 5041, 40321

- Each successive term is about n times the previous
- $a_n = n! + 1$
- My solution: $a_n = a_{n-1} * n - n + 1$

OEIS: Online Encyclopedia of Integer Sequences

- Online at http://www.research.att.com/~njas/sequences/

Useful sequences

- n^2 = 1, 4, 9, 16, 25, 36, …
- n^3 = 1, 8, 27, 64, 125, 216, …
- n^4 = 1, 16, 81, 256, 625, 1296, …
- 2^n = 2, 4, 8, 16, 32, 64, …
- 3^n = 3, 9, 27, 81, 243, 729, …
- $n!$ = 1, 2, 6, 24, 120, 720, …

Summations

- A summation:

$$\sum_{j=m}^{n} a_j \quad \text{or} \quad \textstyle\sum_{j=m}^{n} a_j$$

upper limit

lower limit

index of summation

- is like a for loop:

```
int sum = 0;
for ( int j = m; j <= n; j++ )
    sum += a(j);
```

Evaluating sequences

$$\sum_{k=1}^{5}(k+1)$$

- 2 + 3 + 4 + 5 + 6 = 20

$$\sum_{j=0}^{4}(-2)^{j}$$

- $(-2)^0 + (-2)^1 + (-2)^2 + (-2)^3 + (-2)^4 = 11$

$$\sum_{i=1}^{10}3$$

- 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 = 30

$$\sum_{j=0}^{8}\left(2^{j+1}-2^{j}\right)$$

- $(2^1-2^0) + (2^2-2^1) + (2^3-2^2) + \ldots (2^{10}-2^9) = 511$
  - Note that each term (except the first and last) is cancelled by another term

Evaluating sequences

- Let S = { 1, 3, 5, 7 }

- What is $\Sigma_{j \in S} j$
  - $1 + 3 + 5 + 7 = 16$
- What is $\Sigma_{j \in S} j^2$
  - $1^2 + 3^2 + 5^2 + 7^2 = 84$
- What is $\Sigma_{j \in S} (1/j)$
  - $1/1 + 1/3 + 1/5 + 1/7 = 176/105$
- What is $\Sigma_{j \in S} 1$
  - $1 + 1 + 1 + 1 = 4$

Summation of a geometric series

- Sum of a geometric series:

$$\sum_{j=0}^{n} ar^{j} = \begin{cases} \dfrac{ar^{n+1}-a}{r-1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases}$$

- Example:

$$\sum_{j=0}^{10} 2^{n} = \frac{2^{10+1}-1}{2-1} = \frac{2048-1}{1} = 2047$$

$$S = \sum_{j=0}^{n} ar^{j}$$

Proof of last s

- If $r = 1$, then the sum is:

$$S = \sum_{j=0}^{n} a = (n+1)a$$

Double summations

- Like a nested for loop

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij$$

- Is equivalent to:

```
int sum = 0;
for ( int i = 1; i <= 4; i++ )
    for ( int j = 1; j <= 3; j++ )
        sum += i*j;
```

Useful summation formulae

- Well, only 1 really important one:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Cardinality

- For finite (only) sets, cardinality is the number of elements in the set

- For finite and infinite sets, two sets *A* and *B* have the same cardinality if there is a one-to-one correspondence from *A* to *B*

Cardinality

- Example on finite sets:
  - Let $S = \{ 1, 2, 3, 4, 5 \}$
  - Let $T = \{ a, b, c, d, e \}$
  - There is a one-to-one correspondence between the sets
- Example on infinite sets:
  - Let S = **Z**+
  - Let T = { x | $x = 2k$ and $k \in$ **Z**+ }
  - One-to-one correspondence:

    1 ↔ 2 2 ↔ 4 3 ↔ 6 4 ↔ 2

    5 ↔ 10 6 ↔ 12 7 ↔ 14 8 ↔ 16

    Etc.
    - Note that here the '↔' symbol means that there is a correspondence between them, not the biconditional

More definitions

- Countably infinite: elements can be listed
  - Anything that has the same cardinality as the integers
  - Example: rational numbers, ordered pairs of integers

- Uncountably infinite: elements cannot be listed
  - Example: real numbers

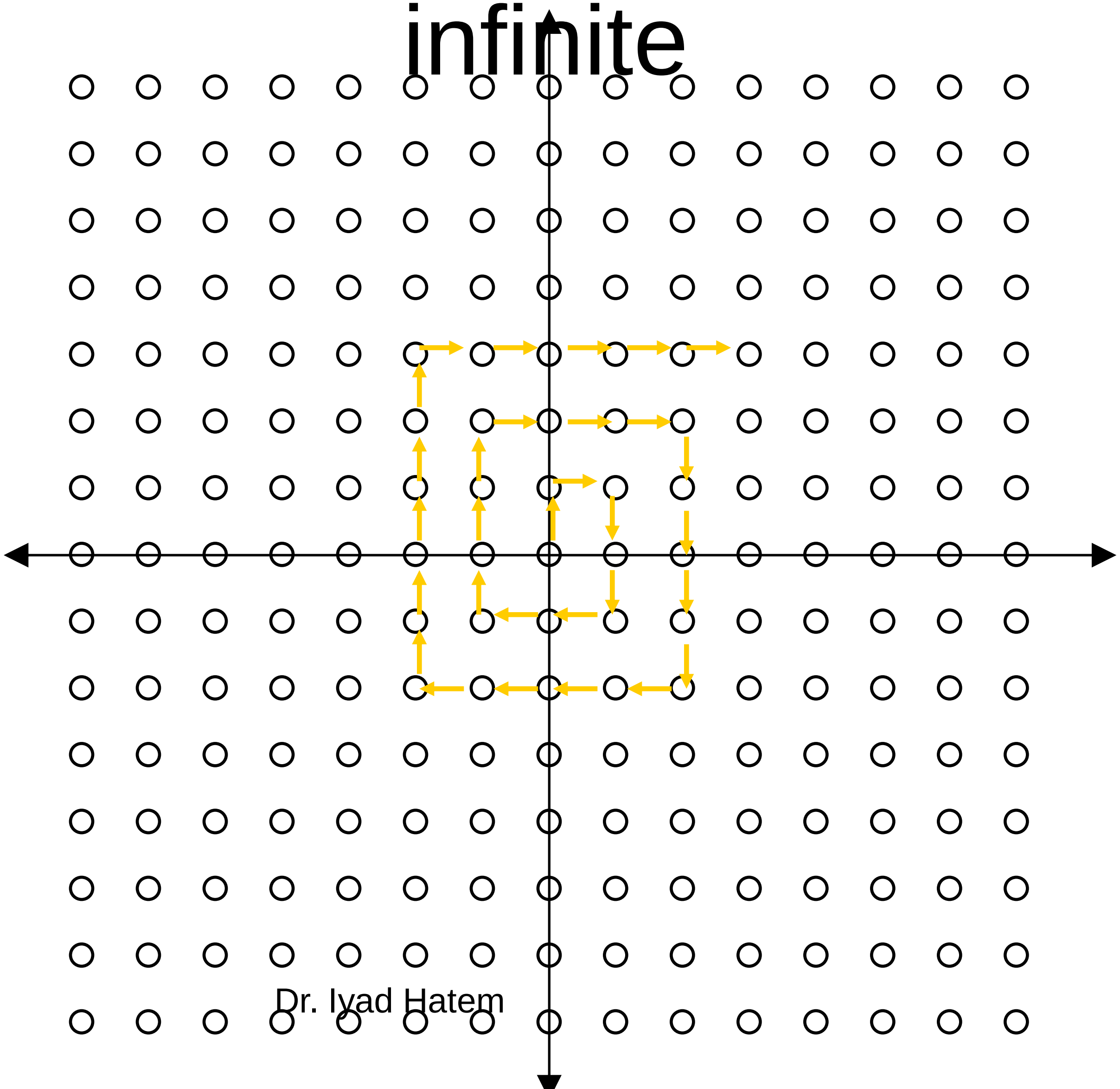
Showing a set is countably infinite

- Done by showing there is a one-to-one correspondence between the set and the integers

- Examples
  - – Even numbers
    - Shown two slides ago
  - – Rational numbers
  - – Ordered pairs of integers
    - Shown next slide

Showing ordered pairs of integers are countably infinite

A one-to-one correspondence



Show that the rational numbers are countably infinite

- First, let's show the positive rationals are countable
- See diagram:
- Can easily add 0 (add one column to the left)
- Can add negative rationals as well