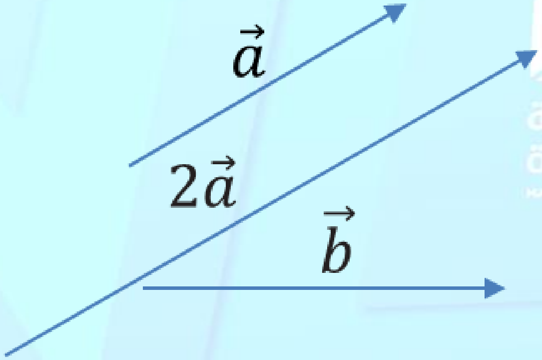
Vectors

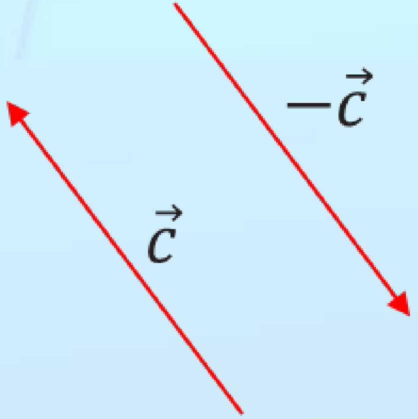
INTRODUCTION



Vector involves both magnitude and direction.

Two vectors are equal if they have the same magnitude and direction.

Vectors $\vec{a}$ and $\vec{b}$ are not equal even if they have same magnitude.



Vector $-\vec{c}$ is defined as having same magnitude but reverse direction as $\vec{c}$.

Multiplying a vector by a scalar changes its magnitude but keeps its direction

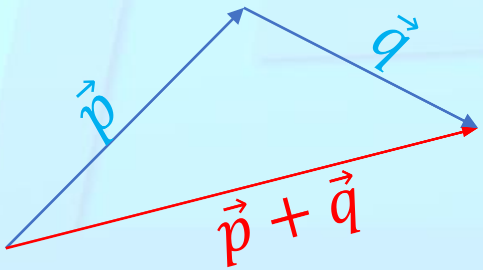
By default vectors are “free” i.e. shifting does not change their magnitude or direction.

Vectors can also be “bound” i.e. cannot be shifted e.g. position vector of point P with respect to origin O.

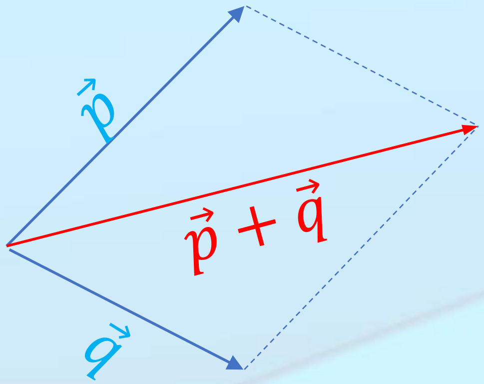
Vector addition implies finding the resultant of two vectors.

There are two methods to do this, both of which are essentially equivalent.

The triangle rule states that, if $\vec{p}$ & $\vec{q}$ represent two sides of a tringle, then $\vec{p}+\vec{q}$ is given by the third side



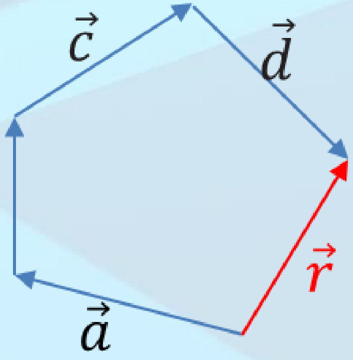
The parallelogram rule states that, if $\vec{p}$ & $\vec{q}$ represent two adjacent sides of a parallelogram, then $\vec{p}+\vec{q}$ is given by its diagonal.



If there are more than two vectors, then we use the polygon rule,

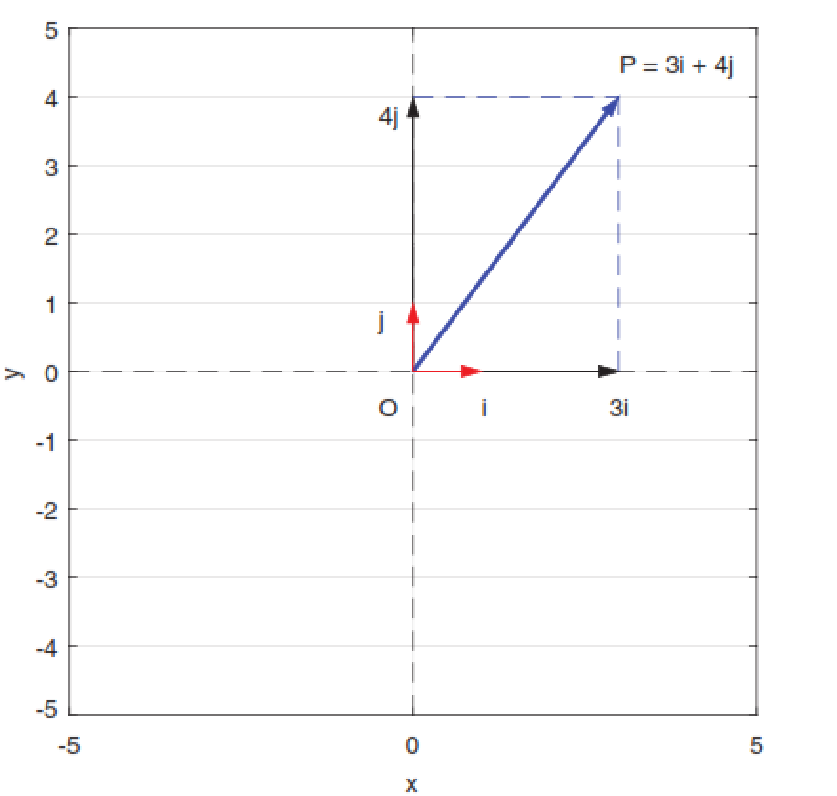
Which says that addition of any number of vectors is obtained by arranging them end to end and closing the final side of resulting polygon:

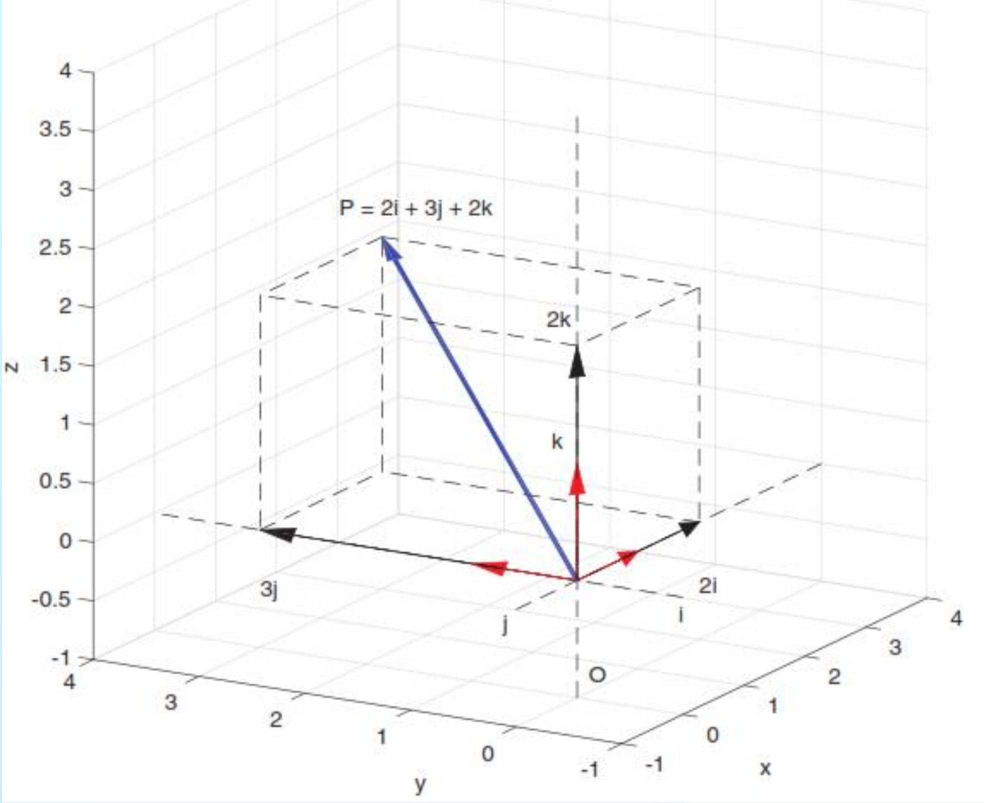
$\vec{r}=\vec{a}+\vec{b}+\vec{c}+\vec{d}$



For 3D vectors, they are to be joined end to end in 3D space.

Vectors can be represented in terms of some chosen reference components. In practice reference vectors are chosen to be orthogonal (perpendicular) and of unit length.

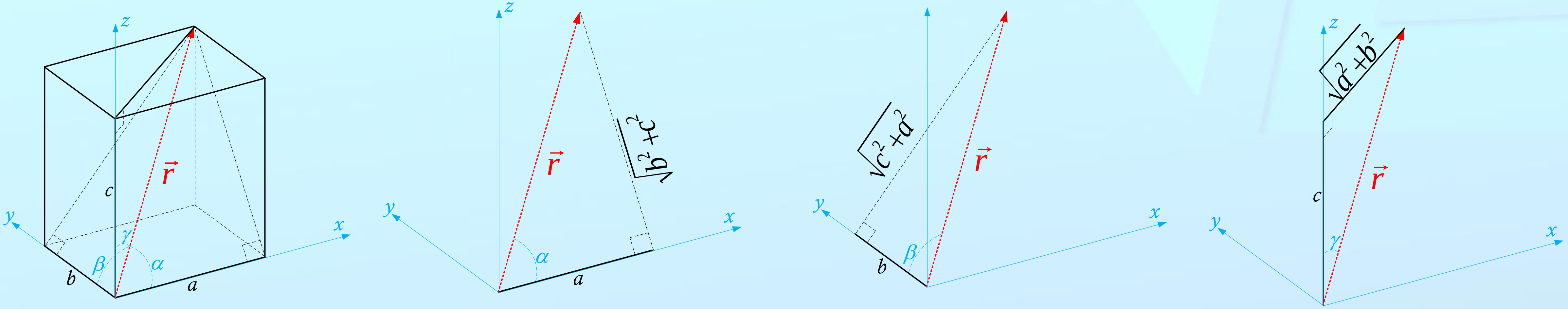




Standard notation for the unit vectors are $\vec{i}$ along *X*-axis, $\vec{j}$ along *Y*-axis, & $\vec{k}$ along the *Z*-axis.

DIRECTION COSINES

Let $\alpha, \beta, \& \gamma$ be the angles made by a vector $\vec{r} = a\vec{i} + b\vec{j} + c\vec{k}$, with the three primary axes. The cosines of these three angles $\cos\alpha, \cos\beta, \&\ \cos\gamma$ are known as direction cosines



$$\cos\alpha = \frac{a}{\sqrt{a^2+b^2+c^2}}, \quad \cos\beta = \frac{b}{\sqrt{a^2+b^2+c^2}}, \quad \cos\gamma = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

It follows from the above that: $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

Example: Sketch the vector $\vec{r} = 5\vec{i} + \sqrt{3}\vec{j} - \sqrt{6}\vec{k}$, and find its length, the direction cosines and the angles it makes with coordinate axes.

DOT PRODUCT or SCALAR PRODUCT

Dot product of two vectors: $\vec{u} = a\vec{i} + b\vec{j} + c\vec{k}$ & $\vec{v} = p\vec{i} + q\vec{j} + r\vec{k}$, with an angle θ between them is given by the following $\vec{u} \cdot \vec{v} = |\vec{u}|\,|\vec{v}| \cos\theta$

Also angle between the vectors: $$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|\,|\vec{v}|}$$

Even it is a product of two vector quantities, the product itself is a scalar number.

$$\vec{u} \cdot \vec{v} = (a\vec{i} + b\vec{j} + c\vec{k}).(p\vec{i} + q\vec{j} + r\vec{k}) = ap+bq+cr$$

The above expression is true because

$$\vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1 \text{ and } \vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{i} = 0$$

Corollary 1: If vectors are parallel then $\theta = 0$ hence $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|$

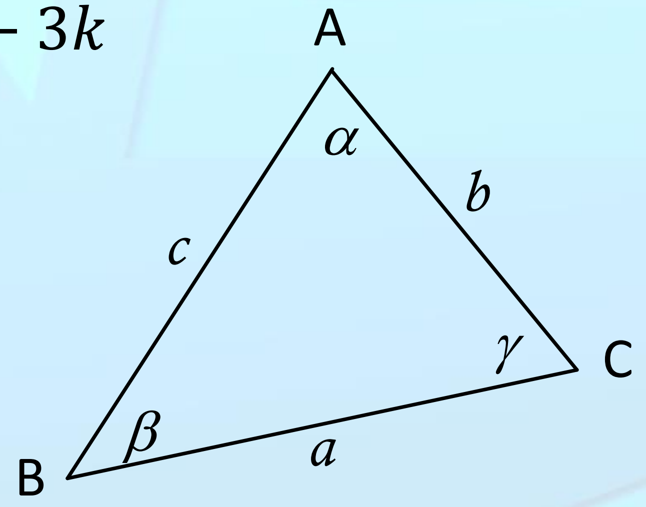
Corollary 2: If vectors are perpendicular then $\theta = 90°$ hence $\vec{u} \cdot \vec{v} = 0$

Example: Sketch the two vectors in the following three cases. Then

(a) Find if vectors $\vec{u} = 3\vec{i} + 5\vec{j} - 2\vec{k}$, and $\vec{v} = 2\vec{i} - 2\vec{j} - 2\vec{k}$, are perpendicular to each other

(b) Find if vectors $\vec{u} = 3\vec{i} + 5\vec{j} - 2\vec{k}$, and $\vec{v} = 6\vec{i} + 10\vec{j} - 4\vec{k}$, are parallel to each other

(c) Find the angle between vectors $\vec{u} = 2\vec{i} - 3\vec{j} + \vec{k}$, and $\vec{v} = 4\vec{i} + \vec{j} - 3\vec{k}$



Cosine Formula in the Triangle

Given a tringle with vertices (singular, vertex): A, B and C.

With α, β and γ as the corresponding angles.

And a, b and c , as the corresponding sides. From the figure.

$\overrightarrow{AC} - \overrightarrow{AB} = \overrightarrow{BC}$, then $(\overrightarrow{BC})^2 = (\overrightarrow{AC} - \overrightarrow{AB})^2$, which gives the general cosine formula

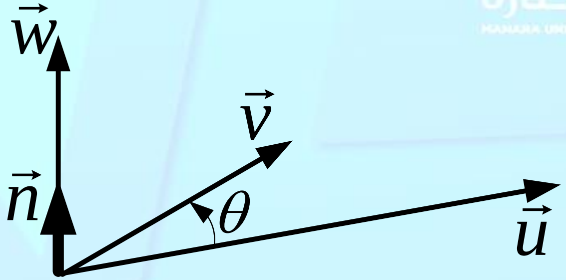
$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

Ex. Write the two other similar formula.

CROSS PRODUCT or VECTOR PRODUCT

Cross (vector) product of two vectors $\vec{u} = a\vec{i} + b\vec{j} + c\vec{k}$ & $\vec{v} = p\vec{i} + q\vec{j} + r\vec{k}$, with an angle θ between them is given by the following

$$\vec{w} = \vec{u} \times \vec{v} = (|\vec{u}|\,|\vec{v}|\,\sin\theta)\vec{n}$$

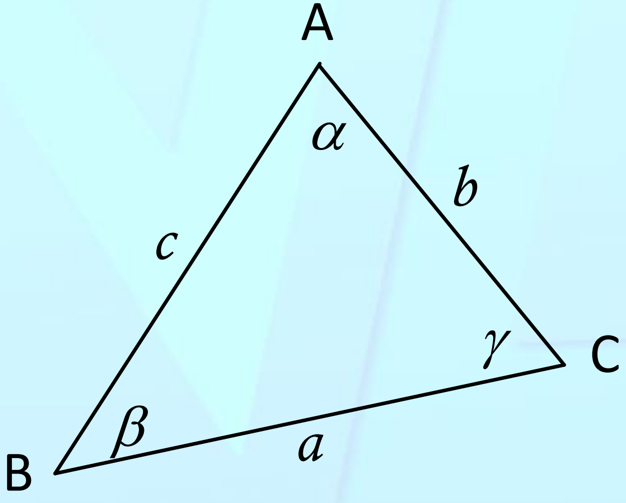


The result is a vector in the direction of $\vec{n}$: unit vector perpendicular to both $\vec{u}$ and $\vec{v}$.

Its direction is governed by the right-handed rule.

Example: Find a unit vector perpendicular to both: $\vec{u} = 3\vec{i} + 5\vec{j} - 2\vec{k}$ & $\vec{v} = 2\vec{i} - 2\vec{j} - 2\vec{k}$

Sine Formula in the Triangle



$$\text{AREA} = \left|\tfrac{1}{2}\overrightarrow{AB}\times\overrightarrow{AC}\right| = \left|\tfrac{1}{2}\overrightarrow{BC}\times\overrightarrow{BA}\right| = \left|\tfrac{1}{2}\overrightarrow{CA}\times\overrightarrow{CB}\right|$$

Then: $\tfrac{1}{2}cb\sin\alpha = \tfrac{1}{2}ac\sin\beta = \tfrac{1}{2}ba\sin\gamma$

Multiply by 2 and divide by abc, to get

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

Inverting the fractions to get

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$