



Calculus 2

**Exercices 2**

**Improper Integrals**

جامعة  
المنارة

MAHARA UNIVERSITY

# Improper Integrals

The integral  $\int_{-1}^1 \frac{dx}{x^3}$  diverges.

Trying to compute that integral by the Fundamental Theorem of Calculus, one gets

$$\int_{-1}^1 \frac{dx}{x^3} = \frac{1}{-2} x^{-2} \Big|_{-1}^1 = -\frac{1}{2} - \left( -\frac{1}{2} \right) = 0.$$

**This is an incorrect computation.**

For what values of  $p$  is the integral  
convergent?

$$\int_1^{+\infty} \frac{dx}{x^p}$$

## Solution

$$\int_1^{\infty} \frac{dx}{x^P} \quad P > 0$$

$$\lim_{b \rightarrow \infty} \frac{b^{-P+1}}{-P+1} - \frac{1^{-P+1}}{-P+1}$$

$$\int_1^{\infty} x^{-P} dx \quad (P \text{ is a constant.})$$

What happens here?

If  $P < 1$  then  $b^{-P+1}$  gets bigger and bigger as  $b \rightarrow \infty$ , therefore the integral diverges.

If  $P > 1$  then  $b$  has a negative exponent and  $b^{-P+1} \rightarrow 0$ , therefore the integral converges.

$$\lim_{b \rightarrow \infty} \int_1^b x^{-P} dx$$
$$\lim_{b \rightarrow \infty} \frac{1}{-P+1} x^{-P+1} \Big|_1^b$$

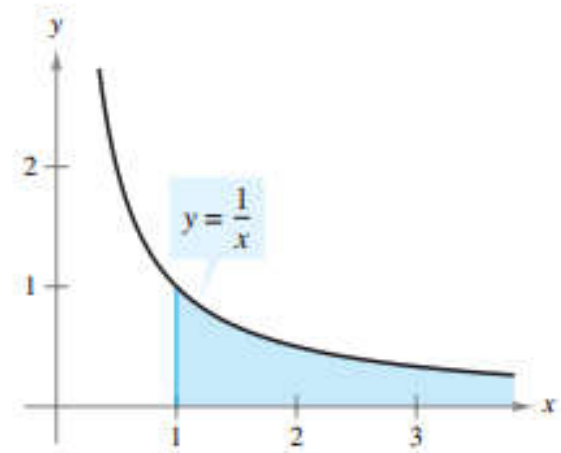
When  $p \neq 1$ ,

$$\int_1^{+\infty} \frac{dx}{x^p} = \lim_{b \rightarrow \infty} \frac{b^{-P+1} - 1^{-P+1}}{-P+1} = \begin{cases} +\infty & p < 1 \\ \frac{1}{p-1} & p > 1 \end{cases}$$

When  $p = 1$ ,

$$\int_1^{+\infty} \frac{dx}{x} = \lim_{b \rightarrow \infty} \ln x \Big|_1^b = \lim_{b \rightarrow \infty} (\ln b - \ln 1) = +\infty$$

Thus, the integral diverges.



Diverges (infinite area)

**Evaluate**  $\lim_{b \rightarrow 1^-} \int_0^b \sqrt{\frac{1+x}{1-x}} dx$

**Solution**  $\int \sqrt{\frac{1+x}{1-x}} \frac{\sqrt{1+x}}{\sqrt{1+x}} dx$

Rationalize the numerator.

$$\int \frac{1+x}{\sqrt{1-x^2}} dx$$

$$\int \frac{1}{\sqrt{1-x^2}} dx + \int \frac{x}{\sqrt{1-x^2}} dx$$

$$\sin^{-1} x - \frac{1}{2} \int u^{-\frac{1}{2}} du$$

$$u = 1 - x^2$$

$$du = -2x dx$$

$$-\frac{1}{2} du = x dx$$

$$\int \frac{1}{\sqrt{1-x^2}} dx + \int \frac{x}{\sqrt{1-x^2}} dx$$

$$\sin^{-1} x - \frac{1}{2} \int u^{-\frac{1}{2}} du$$

$$\sin^{-1} x - u^{\frac{1}{2}}$$

$$u = 1 - x^2$$

$$du = -2x dx$$

$$-\frac{1}{2} du = x dx$$

This integral converges because it approaches a solution.

$$\lim_{b \rightarrow 1^-} \left. \sin^{-1} x - \sqrt{1-x^2} \right|_0^b$$

$$\lim_{b \rightarrow 1^-} \left( \sin^{-1} b - \sqrt{1-b^2} \right) - \left( \sin^{-1} 0 - \sqrt{1} \right) = \frac{\pi}{2} + 1$$

# Evaluating an integral on $(-\infty, \infty)$

$$\int_{-\infty}^{\infty} e^{-|x|} dx$$

## Solution

$$\begin{aligned}
 \int_{-\infty}^{\infty} e^{-|x|} dx &= \int_{-\infty}^0 e^x dx + \int_0^{\infty} e^{-x} dx \\
 &= \lim_{a \rightarrow -\infty} \int_a^0 e^x dx + \lim_{a \rightarrow \infty} \int_0^a e^{-x} dx \\
 &= \lim_{a \rightarrow -\infty} (1 - e^a) + \lim_{a \rightarrow \infty} (-e^{-a} + 1) \\
 &= 1 + 1 = 2
 \end{aligned}$$



evaluate the integral or state that it diverges.

$$\int_0^2 \frac{dx}{1-x^2}$$

## Solution

$\int_0^2 \frac{dx}{1-x^2}$  has an infinite discontinuity at  $x = 1$

$$= \int_0^1 \frac{dx}{1-x^2} + \int_1^2 \frac{dx}{1-x^2}$$

$$= \lim_{c \rightarrow 1^-} \int_0^c \frac{dx}{1-x^2} + \lim_{c \rightarrow 1^+} \int_c^2 \frac{dx}{1-x^2}$$

$$= \lim_{c \rightarrow 1^-} \left( -\frac{1}{2} \ln \left( \frac{x-1}{x+1} \right) \right)_0^c + \lim_{c \rightarrow 1^+} \left( -\frac{1}{2} \ln \left( \frac{x-1}{x+1} \right) \right)_c^2$$

$= \infty$  , the integral diverges

Find  $\int_0^a \frac{dx}{\sqrt{a^2 - x^2}} (a > 0)$ .

## Solution

$$\int_0^a \frac{dx}{\sqrt{a^2 - x^2}} = \left[ \arcsin \frac{x}{a} \right]_0^a = \frac{\pi}{2}.$$

**Use the comparison test to determine if the following integral are convergent or divergent**

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{x}} dx .$$

$$(c) \int_0^{\pi/2} \frac{\cos x}{\sqrt{x}} dx$$

converges because

$$0 \leq \frac{\cos x}{\sqrt{x}} \leq \frac{1}{\sqrt{x}} \quad \text{on} \quad \left[0, \frac{\pi}{2}\right], \quad 0 \leq \cos x \leq 1 \quad \text{on} \quad \left[0, \frac{\pi}{2}\right]$$

and

$$\begin{aligned} \int_0^{\pi/2} \frac{dx}{\sqrt{x}} &= \lim_{a \rightarrow 0^+} \int_a^{\pi/2} \frac{dx}{\sqrt{x}} \\ &= \lim_{a \rightarrow 0^+} \left[ \sqrt{4x} \right]_a^{\pi/2} \quad 2\sqrt{x} = \sqrt{4x} \\ &= \lim_{a \rightarrow 0^+} (\sqrt{2\pi} - \sqrt{4a}) = \sqrt{2\pi} \quad \text{converges.} \end{aligned}$$

Evaluate the improper integral or state that it diverges.

$$\int_1^{\infty} \frac{dx}{x^{\frac{3}{2}}}$$

$$\int_1^{\infty} \frac{dx}{e^x + e^{-x}}$$

$$\int_1^{\infty} \frac{dx}{x^4}$$

Evaluate the improper integral or state that it diverges.

$$1. \int_1^{\infty} \frac{1}{x^3} dx$$

$$2. \int_1^{\infty} \frac{3}{\sqrt[3]{x}} dx$$

$$3. \int_{-\infty}^0 xe^{-4x} dx$$

$$4. \int_0^{\infty} x^2 e^{-x} dx$$

$$5. \int_4^{\infty} \frac{1}{x(\ln x)^3} dx$$

$$6. \int_0^{\infty} \frac{e^x}{1+e^x} dx$$

$$7. \int_0^{\infty} \sin \frac{x}{2} dx$$

$$8. \int_0^{\infty} e^{-x} \cos x dx$$

Use the comparison test to determine if the following integrals are convergent or divergent

$$\int_1^{\infty} \frac{1}{x^2 + x + 1} dx$$

$$\int_1^{\infty} \frac{1}{x - \frac{1}{2}} dx$$

$$\int_0^{\pi} \frac{\cos^2 x}{\sqrt{x}} dx$$

$$\int_0^{\infty} \frac{e^{-x}}{1 + \sin^2 x} dx$$



**TABLE 8.1** Basic integration formulas

$$1. \int k \, dx = kx + C \quad (\text{any number } k)$$

$$2. \int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$3. \int \frac{dx}{x} = \ln |x| + C$$

$$4. \int e^x \, dx = e^x + C$$

$$5. \int a^x \, dx = \frac{a^x}{\ln a} + C \quad (a > 0, a \neq 1)$$

$$6. \int \sin x \, dx = -\cos x + C$$

$$7. \int \cos x \, dx = \sin x + C$$

$$8. \int \sec^2 x \, dx = \tan x + C$$

$$8. \int \csc^2 x \, dx = -\cot x + C$$

$$10. \int \sec x \tan x \, dx = \sec x + C$$

$$11. \int \csc x \cot x \, dx = -\csc x + C$$

$$12. \int \tan x \, dx = \ln |\sec x| + C$$

$$13. \int \cot x \, dx = \ln |\sin x| + C$$

$$14. \int \sec x \, dx = \ln |\sec x + \tan x| + C$$

$$15. \int \csc x \, dx = -\ln |\csc x + \cot x| + C$$

$$16. \int \sinh x \, dx = \cosh x + C$$

$$17. \int \cosh x \, dx = \sinh x + C$$

$$18. \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$19. \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$$

$$20. \int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1}\left|\frac{x}{a}\right| + C$$

$$21. \int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C \quad (a > 0)$$

$$22. \int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C \quad (x > a > 0)$$

**Thank you for your attention**

جامعة  
المنارة  
MANARA UNIVERSITY