Calculus 2

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Lecture 1

Indeterminate forms

Indeterminate forms

This lecture is divided into 2 parts; they are

1 The indeterminate forms of type 0/0 and ∞/∞
  1.1 How to apply L'Hôpital's rule to these types
  1.2 Solved examples of these two indeterminate types
2 More complex indeterminate types
  2.1 How to convert the more complex indeterminate types to 0/0 and ∞/∞ forms
  2.2 Solved examples of such types

What are Indeterminate Forms?

When working with limits, the following forms are **indeterminate** in that the value of the limit is not "obvious."

$$\frac{0}{0},\quad \frac{\infty}{\infty},\quad 0\cdot\infty,\quad \infty-\infty,\quad 0^0,\quad \infty^0,\quad 1^\infty$$

Indeterminate forms

Consider: $\lim_{x\to 2}\frac{x^2-4}{x-2}$ **or** $\lim_{x\to 1}\frac{\ln x}{x-1}$

If we try to evaluate by direct substitution, we get: $\frac{0}{0}$

Zero divided by zero can not be evaluated. The limit may or may not exist, and is called an indeterminate form.

In the case of the first limit, we can evaluate it by factoring and canceling:

$$\lim_{x\to 2}\frac{x^2-4}{x-2} == \lim_{x\to 2}\frac{(x+2)\cancel{(x-2)}}{\cancel{x-2}} = \lim_{x\to 2}(x+2) = 4$$

This method does not work in the case of the second limit
For limits of this type, L'Hôpital's rule is useful

L'Hôpital's Rule

Theorem: Suppose f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

or

$$\lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type 0/0 or ∞/∞.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right hand side exists (or is −∞ or ∞).

Note that : "x $\rightarrow$ a" can be replaced by any of the symbols $x \rightarrow a^-, x \rightarrow a^+, x \rightarrow \infty, x \rightarrow -\infty$.

When to Apply L'Hôpital's Rule?

- An important point to note is that L'Hôpital's rule is only applicable when the limit produce an indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ **For example**

$$\lim_{x\to 0}\frac{\sin 2x}{x+1} \quad \textit{Cannot apply L'Hospital's rule as it's not } \frac{0}{0} \textit{ form}$$

$$\lim_{x\to 0}\frac{\sin 2x}{x} \quad \textit{Can apply the rule as it's } \frac{0}{0} \textit{ form}$$

$$\lim_{x\to +\infty}\frac{e^x}{x} \quad \textit{Can apply the rule as it's } \frac{\infty}{\infty} \textit{ form}$$

$$\lim_{x\to +\infty}\frac{e^x}{\frac{1}{1+x}} \quad \textit{Cannot apply L'Hospital's rule as it's not } \frac{\infty}{\infty} \textit{ form}$$

Examples of 0/0 and ∞/∞

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to 2}\frac{x^2-4}{x-2} = \lim_{x\to 2}\frac{\frac{d}{dx}\left(x^2-4\right)}{\frac{d}{dx}\left(x-2\right)} = \lim_{x\to 2}\frac{2x}{1} = 4$$

$$\lim_{x\to a}\frac{f(x)}{g(x)} \quad \lim_{x\to 1}\frac{\ln x}{x-1} = \lim_{x\to 1}\frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x-1)} = \lim_{x\to 1}\frac{1/x}{1} = \lim_{x\to 1}\frac{1}{x} = 1$$

$$\lim_{x\to\infty}\frac{e^x}{x} = \lim_{x\to\infty}\frac{\frac{d}{dx}(e^x)}{\frac{d}{dx}(x)} = \lim_{x\to\infty}\frac{e^x}{1} = \infty$$

On the other hand, you can apply L'Hôpital's rule **as many times** as necessary as long as the fraction is still indeterminate:

Example : Evaluate $\lim_{x\to 0}\frac{1-\cos x}{x^2}$.

Solution:

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{0}{0}$$

we use L'Hôpital's Rule to find the answer.

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sin x}{2x}=\frac{0}{0}$$

We apply L'Hôpital's Rule again.

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sin x}{2x}=\lim_{x\to 0}\frac{\cos x}{2}=\frac{1}{2}$$

L'Hôpital's rule for other indeterminate forms.

Note: For some functions where the limit **does not** initially appear to as an indeterminant $\frac{0}{0}$ or $\mp\frac{\infty}{\infty}$ It may be possible to use algebraic techniques to **convert** the function one of the indeterminants $\frac{0}{0}$ or $\mp\frac{\infty}{\infty}$

EXAMPLES

Indeterminate Form $0.\infty$

EXAMPLE Evaluate $$\lim_{x\to\infty}\left(x\sin\frac{1}{x}\right)$$

Solution

$$\lim_{x\to\infty}\left(x\sin\frac{1}{x}\right)$$ ← This approaches $\infty\cdot 0$

Rewrite as a ratio!

$$\lim_{x\to\infty}\frac{\sin\frac{1}{x}}{\frac{1}{x}}$$ ← This approaches $\frac{0}{0}$

We apply L'Hôpital's rule

$$\lim_{x\to\infty}\frac{\sin\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty}\frac{\cos\left(\frac{1}{x}\right)\cdot\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty}\cos\left(\frac{1}{x}\right) = \cos(0) = 1$$

Indeterminate Differences $\infty - \infty$

EXAMPLE Evaluate $\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$

Solution

$$\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$$

This is indeterminate form $\infty - \infty$

Rewrite as a ratio!

If we find a common denominator and subtract, we get:

$$\lim_{x\to 1}\left(\frac{x-1-\ln x}{(x-1)\ln x}\right)$$

Now it is in the form $\frac{0}{0}$

$$\lim_{x\to 1}\left(\frac{1-\frac{1}{x}}{\frac{x-1}{x}+\ln x}\right)$$

L'Hôpital's rule applied once.

Indeterminate Differences $\infty - \infty$

$$\lim_{x\to 1}\left(\frac{x-1}{x-1+x\ln x}\right)$$ ← Fractions cleared. Still $\frac{0}{0}$

$$\lim_{x\to 1}\left(\frac{1}{1+1+\ln x}\right)$$ ← L'Hôpital again.

Answer: $\frac{1}{2}$

Indeterminate Powers

Indeterminate Forms: 1^{∞} 0^{0} ∞^{0}

Evaluating these forms requires a mathematical trick to change the expression into a ratio.

$$u^{n} = e^{n \ln u}$$

Indeterminate Powers ∞^0

EXAMPLE Evaluate $\lim_{x \to +\infty} x^{1/x}$

Solution

$$\lim_{x \to +\infty} x^{1/x}$$

This is indeterminate form ∞^0

$$x^{1/x} = e^{\ln\left(x^{1/x}\right)} = e^{\frac{1}{x}\ln(x)} = e^{\frac{\ln(x)}{x}}$$

$$\lim_{x \to +\infty} \frac{\ln(x)}{x} \qquad \frac{\infty}{\infty}$$

we can apply L'Hôpital's rule

$$\lim_{x \to \infty} \frac{\ln(x)}{x} = \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = 0$$

then

$$\lim_{x \to +\infty} x^{1/x} = e^0 = 1$$

Indeterminate Powers 1^∞

EXAMPLE Evaluate $\lim_{x\to 0^+} (\cos x)^{\frac{1}{x}}$

Solution

$\lim_{x\to 0^+} (\cos x)^{\frac{1}{x}}$ ← This is indeterminate form 1^∞

$$(\cos x)^{\frac{1}{x}} = e^{\ln\left((\cos x)^{\frac{1}{x}}\right)} = e^{\frac{1}{x}.\ln(\cos x)} = e^{\frac{\ln(\cos)}{x}}$$

$\lim_{x\to 0^+} \frac{\ln(\cos x)}{x}$ ← $\frac{0}{0}$ we can apply L'Hôpital's rule

$$\lim_{x\to 0^+} \frac{\ln(\cos x)}{x} = \lim_{x\to 0^+} \frac{-\frac{\sin x}{\cos x}}{1} = 0$$

then $\lim_{x\to 0^+} (\cos x)^{\frac{1}{x}} = e^0 = 1$

Indeterminate Powers 0^0

EXAMPLE Evaluate $\lim_{x \to 1} (x-1)^{\ln x}$

Solution

$$\lim_{x \to 1} (x-1)^{\ln x}$$ ← This is indeterminate form 0^0

$$(x-1)^{\ln x} = e^{\ln\left((x-1)^{\ln x}\right)} = e^{\ln(x).\ln(x-1)}$$

$$\lim_{x \to 1} \ln(x).\ln(x-1)$$ ← $0.\infty$ Rewrite as a ratio!

$$\lim_{x \to 1} \ln(x).\ln(x-1) = \lim_{x \to 1} \frac{\ln(x-1)}{\frac{1}{\ln(x)}}$$ ← $\frac{0}{0}$

we can apply L'Hôpital's rule

$$\lim_{x \to 1} \ln(x).\ln(x-1) = \lim_{x \to 1} \frac{\ln(x-1)}{\frac{1}{\ln(x)}} \overset{H}{=} \lim_{x \to 1} \frac{\frac{d}{dx}\ln(x-1)}{\frac{d}{dx}\left(\frac{1}{\ln(x)}\right)}$$

$$= \lim_{x \to 1} \frac{\frac{1}{x-1}}{\frac{-1}{x[\ln(x)]^2}} = \lim_{x \to 1} \frac{x[\ln(x)]^2}{x-1} = \frac{0}{0}$$

then

we can apply L'Hôpital's rule

$$\lim_{x \to 1} (x-1)^{\ln x} = e^{-1} = \frac{1}{e}$$

$$\lim_{x \to 1} \ln(x).\ln(x-1) = \lim_{x \to 1} \frac{x\left[\ln(x)\right]^2}{x-1} \overset{H}{=} \lim_{x \to 1} \frac{\left[\ln(x)\right]^2 + 2\ln(x)}{1} = 0$$

then

$$\lim_{x \to 1} (x-1)^{\ln x} = e^0 = 1$$

EXERCISES Use L'Hôpital's rule to evaluate)

$$1- \lim_{x \to \frac{\pi}{4}} \frac{\sin x - \cos x}{x - \frac{\pi}{4}}$$

$$2 - \lim_{x \to +\infty} \frac{x}{\ln(1 + 3e^{x})}$$

$$3 - \lim_{x \to 0} \frac{1 - \cos x}{x^2}$$

$$4 -- \lim_{x \to 0^+} \sqrt{x}\,.\ln x$$

$$5 - \lim_{x \to \infty} x^2 e^{1-x}$$

$$6 - \lim_{x \to 1} (2 - x)^{\tan \frac{\pi}{2} x}$$

$$\lim_{x \to 0} \frac{\sqrt{1+x} - 1 - \frac{x}{2}}{x^2}$$

$$\lim_{x \to 0^+} (e^x + x)^{\frac{1}{x}}$$

EXERCISES (Use L'Hôpital's rule to evaluate)

$$\lim_{x\to 0^-}\frac{\tan x}{x^2}$$

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}$$

$$\lim_{x\to +\infty}\frac{x^{-4/3}}{\sin(1/x)}$$

$$\lim_{x\to 0^+} x\ln x$$

$$\lim_{x\to \frac{\pi}{2}} \tan x \ln(\sin x)$$

$$\lim_{x\to \pi/4}\frac{1-\tan x}{\cos 2x}$$

$$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$$

Thank you for your attention