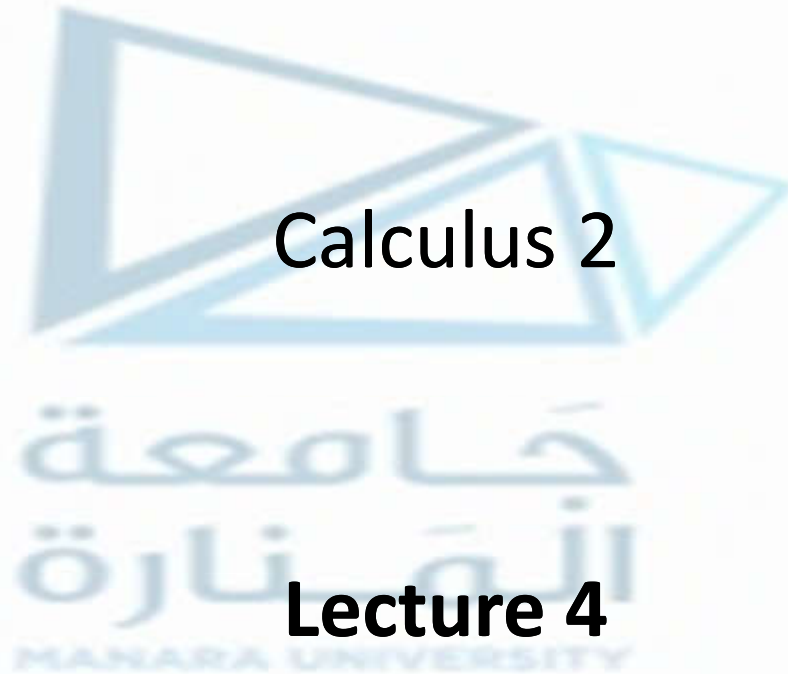


Calculus 2

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Infinite Series



Infinite Series

An *infinite series* is the sum of an infinite sequence of numbers

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots + a_n + \dots \quad \text{Infinite series}$$

The sum of the first n terms

$$s_n = a_1 + a_2 + a_3 + \dots + a_n$$

a_1, a_2, \dots are terms of the series.

a_n is the n^{th} term.



To find the sum of an infinite series, consider the sequence of partial sums listed below.

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

$$S_4 = a_1 + a_2 + a_3 + a_4$$

$$S_5 = a_1 + a_2 + a_3 + a_4 + a_5$$

⋮

$$S_n = a_1 + a_2 + a_3 + a_4 + a_5 + \cdots + a_n$$

$$S_n = \sum_{k=1}^n a_k$$

n^{th} partial sum

If S_n has a limit as $n \rightarrow \infty$, then the series converges, otherwise it diverges.

The series in Example is a **telescoping series** of the form



The n th partial sum of the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \dots$$

is

$$S_n = 1 - \frac{1}{n+1}$$

Find the sum of the series $\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1}$.

Because the limit of S_n is 1, the series converges and its sum is 1.



The series

$$\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 + \dots$$

diverges because $S_n = n$ and the sequence of partial sums diverges.

Telescoping Series:

$$\sum_{n=1}^{\infty} (b_n - b_{n+1})$$

$$s_n = b_1 - b_{n+1}$$

$$s = \sum_{n=1}^{\infty} (b_n - b_{n+1}) = b_1 - \lim_{n \rightarrow \infty} b_{n+1}$$

$$\sum_{n=1}^{\infty} (b_n - b_{n+1}) \text{ is convergent} \iff \{b_n\} \text{ is convergent}$$

Example:

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

Example

Find the sum of the series $\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1}$.

Solution

Using partial fractions, you can write

$$a_n = \frac{2}{4n^2 - 1} = \frac{2}{(2n - 1)(2n + 1)} = \frac{1}{2n - 1} - \frac{1}{2n + 1}.$$

From this telescoping form, you can see that the n th partial sum is

$$S_n = \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{2n - 1} - \frac{1}{2n + 1}\right) = 1 - \frac{1}{2n + 1}.$$

So, the series converges and its sum is 1. That is,

$$\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1} = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{2n + 1}\right) = 1.$$

In a geometric series, each term is found by multiplying the preceding term by the same number, r .

$$a + ar + ar^2 + ar^3 + \dots + ar^{n-1} + \dots = \sum_{n=1}^{\infty} ar^{n-1}$$

This converges to $\frac{a}{1-r}$ if $|r| < 1$, and diverges if $|r| \geq 1$.

$-1 < r < 1$ is the interval of convergence.

EXAMPLE 1 The geometric series with $a = 1/9$ and $r = 1/3$ is

$$\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \cdots = \sum_{n=1}^{\infty} \frac{1}{9} \left(\frac{1}{3}\right)^{n-1} = \frac{1/9}{1 - (1/3)} = \frac{1}{6}.$$

EXAMPLE 2 The series

$$\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = 5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots$$

is a geometric series with $a = 5$ and $r = -1/4$. It converges to

$$\frac{a}{1 - r} = \frac{5}{1 + (1/4)} = 4.$$

THEOREM **Properties of Infinite Series**

Let $\sum a_n$ and $\sum b_n$ be convergent series, and let A , B , and c be real numbers. If $\sum a_n = A$ and $\sum b_n = B$, then the following series converge to the indicated sums.

1.
$$\sum_{n=1}^{\infty} ca_n = cA$$

2.
$$\sum_{n=1}^{\infty} (a_n + b_n) = A + B$$

3.
$$\sum_{n=1}^{\infty} (a_n - b_n) = A - B$$

THEOREM **Limit of the n th Term of a Convergent Series**

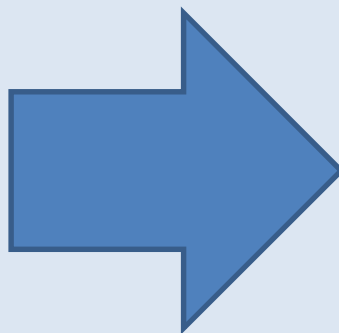
If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n \rightarrow \infty} a_n = 0$.

n th-Term Test for Divergence

THEOREM:

$$\sum_{i=1}^{\infty} a_i$$

Convergent

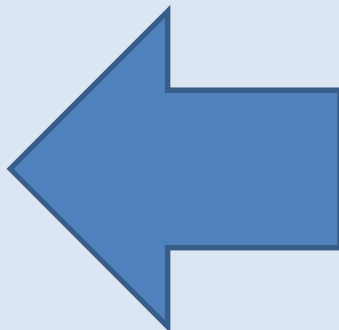


$$\lim_{n \rightarrow \infty} a_n = 0$$

THEOREM: THE TEST FOR DIVERGENCE

$$\sum_{i=1}^{\infty} a_i$$

Divergent



$$\lim_{n \rightarrow \infty} a_n \neq 0$$

or

$$\lim_{n \rightarrow \infty} a_n \text{ DNE}$$

Example

(a) $\sum_{n=1}^{\infty} n^2$ diverges because $n^2 \rightarrow \infty$.

(b) $\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges because $\frac{n+1}{n} \rightarrow 1$. $\lim_{n \rightarrow \infty} a_n \neq 0$

(c) $\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges because $\lim_{n \rightarrow \infty} (-1)^{n+1}$ does not exist.

(d) $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ diverges because $\lim_{n \rightarrow \infty} \frac{-n}{2n+5} = -\frac{1}{2} \neq 0$.

n th-Term Test for Divergence

For the series $\sum_{n=1}^{\infty} \frac{1}{n}$, you have

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0.$$

Because the limit of the n th term is 0, the n th-Term Test for Divergence does *not* apply and you can draw no conclusions about convergence or divergence.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots$$

p -series

is a p -series, where p is a positive constant. For $p = 1$, the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

Harmonic series

is the harmonic series.

Convergence of p -Series

The p -series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots$$

converges for $p > 1$ and diverges for $0 < p \leq 1$.

THEOREM **Direct Comparison Test**

Let $0 < a_n \leq b_n$ for all n .

1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Example Determine the convergence or divergence

$$\sum_{n=1}^{\infty} \frac{1}{2 + 3^n}$$

Solution

$$\sum_{n=1}^{\infty} \frac{1}{2 + 3^n}$$

Solution This series resembles

$$\sum_{n=1}^{\infty} \frac{1}{3^n} \quad \text{Convergent geometric series}$$

Term-by-term comparison yields

$$a_n = \frac{1}{2 + 3^n} < \frac{1}{3^n} = b_n, \quad n \geq 1.$$

So, by the Direct Comparison Test, the series converges.

THEOREM Limit Comparison Test

If $a_n > 0$, $b_n > 0$, and

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L$$

where L is *finite and positive*, then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \sum_{n=1}^{\infty} b_n$$

either both converge or both diverge.

Example

Given Series

$$\sum_{n=1}^{\infty} \frac{1}{3n^2 - 4n + 5}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n - 2}}$$

$$\sum_{n=1}^{\infty} \frac{n^2 - 10}{4n^5 + n^3}$$

Comparison Series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

$$\sum_{n=1}^{\infty} \frac{n^2}{n^5} = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

Conclusion

Both series converge.

Both series diverge.

Both series converge.

Limit Comparison Test

Example

Determine the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{n2^n}{4n^3 + 1}.$$

Solution

$$\sum_{n=1}^{\infty} \frac{2^n}{n^2}.$$

Divergent series

Note that this series diverges by the n th-Term Test. From the limit

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \left(\frac{n2^n}{4n^3 + 1} \right) \left(\frac{n^2}{2^n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{4 + (1/n^3)} \\ &= \frac{1}{4} \end{aligned}$$

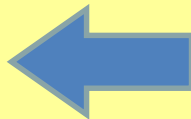
you can conclude that the series diverges.

summary

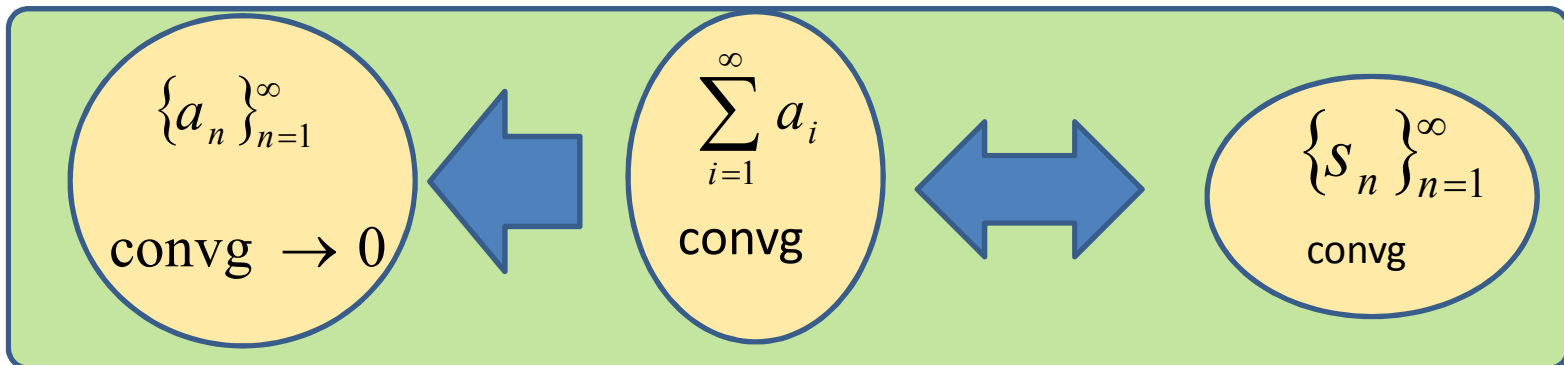
	Geometric	Telescoping	General
When convg	$ r < 1$	$\{b_{n+1}\} \text{ convg}$	$\{s_n\} \text{ convg}$
sum	$\frac{a}{1-r}$	$b_1 - \lim_{n \rightarrow \infty} b_{n+1}$	$\lim_{n \rightarrow \infty} s_n$
nth partial sum	$s_n = a \frac{1-r^n}{1-r}$	$s_n = b_1 - b_{n+1}$	$s_n = a_1 + \dots + a_n$ $a_n = s_n - s_{n-1}$

THEOREM: THE TEST FOR DIVERGENCE

$$\sum_{i=1}^{\infty} a_i \text{ Divergent}$$



$$\lim_{n \rightarrow \infty} a_n \neq 0 \quad \text{or} \quad \lim_{n \rightarrow \infty} a_n \text{ DNE}$$





Thank you for your attention

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