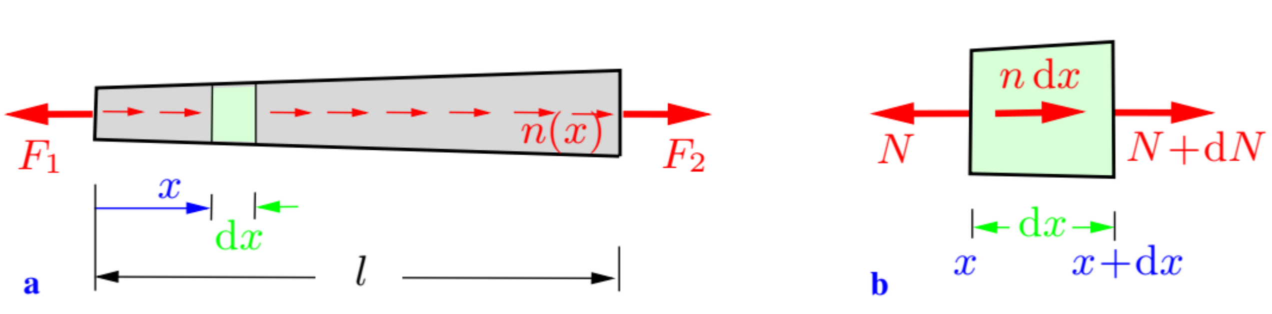
4 Single Bar under Tension or Compression

There are three different types of equations that allow us to determine the stresses & the strains in a bar: the equilibrium condition, the kinematic relation and Hooke's law.

Depending on the problem, the equilibrium condition may be formulated for the entire bar, a portion of the bar or for an element of the bar.

We will derive the equilibrium condition for an element. For this purpose we consider a bar which is subjected to two forces F_1 & F_2 at its ends and to a line load $n = n(x)$, see Fig.a.



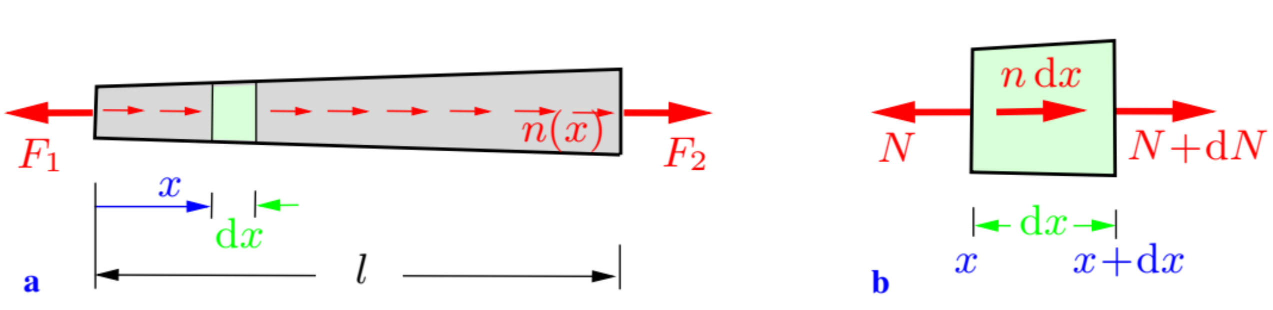
The forces are assumed to be in equilibrium. We imagine a slice element of infinitesimal length dx separated from the bar as shown in Fig.b.

The F. B. D. shows the normal forces N and $N + dN$, respectively, at the ends of the element; the line load is replaced by its resultant ndx (note that n may be considered to be constant over the length dx). Equilibrium of the forces in the direction of the axis of the bar

$\rightarrow: N + dN + n\,dx - N = 0$

yields the *equilibrium condition*

$$\frac{dN}{dx} + n = 0$$



In the special case of a vanishing line load ($n \equiv 0$) $F_1 = F_2 = N$

The *kinematic relation* for the bar is $\varepsilon = \frac{du}{dx}$ and Hooke's law is given by $\varepsilon = \frac{\sigma}{E}$

If we insert the kinematic relation and $\sigma = N/A$ into Hooke's law we obtain

$$\varepsilon = \frac{du}{dx} = \frac{N}{EA}$$

This equation relates the displacements $u(x)$ of the cross sections and the normal force $N(x)$. It may be called the *constitutive law for the bar*.

The displacement u of a cross section is found through integration of the strain:

$$\varepsilon = \frac{du}{dx} \rightarrow \int du = \int \varepsilon dx \rightarrow u(x) - u(0) = \int_0^x \varepsilon d\bar{x}\,.$$

The elongation Δl follows as the difference of the displacements at the ends $x = l$ and $x = 0$ of the bar:

$$\Delta l = u(l) - u(0) = \int_0^l \varepsilon dx$$

With $\varepsilon = du/dx$ and the *constitutive law for the bar* this yields

$$\Delta l = \int_0^l \frac{N}{EA} dx$$

In the special case of a bar (length l) with constant axial rigidity ($EA = const$) which is subjected only to forces at its end ($n \equiv 0, N = F$) the elongation is given by

$$\Delta l = \frac{l}{EA} F \Leftrightarrow F = \frac{EA}{l} \Delta l$$

Quantity $\frac{EA}{l}$ is the *axial rigidity* (*Stiffness*) *of the bar*. The Inverse $\frac{l}{EA}$ is the axial *flexibility* of the bar

If we want to apply these equations to specific problems, we have to distinguish between *statically determinate* and *statically indeterminate* problems.

In a *statically determinate* system we can always calculate the normal force $N(x)$ with the aid of the equilibrium condition.

Subsequently, the strain $\varepsilon(x)$ follows from $\sigma = N/A$ and Hooke's law $\varepsilon = \sigma/E$. Finally, integration yields the displacement $u(x)$ and the elongation Δl.

In a *statically indeterminate* problem, with the equilibrium condition alone the normal force cannot be calculated.

In such problems the basic equations (equilibrium condition, kinematic relation and Hooke´s law) are a system of *coupled* equations and have to be solved simultaneously.

Finally we will reduce the basic equations to a single equation for the displacement u.

By combining the two equations: $$\boldsymbol{\varepsilon = \frac{du}{dx} = \frac{N}{EA}}$$ $$\boldsymbol{\frac{dN}{dx} + n = 0}$$ To get:

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right) = \frac{d}{dx}(N) = -n$$

With, the primes denoting derivatives with respect to x

$$(EAu')' = -n$$

If the functions $EA(x)$, $n(x)$, are given, the equation $(EAu')' = -n$

the displacement $u(x)$ of an arbitrary cross section can be determined by integration. The constants of integration are calculated from the boundary conditions.

If, for example, one end of the bar is fixed then $u = 0$ at this end.

If, on the other hand, one end of the bar can move and is subjected to a force F_0, then applying and $N = F_0$ yields the boundary condition $u' = F_0/EA$.

This reduces to the boundary condition $u' = 0$ in the special case of a stress-free end ($F_0 = 0$) of a bar.

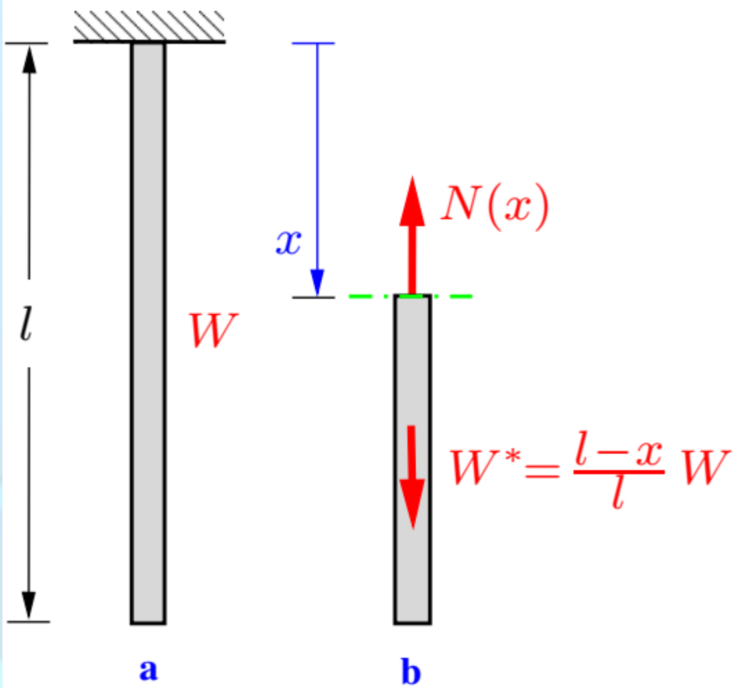
illustrative example As a statically determinate system let us consider a slender bar (weight *W*, cross-sectional area *A*) that is suspended from the ceiling (Fig.a).

First we determine the normal force caused by the weight of the bar. We cut the bar at an arbitrary position x (Fig.b).

The normal force N = the weight W^* of the portion of the bar below the imaginary cut. Thus, it is given by $N(x) = W^*(x) = W(l - x)/l$. Then the normal stress is

$$\sigma(x) = \frac{N(x)}{A} = \frac{W}{A}\left(1 - \frac{x}{l}\right)$$

Accordingly, the normal stress in the bar varies linearly; it decreases from the value $\sigma(0) = W/A$ at the upper end to $\sigma(l) = 0$ at the free end.



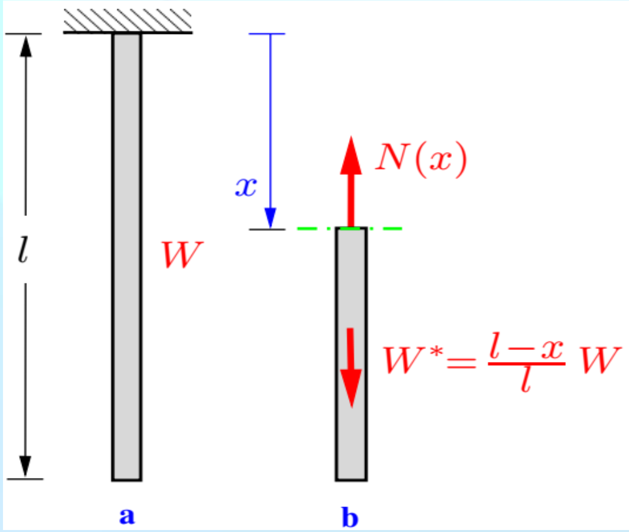
illustrative example As a statically determinate system let us consider a slender bar (weight W, cross-sectional area A) that is suspended from the ceiling (Fig.a).

First we determine the normal force caused by the weight of the bar. We cut the bar at an arbitrary position x (Fig.b).

The normal force N = the weight W^* of the portion of the bar below the imaginary cut.

Thus, it is given by $N(x) = W^*(x) = W(l - x)/l$. Then the normal stress is

$$\sigma(x) = \frac{N(x)}{A} = \frac{W}{A}\left(1 - \frac{x}{l}\right)$$



Accordingly, the normal stress in the bar varies linearly; it decreases from the value $\sigma(0) = W/A$ at the upper end to $\sigma(l) = 0$ at the free end.

The elongation Δl of the bar due to its own weight is obtained from

$$\Delta l = \int_0^l \frac{N}{EA} dx = \frac{W}{EA}\int_0^l \left(1 - \frac{x}{l}\right) dx = \frac{1}{2}\frac{Wl}{EA}$$

It is half the elongation of a bar with negligible weight which is subjected to the force W at the free end.

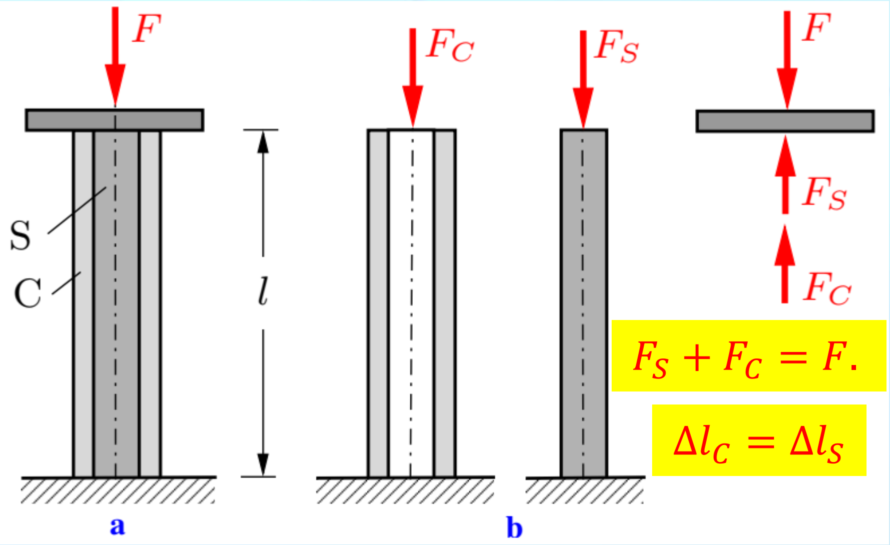
We may also solve the problem by applying the differential equation $(EAu')' = -n$ for the displacements $u(x)$ of the cross sections of the bar. Integration with the constant line load $n = W/l$, yields

$$EAu'' = -w/l \qquad \Rightarrow EAu' = -(w/l)x + C_1 \qquad \Rightarrow EAu = -(w/2l)x^2 + C_1 x + C_2$$

C_1 & C_2, constants of integration, can be determined from the boundary conditions. The displacement of the cross section at the upper end of the bar is equal to zero: $u(0) = 0$. Since the stress σ vanishes at the free end, we have $u'(l) = 0$. This leads to $C_2 = 0$ and $C_1 = W$. Thus, the displacement and the normal force are given by

$u(x) = \frac{1}{2}\frac{Wl}{EA}\left(2\frac{x}{l} - \frac{x^2}{l^2}\right)$ The bar elongation $\Delta l = u(l) = \frac{1}{2}\frac{Wl}{EA}$ and the normal force $N(x) = EAu' = W(1 - \frac{x}{l})$

illustrative example As an illustrative example of a statically indeterminate system let us consider a solid circular steel cylinder (cross-sectional area A_S, modulus of elasticity E_S, length l) is placed inside a copper tube (cross-sectional area A_C, modulus of elasticity E_C, length l). The assembly is compressed between a rigid plate and the rigid floor by a force F (Fig.a). Calculate the shortening of the assembly and Determine the normal Forces in the cylinder and in the tube..



$F_S + F_C = F.$

$\Delta l_C = \Delta l_S$

Solution: 4 unknowns,

Denote the compressive forces in the steel cylinder and in the copper tube by F_S and F_C, respectively (Fig.b). Equilibrium at the F. B. D. of the plate yields

$$F_S + F_C = F.$$

Since equilibrium furnishes only one equation for the two unknown forces F_s and F_c, the problem is statically indeterminate.

obtain a second equation by taking into account the deformation of the system. The shortenings (here counted positive) of the two parts are given according to $\Delta l = (l/EA)F$, by

$$\Delta l_C = \frac{lF_C}{E_C A_C} \text{ and } \Delta l_S = \frac{lF_S}{E_S A_S}$$

The plate and the floor are assumed to be rigid. Therefore the geometry of the problem requires that the shortenings of the copper tube and of the steel cylinder coincide. This gives the compatibility condition $\Delta l_C = \Delta l_S = \Delta l$

Sub. in the two last equations gives $F_S = E_S A_S \Delta l/l$ and $F_C = E_C A_C \Delta l/l$ Sub. these into the equilibrium Eq. gives

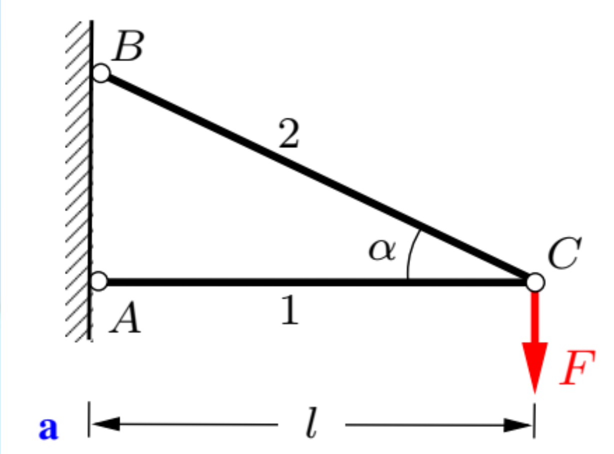
$$\Delta l = \frac{Fl}{E_S A_S + E_C A_C} \quad F_S = \frac{E_S A_S}{E_S A_S + E_C A_C} F \quad \sigma_S = \frac{E_S}{E_S A_S + E_C A_C} F \quad F_C = \frac{E_C A_C}{E_S A_S + E_C A_C} F \quad \sigma_C = \frac{E_C}{E_S A_S + E_C A_C} F$$

5 Statically Determinate Systems of Bars

In the preceding section we calculated the stresses and deformations of single slender bars. We will now extend the investigation to trusses and to structures which consist of bars and rigid bodies.

In this section we will restrict ourselves to statically determinate systems where we can first calculate the forces in the bars with the aid of the equilibrium conditions.

Subsequently, the stresses in the bars and the elongations are determined. Finally, the displacements of arbitrary points of the structure can be found. Since it is assumed that the elongations are small as compared with the lengths of the bars, we can apply the equilibrium conditions to the *undeformed* system.



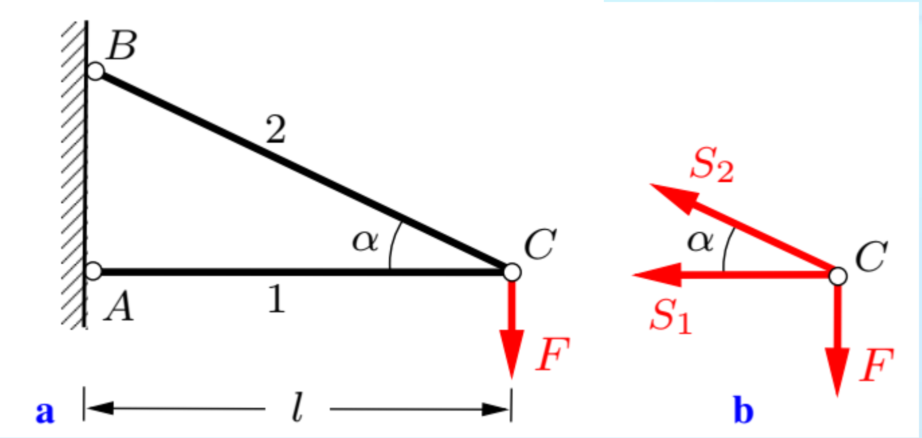
As an illustrative example let us consider the truss in Fig. a. Both bars have the axial rigidity EA. We want to determine the displacement of pin C due to the applied force F . First we calculate the forces S_1 and S_2 in the bars. The equilibrium conditions, applied to the free-body diagram (Fig. b), yield

$\uparrow$: $S_2 \sin\alpha - F = 0$ and $\leftarrow$: $S_1 + S_2\cos\alpha = 0 \;\Rightarrow\; S_1 = -F/\tan\alpha$ and $S_2 = F/\sin\alpha$

$$\Delta l_1 = \frac{l_1}{EA} S_1 = -\frac{Fl}{EA\tan\alpha} \quad \& \quad \Delta l_2 = \frac{l_2}{EA} S_2 = \frac{Fl}{EA\sin\alpha\cos\alpha}$$

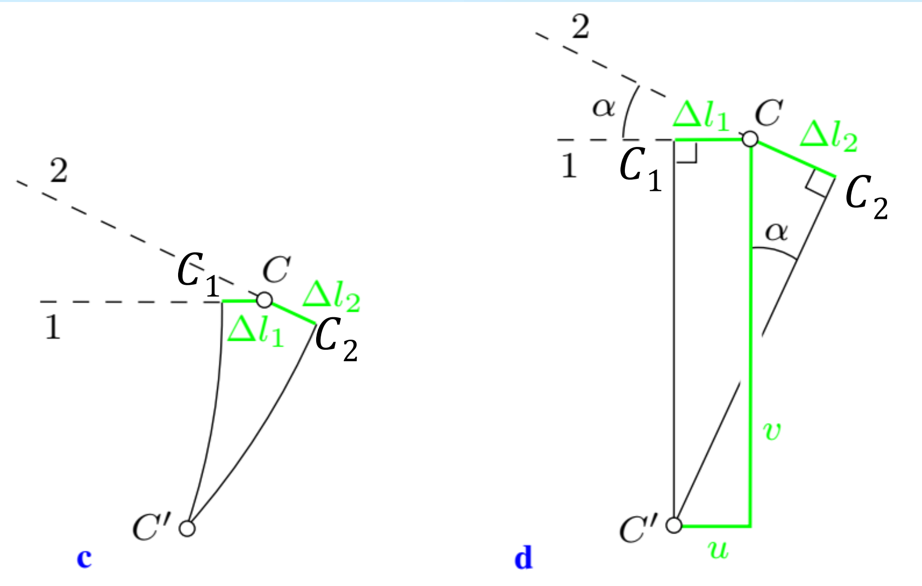
Bar 1 becomes shorter (compression) and bar 2 becomes longer (tension).

The new position C' of pin C can be found as follows. We consider the bars to be disconnected at C. Then the system becomes movable: bar 1 can rotate about point A; bar 2 can rotate about point B.



The End of bar 1 makes a circle of radius $l + \Delta l_1$ and the end of bar 2 makes a circle of radius $(l/\cos\alpha) + \Delta l_2$ the two circles intersect at C', (Fig.c)

The two arcs C_1C' & C_2C' and are very small compared to l, so the can be approximated by the two tangents as in (Fig.d).



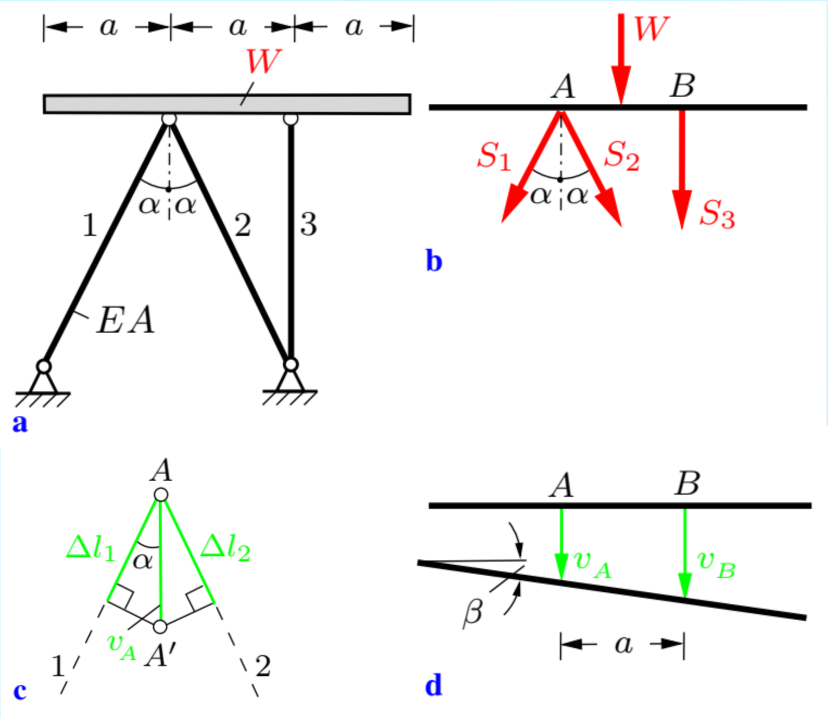
$$\overrightarrow{CC'} = u\vec{\imath} + v\vec{\jmath} \;\rightarrow\; \overrightarrow{CC'}\cdot\vec{e}_1 = \Delta l_1 = u \quad \& \quad \overrightarrow{CC'}\cdot\vec{e}_2 = \Delta l_2 = u\cos\alpha + v\sin\alpha$$

$$u = -Fl/(EA\tan\alpha)$$

$$v\sin\alpha = \Delta l_2 - u\cos\alpha \Rightarrow v = Fl(1+\cos^3\alpha)/(EA\sin^2\alpha\cos\alpha)$$

Example 1. A rigid beam (weight W) is mounted on three elastic bars (axial rigidity EA) as shown in Fig.a. Determine the angle of slope of the beam that is caused by its weight after the structure has been assembled.

Solution First we calculate the forces in the bars with the aid of the equilibrium conditions (Fig.b): $S_1 = S_2 = -W/4\cos\alpha\,, S_3 = -W/2$



With $l_1 = l_2 = a/\sin\alpha$ & $l_3 = a/\tan\alpha$, the elongations are:

$$\Delta l_1 = \Delta l_2 = \frac{l_1 S_1}{EA} = \frac{l_2 S_2}{EA} = -\frac{Wa}{4EA\sin\alpha\cos\alpha}, \Delta l_3 = \frac{l_3 S_{13}}{EA} = -\frac{Wa}{2EA\tan\alpha}$$

Point B of the beam is displaced downward by $v_B = |\Delta l_3|$. To determine the vertical displacement v_A of point A we sketch the diagram (Fig.c). First we plot the changes Δl_1 & Δl_2 of the lengths in the direction of each bar. The lines perpendicular to these directions intersect at the displaced position A' of point A. So, its vertical displacement is $v_A = |\Delta l_1|/\cos\alpha$

Since v_A and v_B do not coincide, the beam does not stay horizontal.

The angle of slope β is obtained with the approximation $\tan\beta \approx \beta$ (small deformations) (Fig.d)

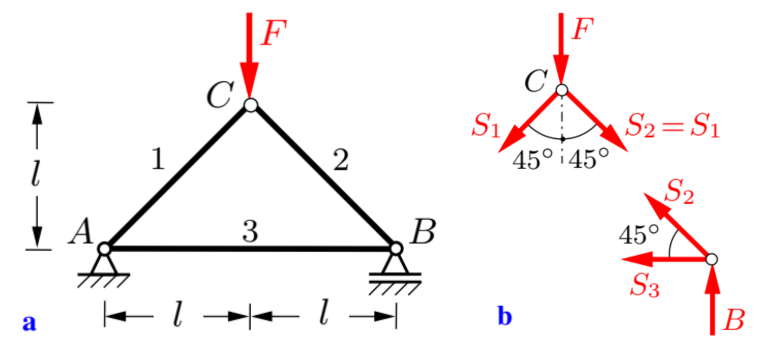
$$\beta = \frac{v_B - v_A}{a} = \frac{2\cos^3\alpha - 1}{4\cos^3\alpha}\frac{W\cot\alpha}{EA}.$$

$\cos^3\alpha = \frac{1}{2}$, the beam stays horizontal. $\Rightarrow \alpha = 37.5^0$

$\cos^3\alpha > \frac{1}{2}$ inclined to right

$\cos^3\alpha < \frac{1}{2}$ inclined to left

Example 2. (Design problem) The truss in Fig.a is under the action of the force $F = 20$ kN. If $E = 200$ Gpa (200000 MPa[≡N/mm²]). Determine the required cross-sectional areas of the three members so that the stresses do not exceed the allowable stress $\sigma_{allow} = 150$ MPa and the displacement of B is smaller than 0.5% of the length of bar 3.



Solution

- First we calculate the (reactions if necessary) and forces in the members. The equilibrium conditions for the free-body diagrams of pin C and support B (Fig.b) yield:

At C; $\rightarrow$: $S_1 = S_2$; $\uparrow$: $S_1 = S_2 = -F/\sqrt{2}$ At B; $\rightarrow$: $-S_3 - S_2 \cos 45^0 = 0 \Rightarrow S_3 = F/2$

- Then we establish the design requirements (Conditions) متطلبات أو شروط التصميم

➢ Stress requirements: $\sigma_i = \frac{S_i}{A_i} \Rightarrow (A_i)_{\min} \geq \frac{S_i}{\sigma_{allow}}, i = 1,2,3.$

➢ Displacemen requirement: $u_B = (\Delta l)_3 = \frac{l_3 S_3}{E A_3} \leq 0.5 \times 10^{-2}(2l) \Rightarrow (A_3)_{\min} \geq \frac{(2l) S_3}{E(\Delta l)_3} = \frac{S_3}{0.5\times10^{-2} E}$

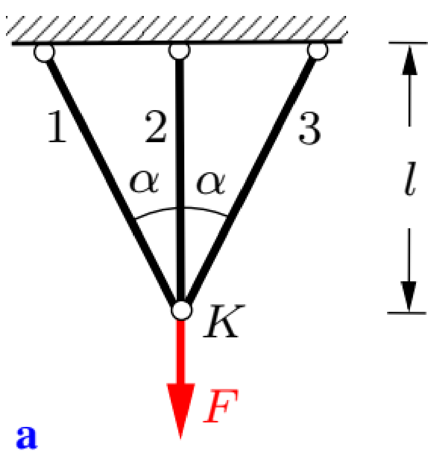
Member (length)	Normal Force: S[kN]	Min. Area [mm²] $A_{\min} \geq S/\sigma_{all}$	Min. Area [mm²] Displacement of B	Req. Min. Areas [mm²]
1 ($l\sqrt{2}$)	$-F/\sqrt{2} = -14.1$	95	–	95
2 ($l\sqrt{2}$)	$-F/\sqrt{2} = -14.1$	95	–	95
3 ($2l$)	$F/2 = 10$	67	10	67

6 Statically Indeterminate Systems of Bars

We will now investigate statically indeterminate systems for which the forces in the bars cannot be determined with the aid of the equilibrium conditions alone since the number of the unknown quantities exceeds the number of the equilibrium conditions.

In such systems the basic equations (1) Equilibrium conditions. (2) Kinematic equations (compatibility) & (3) Material behavior(Hooke's law), are coupled.

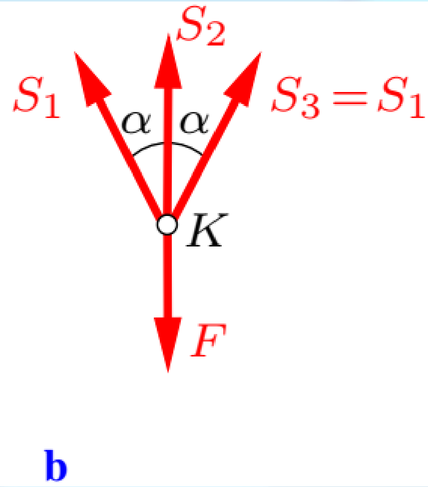
Let us consider the symmetrical truss shown in (Fig.a) It is stress-free before the load is applied. The axial rigidities $EA_1, EA_2, EA_3 = EA_1$ are given; the forces in the members are unknown.



The system is statically indeterminate to the first degree: The two equilibrium conditions applied to the free-body diagram of pin K (Fig.b) yield

$$\rightarrow: -S_1 \sin\alpha + S_3 \sin\alpha = 0 \Rightarrow S_1 = S_3,$$

$$\uparrow: S_1 \cos\alpha + S_2 + S_3 \cos\alpha - F = 0 \Rightarrow S_1 = S_3 = \frac{F - S_2}{2\cos\alpha}$$

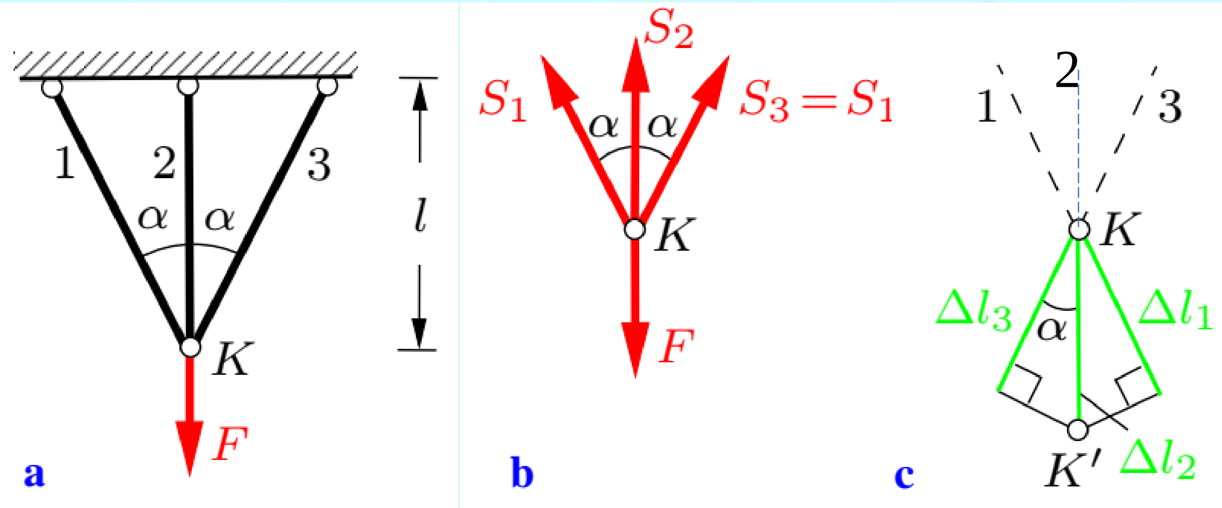


Number of static (force) unknowns is 3. Number of equilibrium equations is 2. So the number of indeterminacy is: 3-2=1.

The system is indeterminate to the first degree.

Hook's law gives the elongations of the bars by:

$$\Delta l_1 = \Delta l_3 = \frac{l_1}{EA_1} S_1 \qquad \Delta l_2 = \frac{l_2}{EA_2} S_2$$



Kinematic (compatibility) condition is found by displacement diagram (Fig.c):

$$\Delta l_1 \equiv \Delta l_3 = \Delta l_2 \cos\alpha$$

Substituting the material equations (Hook's law) in this compatibility equation, we write it in terms of the unknowns forces

$$\frac{l_1}{EA_1} S_1 = \frac{l_2}{EA_2} S_2 \cos\alpha$$

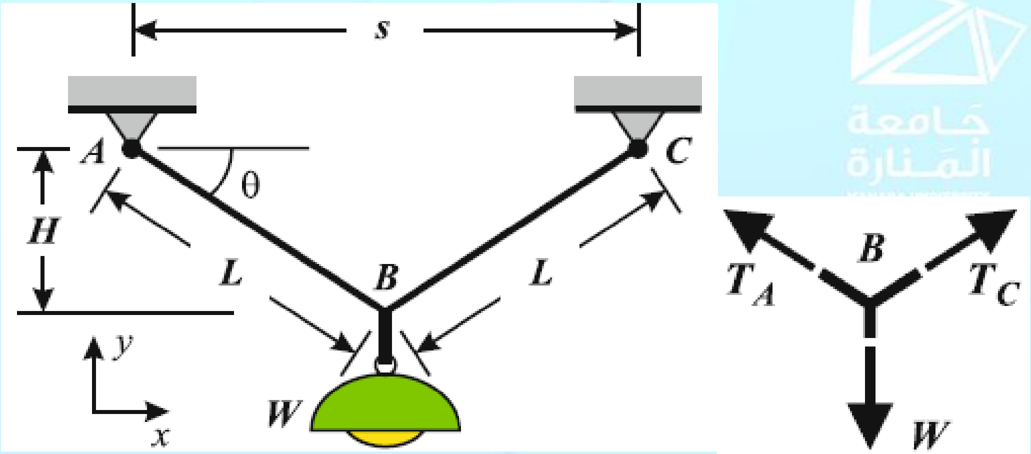
With the combination of the two equilibrium equations $S_1 = S_3 = \dfrac{F - S_2}{2\cos\alpha}$

And the geometric evidence: $l_1 = l/\cos\alpha$ and $l_2 = l$

We obtain the three unknowns forces, then the elongations and displacements of *K*.

Example: 1. Hanging Lamp

Given: A lamp weighing W=60 N is supported by two wires, both of length L=1.5 m & diameter D=2.5 mm. The distance between the two cable mounts is s = 2.4 m so that point B is H=0.9 m below horizontal line AC. The wires are made of steel with modulus E=207 GPa.



Required: Determine

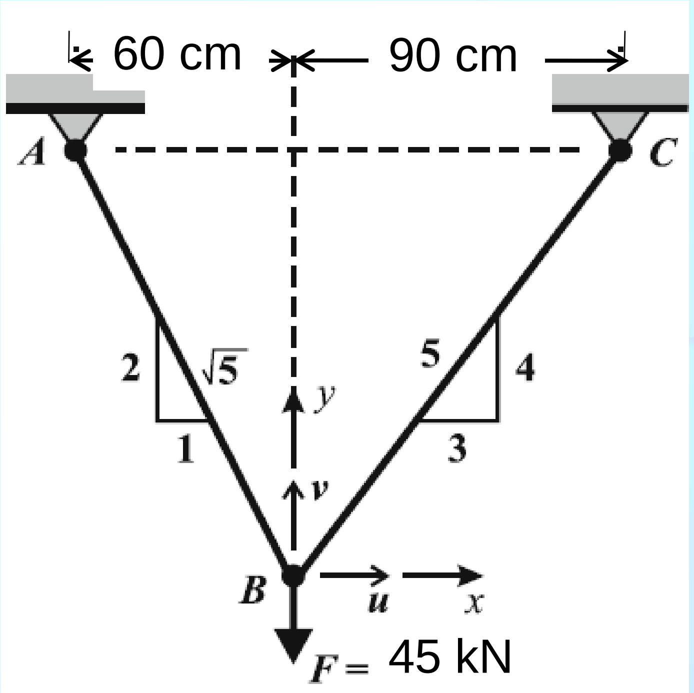
(a) the downward displacement of the lamp v (i.e., of point B) due to its own weight,

(b) the stiffness of the wire assembly in the vertical direction, $K = W/v$.

Example: 2. Truss Deflection

Given: Aluminum truss *ABC* is loaded at joint *B* by a point load of $F = 45$ kN. The cross-sectional areas of the bars are: $A_{AB} = 325$ mm^2 and $A_{BC} = 390$ mm^2. The modulus of aluminum is $E = 70$ GPa.

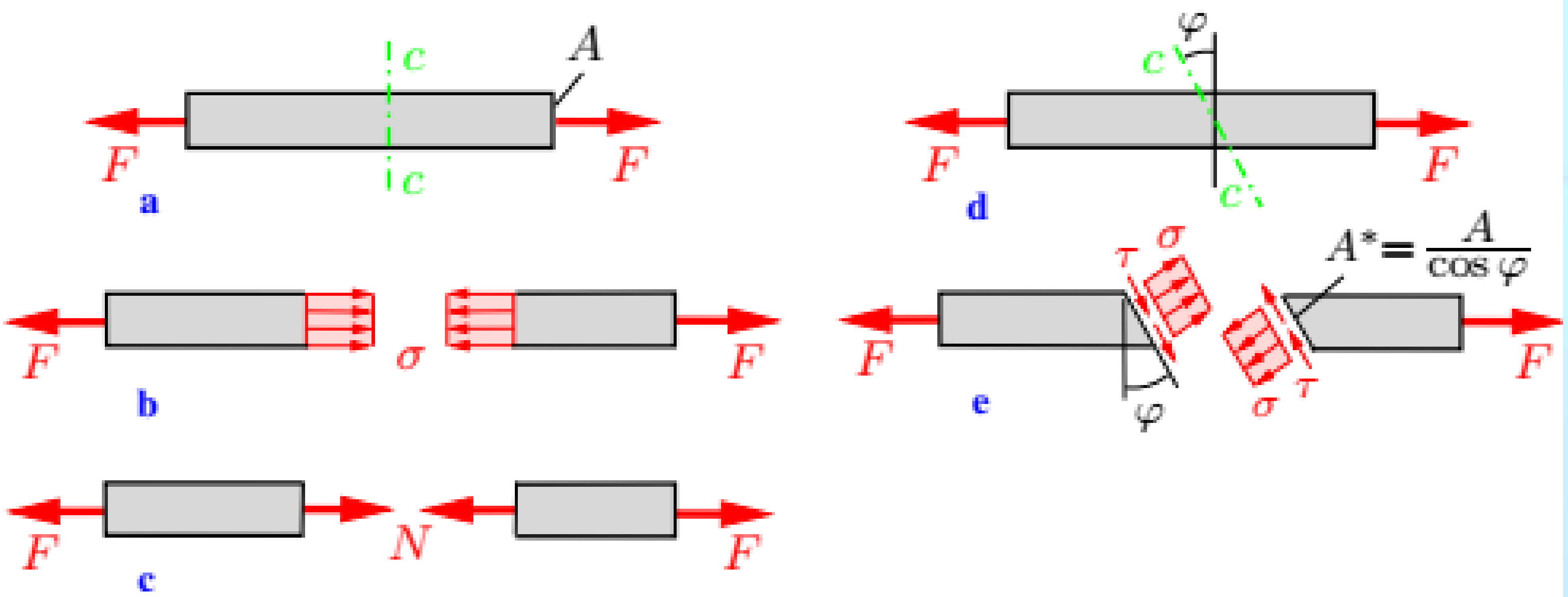
Required: Determine the horizontal and vertical displacements of joint *B, u, and v.*



$$\rightarrow: \quad \sigma A^* \cos\varphi + \tau A^* \sin\varphi - F = 0$$
$$\uparrow: \quad \sigma A^* \sin\varphi + \tau A^* \cos\varphi = 0$$

These Eq. Eqs. are written for the *forces, not* for the *stresses*. With $A^* = A/\cos\varphi$ we obtain

$$\begin{cases} \sigma + \tau \tan\varphi = \frac{F}{A} \\ \sigma \tan\varphi - \tau = 0 \end{cases}$$



Solving yields

$$\begin{cases} \sigma = \frac{1}{1+\tan^2\varphi}\frac{F}{A} \\ \tau = \frac{\tan\varphi}{1+\tan^2\varphi}\frac{F}{A} \end{cases}$$

It is practical to write these equations in a different form. Using the trigonometric relations

$$\frac{1}{1+\tan^2\varphi} = \cos^2\varphi = \tfrac{1}{2}(1+\cos 2\varphi), \qquad \frac{\tan\varphi}{1+\tan^2\varphi} = \sin\varphi\cos\varphi\,, \quad \sin 2\varphi = \frac{2\tan\varphi}{1+\tan^2\varphi}, \qquad \cos 2\varphi = \frac{1-\tan^2\varphi}{1+\tan^2\varphi}$$

and the abbreviation $\sigma_0 = F/A$ (normal stress in a section perpendicular to the axis) we get

$$\sigma = \frac{\sigma_0}{2}(1+\cos 2\varphi), \qquad \tau = \frac{\sigma_0}{2}\sin 2\varphi$$

Stresses depend on the direction of the cut. If σ_0 is known, σ & τ can be calculated for any φ. The maximum value of σ is obtained for $\varphi = 0$, where $\sigma_{max} = \sigma_0$; the maximum value of τ is found for $\varphi = \pi/4$ where $\tau_{max} = \sigma_0/2$.