Mathematical Induction

Epp, chapter 4

How do you climb infinite stairs?

- Not a rhetorical question!

- First, you get to the base platform of the staircase
- Then repeat:
  - From your current position, move one step up

Let's use that as a proof method

- First, show P(x) is true for $x=0$
  - This is the base of the stairs
- Then, show that if it's true for some value n, then it is true for $n+1$
  - Show: $P(n) \rightarrow P(n+1)$
  - This is climbing the stairs
  - Let $n=0$. Since it's true for P(0) (base case), it's true for $n=1$
  - Let $n=1$. Since it's true for P(1) (previous bullet), it's true for $n=2$
  - Let $n=2$. Since it's true for P(2) (previous bullet), it's true for $n=3$
  - Let $n=3$ …
  - And onwards to infinity
- Thus, we have shown it to be true for *all* non-negative numbers

What is induction?

- A method of proof
- It does not generate answers: it only can prove them

- Three parts:
  - Base case(s): show it is true for one element
    - (get to the stair's base platform)
  - Inductive hypothesis: assume it is true for any given element
    - (assume you are on a stair)
    - **Must be clearly labeled!!!**
  - Show that if it true for the next highest element
    - (show you can move to the next stair)

Induction example

- Show that the sum of the first n odd integers is n^2
  - Example: If $n = 5$, $1+3+5+7+9 = 25 = 5^2$
  - Formally, show:

$$\forall n \; P(n) \text{ where } P(n) = \sum_{i=1}^{n} 2i - 1 == n^2$$

- Base case: Show that P(1) is true

$$P(1) = \sum_{i=1}^{1} 2(i) - 1 == 1^2$$

$$= 1 == 1$$

Induction example, continued

- Inductive hypothesis: assume true for *k*
  - Thus, we assume that P(*k*) is true, or that

$$\sum_{i=1}^{k} 2i - 1 == k^2$$

  - Note: we don't yet know if this is true or not!

- Inductive step: show true for *k*+1
  - We want to show that:

$$\sum_{i=1}^{k+1} 2i - 1 == (k+1)^2$$

Induction example, continued

- Recall the inductive hypothesis: $\sum_{i=1}^{k} 2i-1 == k^2$

- Proof of inductive step:

$$\sum_{i=1}^{k+1} 2i-1 == (k+1)^2$$

$$2(k+1)-1+\boxed{\sum_{i=1}^{k} 2i-1} == k^2+2k+1$$

$$2(k+1)-1+\boxed{k^2} == k^2+2k+1$$

$$k^2+2k+1 == k^2+2k+1$$

What did we show

- Base case: P(1)
- If P(*k*) was true, then P(*k*+1) is true
  - i.e., P(*k*) → P(*k*+1)

- We know it's true for P(1)
- Because of P(*k*) → P(*k*+1), if it's true for P(1), then it's true for P(2)
- Because of P(*k*) → P(*k*+1), if it's true for P(2), then it's true for P(3)
- Because of P(*k*) → P(*k*+1), if it's true for P(3), then it's true for P(4)
- Because of P(*k*) → P(*k*+1), if it's true for P(4), then it's true for P(5)
- And onwards to infinity

- Thus, it is true for all possible values of *n*

- In other words, we showed that:

$$\left[P(1) \wedge \forall k \left(P(k) \rightarrow P(k+1)\right)\right] \rightarrow \forall n\, P(n)$$

The idea behind inductive proofs

- Show the base case
- Show the inductive hypothesis
- Manipulate the inductive step so that you can substitute in part of the inductive hypothesis
- Show the inductive step

Second induction example

- Show the sum of the first *n* positive even integers is $n^2 + n$
  - Rephrased:

$$\forall n \; \mathrm{P}(n) \text{ where } \mathrm{P}(n) = \sum_{i=1}^{n} 2i == n^2 + n$$

- The three parts:
  - Base case
  - Inductive hypothesis
  - Inductive step

Second induction example, continued

- Base case: Show P(1):

$$P(1) = \sum_{i=1}^{1} 2(i) == 1^2 + 1$$
$$= 2 == 2$$

- Inductive hypothesis: Assume

$$P(k) = \sum_{i=1}^{k} 2i == k^2 + k$$

- Inductive step: Show

$$P(k+1) = \sum_{i=1}^{k+1} 2i == (k+1)^2 + (k+1)$$

Second induction example, continued

- Recall our inductive hypothesis:

$$P(k) = \sum_{i=1}^{k} 2i == k^2 + k$$

$$\sum_{i=1}^{k+1} 2i == (k+1)^2 + k + 1$$

$$2(k+1) + \boxed{\sum_{i=1}^{k} 2i} == (k+1)^2 + k + 1$$

$$2(k+1) + \boxed{k^2 + k} == (k+1)^2 + k + 1$$

$$k^2 + 3k + 2 == k^2 + 3k + 2$$

Notes on proofs by induction

- We manipulate the *k*+1 case to make part of it look like the *k* case
- We then replace that part with the other side of the *k* case

$$P(k) = \sum_{i=1}^{k} 2i == k^2 + k$$

$$\sum_{i=1}^{k+1} 2i == (k+1)^2 + k + 1$$

$$2(k+1) + \boxed{\sum_{i=1}^{k} 2i} == (k+1)^2 + k + 1$$

$$2(k+1) + \boxed{k^2 + k} == (k+1)^2 + k + 1$$

$$k^2 + 3k + 2 == k^2 + 3k + 2$$

Third induction example

- Show $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

- Base case: $n = 1$
$$\sum_{i=1}^{1} i^2 = \frac{1(1+1)(2+1)}{6}$$
$$1^2 = \frac{6}{6}$$
$$1 = 1$$

- Inductive hypothesis: assume
$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

Third induction example

- Inductive step: show $$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

$$(k+1)^2 + \sum_{i=1}^{k} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$(k+1)^2 + \frac{k(k+1)(2k+1)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$6(k+1)^2 + k(k+1)(2k+1) = (k+1)(k+2)(2k+3)$$

$$2k^3 + 9k^2 + 13k + 6 = 2k^3 + 9k^2 + 13k + 6$$

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

Third induction again: what if your inductive hypothesis was wrong?

- Show: $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+2)}{6}$$
- Base case: $n = 1$: $$\sum_{i=1}^{1} i^2 = \frac{1(1+1)(2+2)}{6}$$ $$1^2 = \frac{7}{6}$$ $$1 \neq \frac{7}{6}$$
- But let's continue anyway…
- Inductive hypothesis: assume $$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+2)}{6}$$

Third induction again: what if your inductive hypothesis was wrong?

- Inductive step: show $\sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+2)}{6}$

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+2)}{6}$$

$$(k+1)^2 + \sum_{i=1}^{k} i^2 = \frac{(k+1)(k+2)(2k+4)}{6}$$

$$(k+1)^2 + \frac{k(k+1)(2k+2)}{6} = \frac{(k+1)(k+2)(2k+4)}{6}$$

$$6(k+1)^2 + k(k+1)(2k+2) = (k+1)(k+2)(2k+4)$$

$$2k^3 + 10k^2 + 14k + 6 \neq 2k^3 + 10k^2 + 16k + 8$$

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+2)}{6}$$

Fourth induction example

- S that $n! < n^n$ for all $n > 1$

- Base case: $n = 2$

  $2! < 2^2$

  $2 < 4$

- Inductive hypothesis: assume $k! < k^k$
- Inductive step: show that $(k+1)! < (k+1)^{k+1}$

$(k+1)!$	$=(k+1)k!$	$<(k+1)k^k$	$<(k+1)(k+1)^k$	$=(k+1)^{k+1}$

Strong induction

- Weak mathematical induction assumes P(k) is true, and uses that (and only that!) to show P(k+1) is true

- Strong mathematical induction assumes P(1), P(2), …, P(k) are all true, and uses that to show that P(k+1) is true.

$$\left[P(1) \wedge P(2) \wedge P(3) \wedge \ldots \wedge P(k)\right] \rightarrow P(k+1)$$

Strong induction example 1

- Show that any number > 1 can be written as the product of one or more primes

- Base case: P(2)
  - 2 is the product of 2 (remember that 1 is not prime!)
- Inductive hypothesis: assume P(2), P(3), …, P(k) are all true
- Inductive step: Show that P(k+1) is true

Strong induction example 1

- Inductive step: Show that P(k+1) is true
- There are two cases:
  - k+1 is prime
    - It can then be written as the product of k+1
  - k+1 is composite
    - It can be written as the product of two composites, a and b, where $2 \leq a \leq b < k+1$
    - By the inductive hypothesis, both P(a) and P(b) are true

Strong induction vs. non-strong induction

- Determine which amounts of postage can be written with 5 and 6 cent stamps
  - Prove using both versions of induction
- Answer: any postage ≥ 20

Answer via mathematical induction

- Show base case: P(20):
  - $20 = 5 + 5 + 5 + 5$
- Inductive hypothesis: Assume P(*k*) is true
- Inductive step: Show that P(*k*+1) is true
  - If P(*k*) uses a 5 cent stamp, replace that stamp with a 6 cent stamp
  - If P(*k*) does not use a 5 cent stamp, it must use only 6 cent stamps
    - Since $k > 18$, there must be four 6 cent stamps
    - Replace these with five 5 cent stamps to obtain $k+1$

Answer via strong induction

- Show base cases: P(20), P(21), P(22), P(23), and P(24)
  - 20 = 5 + 5 + 5 + 5
  - 21 = 5 + 5 + 5 + 6
  - 22 = 5 + 5 + 6 + 6
  - 23 = 5 + 6 + 6 + 6
  - 24 = 6 + 6 + 6 + 6
- Inductive hypothesis: Assume P(20), P(21), …, P(k) are all true
- Inductive step: Show that P(k+1) is true
  - We will obtain P(k+1) by adding a 5 cent stamp to P(k+1-5)
  - Since we know P(k+1-5) = P(k-4) is true, our proof is complete

Strong induction vs. non-strong induction, take 2

- Show that every postage amount 12 cents or more can be formed using only 4 and 5 cent stamps

- Similar to the previous example

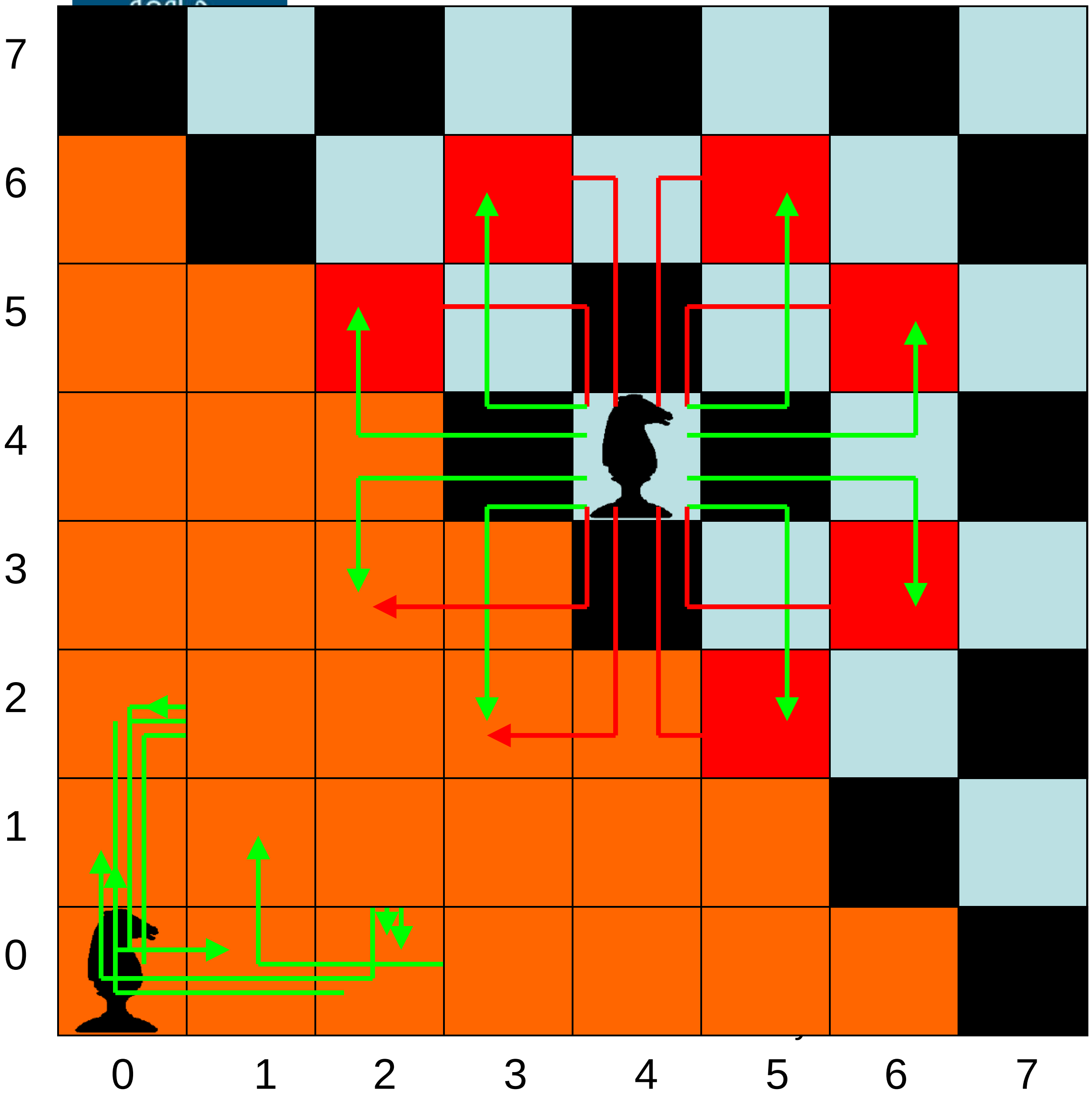
Answer via mathematical induction

- Show base case: P(12):
  - 12 = 4 + 4 + 4
- Inductive hypothesis: Assume P(k) is true
- Inductive step: Show that P(k+1) is true
  - If P(k) uses a 4 cent stamp, replace that stamp with a 5 cent stamp to obtain $P(k+1)$
  - If P(k) does not use a 4 cent stamp, it must use only 5 cent stamps
    - Since $k > 10$, there must be at least three 5 cent stamps
    - Replace these with four 4 cent stamps to obtain $k+1$
- Note that only $P(k)$ was assumed to be true

Answer via strong induction

- Show base cases: P(12), P(13), P(14), and P(15)
  - 12 = 4 + 4 + 4
  - 13 = 4 + 4 + 5
  - 14 = 4 + 5 + 5
  - 15 = 5 + 5 + 5
- Inductive hypothesis: Assume P(12), P(13), …, P(k) are all true
  - For $k \geq 15$
- Inductive step: Show that P(k+1) is true
  - We will obtain P(k+1) by adding a 4 cent stamp to P(k+1-4)
  - Since we know P(k+1-4) = P(k-3) is true, our proof is complete
- Note that P(12), P(13), …, P(k) were all assumed to be true

Chess and induction



Can the knight reach any square in a finite number of moves?

Show that the knight can reach any square (i, j) for which $i+j=k$ where $k > 1$.

Base case: $k = 2$

Inductive hypothesis: assume the knight can reach any square (i, j) for which $i+j=k$ where $k > 1$.

Inductive step: show the knight can reach any square (i, j) for which $i+j=k+1$ where $k > 1$.

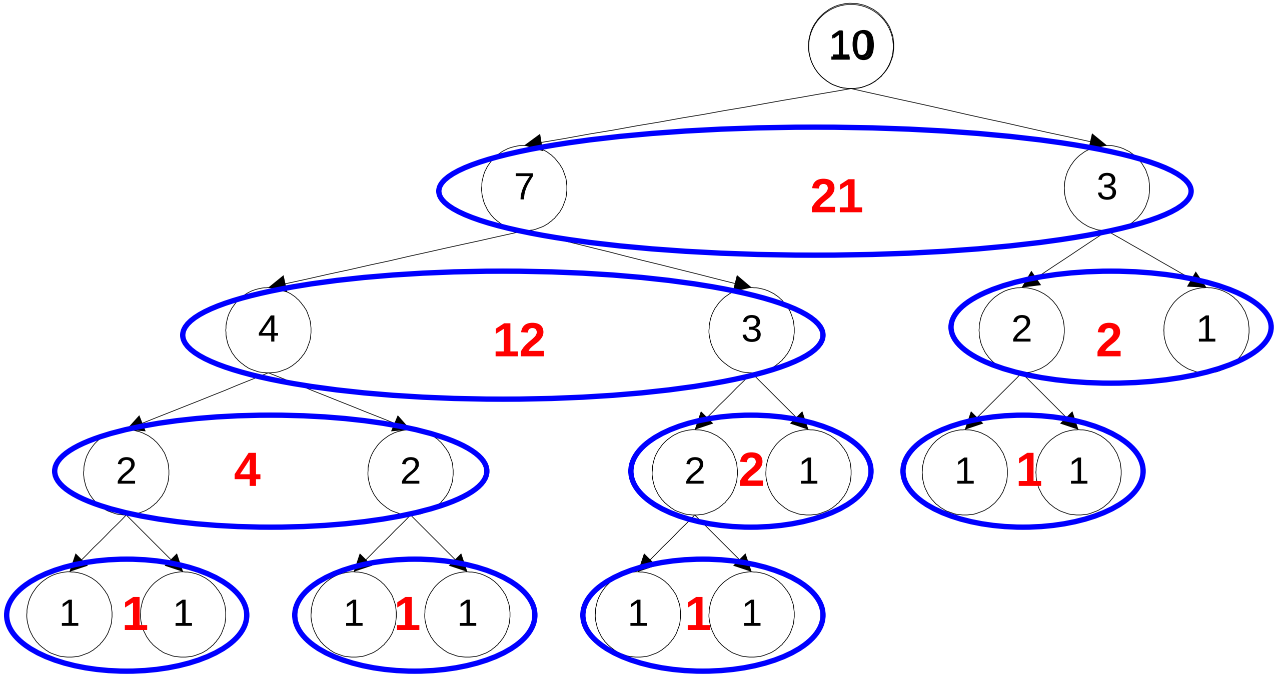
Chess and induction

- Inductive step: show the knight can reach any square (i, j) for which $i+j=k+1$ where $k > 1$.
  - Note that $k+1 \geq 3$, and one of i or j is ≥ 2
  - If $i \geq 2$, the knight could have moved from (i-2, j+1)
    - Since $i+j = k+1$, $i-2 + j+1 = k$, which is assumed true
  - If $j \geq 2$, the knight could have moved from (i+1, j-2)
    - Since $i+j = k+1$, $i+1 + j-2 = k$, which is assumed true

Inducting stones

- Take a pile of *n* stones
  - Split the pile into two smaller piles of size *r* and *s*
  - Repeat until you have *n* piles of 1 stone each
- Take the product of **all** the splits
  - So all the *r*'s and *s*'s from **each** split
- Sum up each of these products
- Prove that this product equals $\frac{n(n-1)}{2}$

Inducting stones



$$\frac{n(n-1)}{2}$$

$$21+12+2+4+2+1+1+1+1=45=\frac{10*9}{2}$$

Inducting stones

- We will show it is true for a pile of k stones, and show it is true for $k+1$ stones
  - So $P(k)$ means that it is true for k stones

- Base case: $n = 1$
  - No splits necessary, so the sum of the products = 0
  - 1*(1-1)/2 = 0
  - Base case proven

Inducting stones

- Inductive hypothesis: assume that P(1), P(2), …, P(k) are all true
  - This is strong induction!

- Inductive step: Show that P(k+1) is true
  - We assume that we split the k+1 pile into a pile of i stones and a pile of k+1-i stones
  - Thus, we want to show that $(i)*(k+1-i) + P(i) + P(k+1-i) = P(k+1)$
  - Since $0 < i < k+1$, both i and k+1-i are between 1 and k, inclusive

Inducting stones

Thus, we want to show that

(i)*(k+1-i) + P(i) + P(k+1-i) = P(k+1)

$$P(i) = \frac{i^2 - i}{2}$$

$$P(k+1-i) = \frac{(k+1-i)(k+1-i-1)}{2} = \frac{k^2 + k - 2ki - i + i^2}{2}$$

$$P(k+1) = \frac{(k+1)(k+1-1)}{2} = \frac{k^2 + k}{2}$$

$$(i)*(k+1-i) + P(i) + P(k+1-i) = P(k+1)$$

$$ki + i - i^2 + \frac{i^2 - i}{2} + \frac{k^2 + k - 2ki - i - i^2}{2} = \frac{k^2 + k}{2}$$

$$2ki + 2i - 2i^2 + i^2 - i + k^2 + k - 2ki - i + i^2 = k^2 + k$$

$$k^2 + k = k^2 + k$$