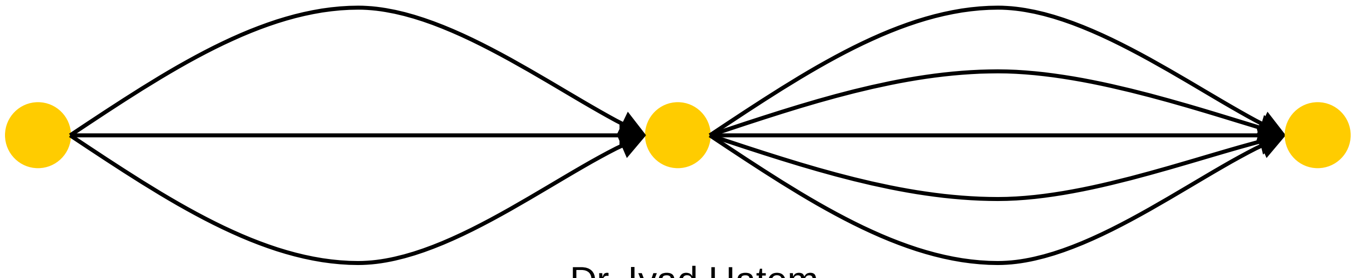
Basics of Counting

Epp sections 6.2 & 6.3

The product rule

- Also called the multiplication rule
- If there are n_1 ways to do task 1, and n_2 ways to do task 2
  - Then there are $n_1 n_2$ ways to do both tasks in sequence
  - This applies when doing the “procedure” is made up of separate tasks
  - We must make one choice AND a second choice



Product rule example

- Sample question
  - There are 18 math majors and 325 CS majors
  - How many ways are there to pick one math major and one CS major?

- Total is 18 * 325 = 5850

Product rule example

More sample questions…

- How many strings of 4 decimal digits…

a) Do not contain the same digit twice?

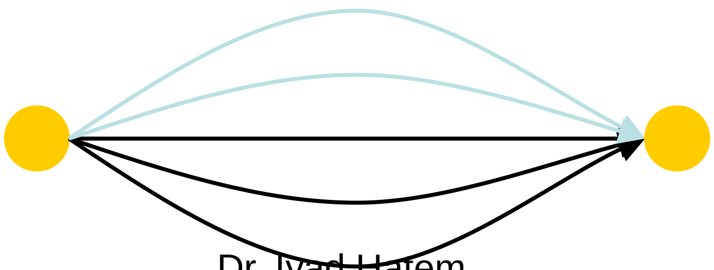
- We want to chose a digit, then another that is not the same, then another…
  - First digit: 10 possibilities
  - Second digit: 9 possibilities (all but first digit)
  - Third digit: 8 possibilities
  - Fourth digit: 7 possibilities
- Total = 10*9*8*7 = 5040

b) End with an even digit?

- First three digits have 10 possibilities
- Last digit has 5 possibilities
- Total = 10*10*10*5 = 5000

The sum rule

- Also called the addition rule
- If there are n_1 ways to do task 1, and n_2 ways to do task 2
  - If these tasks can be done at the same time, then…
  - Then there are n_1+n_2 ways to do one of the two tasks
  - We must make one choice OR a second choice



Sum rule example

- Sample question
  - There are 18 math majors and 325 CS majors
  - How many ways are there to pick one math major or one CS major?

- Total is 18 + 325 = 343

Sum rule example

More sample questions

- How many strings of 4 decimal digits…
- Have exactly three digits that are 9s?
  - The string can have:
    - The non-9 as the first digit
    - OR the non-9 as the second digit
    - OR the non-9 as the third digit
    - OR the non-9 as the fourth digit
    - Thus, we use the sum rule
  - For each of those cases, there are 9 possibilities for the non-9 digit (any number other than 9)
  - Thus, the answer is 9+9+9+9 = 36

More complex counting problems

- We combining the product rule and the sum rule

- Thus we can solve more interesting and complex problems

Wedding pictures example

- Consider a wedding picture of 6 people
  - There are 10 people, including the bride and groom

a) How many possibilities are there if the bride must be in the picture

- Product rule: place the bride AND then place the rest of the party
- First place the bride
  - She can be in one of 6 positions
- Next, place the other five people via the product rule
  - There are 9 people to choose for the second person, 8 for the third, etc.
  - Total = 9*8*7*6*5 = 15120
- Product rule yields 6 * 15120 = 90,720 possibilities

Wedding pictures example

- Consider a wedding picture of 6 people
  - There are 10 people, including the bride and groom

b) How many possibilities are there if the bride and groom must both be in the picture

- Product rule: place the bride/groom AND then place the rest of the party
- First place the bride and groom
  - She can be in one of 6 positions
  - He can be in one 5 remaining positions
  - Total of 30 possibilities
- Next, place the other four people via the product rule
  - There are 8 people to choose for the third person, 7 for the fourth, etc.
  - Total = 8*7*6*5 = 1680
- Product rule yields 30 * 1680 = 50,400 possibilities

Wedding pictures example

- Consider a wedding picture of 6 people
  - There are 10 people, including the bride and groom

c) How many possibilities are there if only one of the bride and groom are in the picture

- Sum rule: place only the bride
  - Product rule: place the bride AND then place the rest of the party
  - First place the bride
    - She can be in one of 6 positions
  - Next, place the other five people via the product rule
    - There are 8 people to choose for the second person, 7 for the third, etc.
      - » We can't choose the groom!
    - Total = 8*7*6*5*4 = 6720
  - Product rule yields 6 * 6720 = 40,320 possibilities
- OR place only the groom
  - Same possibilities as for bride: 40,320
- Sum rule yields 40,320 + 40,320 = 80,640 possibilities

Wedding pictures example

- Consider a wedding picture of 6 people
  - There are 10 people, including the bride and groom

- Alternative means to get the answer

c) How many possibilities are there if only one of the bride and groom are in the picture

- Total ways to place the bride (with or without groom): 90,720
  - From part (a)
- Total ways for both the bride and groom: 50,400
  - From part (b)
- Total ways to place ONLY the bride: 90,720 – 50,400 = 40,320
- Same number for the groom
- Total = 40,320 + 40,320 = 80,640

The inclusion-exclusion principle

- When counting the possibilities, we can't include a given outcome more than once!

- $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$
  - Let A_1 have 5 elements, A_2 have 3 elements, and 1 element be both in A_1 and A_2
  - Total in the union is 5+3-1 = 7, not 8

Inclusion-exclusion example

- How may bit strings of length eight start with 1 or end with 00?

- Count bit strings that start with 1
  - Rest of bits can be anything: $2^7 = 128$
  - This is $|A_1|$
- Count bit strings that end with 00
  - Rest of bits can be anything: $2^6 = 64$
  - This is $|A_2|$
- Count bit strings that both start with 1 and end with 00
  - Rest of the bits can be anything: $2^5 = 32$
  - This is $|A_1 \cap A_2|$

- Use formula $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$
- Total is $128 + 64 – 32 = 160$

Bit string possibilities

- How many bit strings of length 10 contain either 5 consecutive 0s or 5 consecutive 1s?

Bit string possibilities

- Consider 5 consecutive 0s first
- Sum rule: the 5 consecutive 0's can start at position 1, 2, 3, 4, 5, or 6
  - Starting at position 1
    - Remaining 5 bits can be anything: $2^5 = 32$
  - Starting at position 2
    - First bit must be a 1
      - Otherwise, we are including possibilities from the previous case!
    - Remaining bits can be anything: $2^4 = 16$
  - Starting at position 3
    - Second bit must be a 1 (same reason as above)
    - First bit and last 3 bits can be anything: $2^4 = 16$
  - Starting at positions 4 and 5 and 6
    - Same as starting at positions 2 or 3: 16 each
  - Total = 32 + 16 + 16 + 16 + 16 + 16 = 112
- The 5 consecutive 1's follow the same pattern, and have 112 possibilities
- There are two cases counted twice (that we thus need to exclude): 0000011111 and 1111100000
- Total = 112 + 112 – 2 = 222

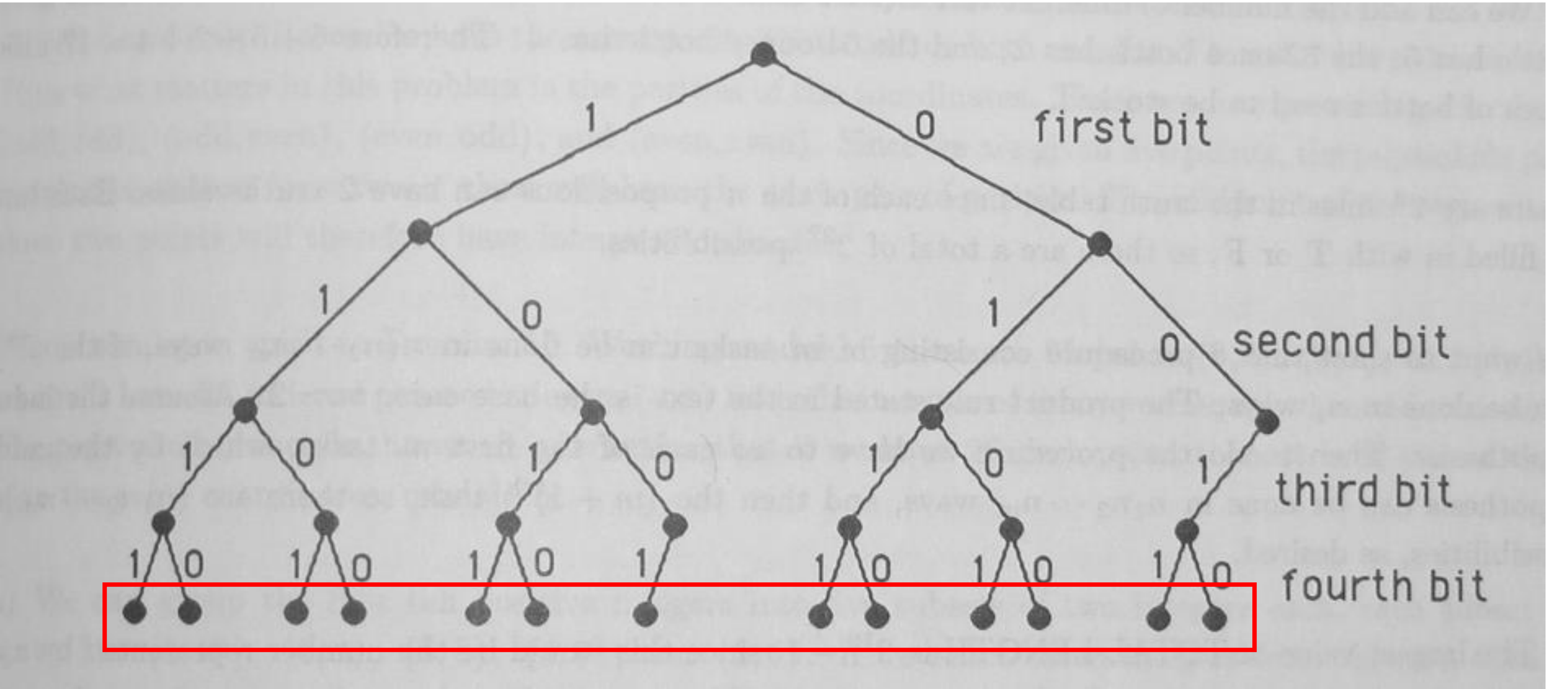
Tree diagrams

- We can use tree diagrams to enumerate the possible choices

- Once the tree is laid out, the result is the number of (valid) leaves

Tree diagrams example

- Use a tree diagram to find the number of bit strings of length four with no three consecutive 0s



An example closer to home…

- How many ways can the Cavs finish the season 9 and 2?
  - This was from fall ’04….