



Calculus 2

Exercices 4

Infinite Series

جامعة
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Evaluate

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \dots$$

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

Solution

$$\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \dots$$

$$0.3 + 0.03 + 0.003 + 0.0003 + \dots = 0.333\dots$$

$$= \frac{1}{3}$$

$$\frac{\frac{3}{10}}{1 - \frac{1}{10}} = \frac{3}{9} = \frac{3}{9} = \frac{1}{3}$$

} *a* } *r*

Solution

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots$$

$$\frac{1 \leftarrow a}{1 - \left(-\frac{1}{2}\right) \leftarrow r} = \frac{1}{1 + \frac{1}{2}} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$$

Solution

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$$

$$\left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4}\right) + \dots$$

$$S_3 = 1 - \frac{1}{4}$$

$$S_n = 1 - \frac{1}{n+1}$$

$$\lim_{n \rightarrow \infty} S_n = 1$$

Use a geometric series to write $0.\overline{08}$ as the ratio of two integers.

Solution

Use a geometric series to write $0.\overline{08}$ as the ratio of two integers.

Solution For the repeating decimal $0.\overline{08}$, you can write

$$\begin{aligned}0.080808 \dots &= \frac{8}{10^2} + \frac{8}{10^4} + \frac{8}{10^6} + \frac{8}{10^8} + \dots \\ &= \sum_{n=0}^{\infty} \left(\frac{8}{10^2}\right) \left(\frac{1}{10^2}\right)^n.\end{aligned}$$

For this series, you have $a = 8/10^2$ and $r = 1/10^2$. So,

$$0.080808 \dots = \frac{a}{1-r} = \frac{8/10^2}{1-(1/10^2)} = \frac{8}{99}.$$

Try dividing 8 by 99 on a calculator to see that it produces $0.\overline{08}$.

EXAMPLE Express the repeating decimal $5.232323 \dots$ as the ratio of two integers.

Solution From the definition of a decimal number, we get a geometric series

$$\begin{aligned}
 5.232323 \dots &= 5 + \frac{23}{100} + \frac{23}{(100)^2} + \frac{23}{(100)^3} + \dots \\
 &= 5 + \frac{23}{100} \left(1 + \frac{1}{100} + \left(\frac{1}{100} \right)^2 + \dots \right) \quad \begin{array}{l} a = 1, \\ r = 1/100 \end{array} \\
 &\quad \quad \quad \underbrace{\hspace{10em}}_{1/(1 - 0.01)} \\
 &= 5 + \frac{23}{100} \left(\frac{1}{0.99} \right) = 5 + \frac{23}{99} = \frac{518}{99}
 \end{aligned}$$



Find the sums of the following series.

(a)
$$\sum_{n=1}^{\infty} \frac{3^{n-1} - 1}{6^{n-1}} :$$

(b)
$$\sum_{n=0}^{\infty} \frac{4}{2^n}$$

Solution

$$\begin{aligned}
 \text{(a)} \quad \sum_{n=1}^{\infty} \frac{3^{n-1} - 1}{6^{n-1}} &= \sum_{n=1}^{\infty} \left(\frac{1}{2^{n-1}} - \frac{1}{6^{n-1}} \right) \\
 &= \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} - \sum_{n=1}^{\infty} \frac{1}{6^{n-1}} \\
 &= \frac{1}{1 - (1/2)} - \frac{1}{1 - (1/6)} \\
 &= 2 - \frac{6}{5} = \frac{4}{5}
 \end{aligned}$$

Difference Rule

Geometric series with $a = 1$ and $r = 1/2, 1/6$

$$\begin{aligned}
 \text{(b)} \quad \sum_{n=0}^{\infty} \frac{4}{2^n} &= 4 \sum_{n=0}^{\infty} \frac{1}{2^n} \\
 &= 4 \left(\frac{1}{1 - (1/2)} \right) \\
 &= 8
 \end{aligned}$$

Constant Multiple Rule

Geometric series with $a = 1, r = 1/2$



Using the n th-Term Test for Divergence

$$\sum_{n=0}^{\infty} 2^n$$

$$\sum_{n=1}^{\infty} \frac{n!}{2n! + 1}$$

Solution

a. For the series $\sum_{n=0}^{\infty} 2^n$, you have

$$\lim_{n \rightarrow \infty} 2^n = \infty.$$

So, the limit of the n th term is not 0, and the series diverges.

b. For the series $\sum_{n=1}^{\infty} \frac{n!}{2n! + 1}$, you have

$$\lim_{n \rightarrow \infty} \frac{n!}{2n! + 1} = \frac{1}{2}.$$

So, the limit of the n th term is not 0, and the series diverges.

Determine the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 + 1}$$

Solution Disregarding all but the highest powers of n in the numerator and the denominator, you can compare the series with

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}. \quad \text{Convergent } p\text{-series}$$

Because

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_n}{b_n} &= \lim_{n \rightarrow \infty} \left(\frac{\sqrt{n}}{n^2 + 1} \right) \left(\frac{n^{3/2}}{1} \right) \\ &= \lim_{n \rightarrow \infty} \frac{n^2}{n^2 + 1} \\ &= 1 \end{aligned}$$

you can conclude by the Limit Comparison Test that the series converges.

Find the sum:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$$

Solution

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$$

$$\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}$$

$$1 = A(n+2) + Bn$$

$$1 = (A+B)n + 2A$$

$$A = \frac{1}{2}, B = -\frac{1}{2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2} \right)$$

$$S_n = \frac{1}{2} \left[\left(1 - \frac{1}{3} \right) + \left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \left(\frac{1}{4} - \frac{1}{6} \right) + \left(\frac{1}{5} - \frac{1}{7} \right) + \dots + \left(\frac{1}{n} - \frac{1}{n+2} \right) \right]$$

$$S_n = \frac{1}{2} \left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right) \rightarrow \frac{1}{2} \left(1 + \frac{1}{2} \right)$$

$$S = \frac{3}{4}$$

The geometric series

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{3}{2^n} &= \sum_{n=0}^{\infty} 3\left(\frac{1}{2}\right)^n \\ &= 3(1) + 3\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \dots\end{aligned}$$

has a ratio of $r = \frac{1}{2}$ with $a = 3$.

Because $0 < |r| < 1$, the series converges and its sum is

$$S = \frac{a}{1 - r} = \frac{3}{1 - (1/2)} = 6.$$

The geometric series

$$\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n = 1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \dots$$

has a ratio of $r = \frac{3}{2}$.

Because $|r| \geq 1$, the series diverges.



$$0.999\dots = 1?$$

Solution

$$\begin{aligned}
 0.999\dots &= \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10000} \dots \\
 &= \frac{9}{10} + \frac{9}{10} \left(\frac{1}{10} \right) + \frac{9}{10} \left(\frac{1}{100} \right) + \frac{9}{10} \left(\frac{1}{1000} \right) + \dots \\
 &= \frac{9}{10} + \frac{9}{10} \left(\frac{1}{10} \right)^1 + \frac{9}{10} \left(\frac{1}{10} \right)^2 + \frac{9}{10} \left(\frac{1}{10} \right)^3 + \dots
 \end{aligned}$$

$$\begin{aligned}
 0.999\dots &= \left(\frac{9}{10} \right) \left(\frac{1}{1 - \frac{1}{10}} \right) \\
 &= \left(\frac{9}{10} \right) \left(\frac{1}{\left(\frac{9}{10} \right)} \right) \\
 &= \left(\frac{9}{10} \right) \left(\frac{10}{9} \right) = 1
 \end{aligned}$$

$$1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \dots = ?$$

$$1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \dots = ?$$

Solution

$$S = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \dots = 1 + \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

$$1 + \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \dots$$

$$1 + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \dots$$

$$1 + 1 = \boxed{2}$$

Exercises

If $\{S_n\}$ is the sequence of partial sums of the series $\sum_{n=2}^{\infty} \frac{2}{n^2 - 1}$,
then $\lim_{n \rightarrow \infty} S_n$

- (a) is equal to $\frac{3}{2}$
- (b) is equal to 1
- (c) is equal to $\frac{1}{2}$
- (d) is equal to 0
- (e) does not exist

If $\{S_n\}_{n=1}^{\infty}$ is the sequence of partial sums of the series $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt[3]{n}} - \frac{1}{\sqrt[3]{n+1}} \right)$, then $S_n =$

(a) $1 - \frac{1}{\sqrt[3]{n}}$

(b) $\frac{1}{\sqrt[3]{n}} - \frac{1}{\sqrt[3]{n+1}}$

(c) $\frac{n}{\sqrt[3]{n}}$

(d) $\frac{1}{\sqrt[3]{2}} - \frac{1}{\sqrt[3]{n+1}}$

(e) $1 - \frac{1}{\sqrt[3]{n+1}}$

The series $\sum_{n=2}^{\infty} 3^{n+1} \cdot 2^{1-2n}$ is

- (a) divergent
- (b) convergent and its sum is $27/2$
- (c) convergent and its sum is $9/4$
- (d) convergent and its sum is $3/2$
- (e) convergent and its sum is $27/8$

The series $\sum_{n=0}^{\infty} \frac{e^{1-2n}}{(\sqrt{2})^{2-2n}}$ is

- (a) Convergent and its sum is $\frac{e^3}{2e^2 - 4}$
- (b) Convergent and its sum is $\frac{1}{e^3}$
- (c) Convergent and its sum is $\frac{e}{2}$
- (d) Convergent and its sum is $\frac{e^2}{e^2 - 2}$
- (e) Divergent

Find the value of b for which

$$\sum_{n=0}^{\infty} e^{nb} = 1 + e^b + e^{2b} + e^{2b} + \dots = 9$$

(a) $\ln\left(\frac{8}{9}\right)$

(b) $\frac{-e}{9}$

(c) $\ln\left(\frac{1}{9}\right)$

(d) $\ln\left(\frac{9}{8}\right)$

(e) $e \ln\left(\frac{1}{9}\right)$

The sum of the series $\sum_{n=1}^{\infty} \left[\frac{3}{n(n+1)} + \frac{1}{2^n} \right]$ is equal to

- (a) 4
- (b) 3
- (c) 2
- (d) 1
- (e) 5

The series $\sum_{n=1}^{+\infty} \frac{2^n + (-1)^{n-1}}{3^n}$

- (a) diverges
- (b) converges and its sum is $\frac{3}{4}$
- (c) converges and its sum is 2
- (d) converges and its sum is $\frac{2}{3}$
- (e) converges and its sum is $\frac{9}{4}$

$$\sum_{n=1}^{\infty} \frac{1 + (-2)^n}{3^n} =$$

(a) 0.001

(b) 0.01

(c) 0.1

(d) 1.1

(e) 1.01

Thank you for your attention



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