2 Stress

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$\rightarrow$: $\sigma A^* \cos\varphi + \tau A^* \sin\varphi - F = 0$

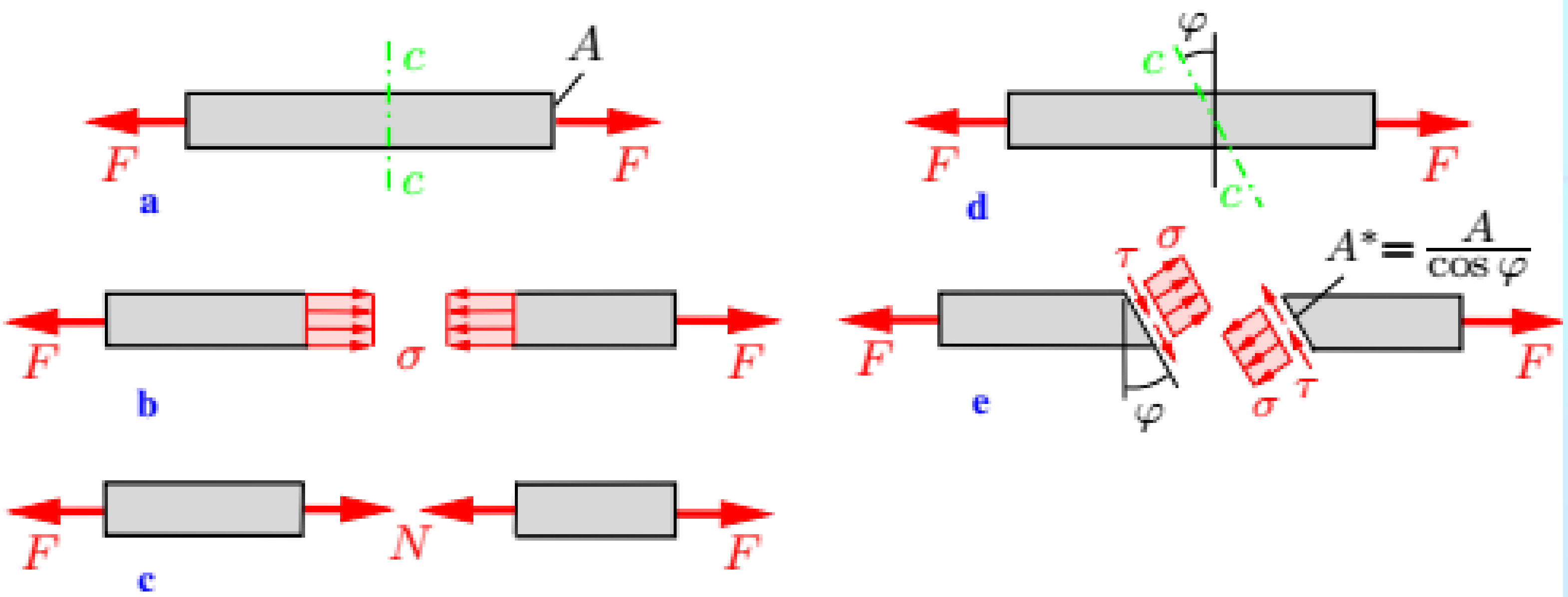
$\uparrow$: $\sigma A^* \sin\varphi + \tau A^* \cos\varphi = 0$

These Eq. Eqs. are written for the *forces, not* for the *stresses*. With $A^* = A/\cos\varphi$ we obtain

$$\begin{cases} \sigma + \tau \tan\varphi = \frac{F}{A} \\ \sigma \tan\varphi - \tau = 0 \end{cases}$$

Solving yields

$$\begin{cases} \sigma = \frac{1}{1+\tan^2\varphi}\frac{F}{A} \\ \tau = \frac{\tan\varphi}{1+\tan^2\varphi}\frac{F}{A} \end{cases}$$



It is practical to write these equations in a different form. Using the trigonometric relations

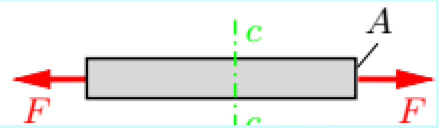
$$\frac{1}{1+\tan^2\varphi} = \cos^2\varphi = \tfrac{1}{2}(1+\cos 2\varphi), \qquad \frac{\tan\varphi}{1+\tan^2\varphi} = \sin\varphi\cos\varphi\,, \quad \sin 2\varphi = \frac{2\tan\varphi}{1+\tan^2\varphi}, \qquad \cos 2\varphi = \frac{1-\tan^2\varphi}{1+\tan^2\varphi}$$

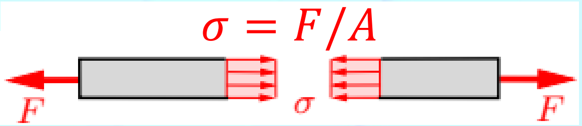
and the abbreviation $\sigma_0 = F/A$ (normal stress in a section perpendicular to the axis) we get

$$\sigma = \frac{\sigma_0}{2}(1+\cos 2\varphi), \qquad \tau = \frac{\sigma_0}{2}\sin 2\varphi$$

Stresses depend on the direction of the cut. If σ_0 is known, σ & τ can be calculated for any φ. The maximum value of σ is obtained for $\varphi = 0$, where $\sigma_{max} = \sigma_0$; the maximum value of τ is found for $\varphi = \pi/4$ where $\tau_{max} = \sigma_0/2$.

2.1 Stress Vector and Stress Tensor





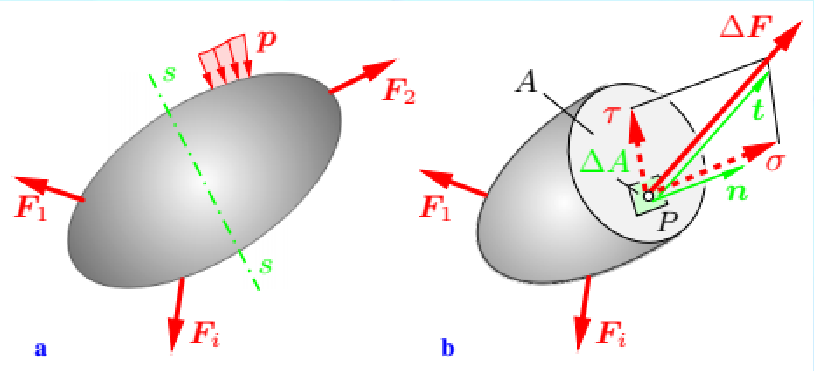
So far, stresses have been calculated only in bars.

To determine stresses also in other structures we must generalize the concept of stress.

Consider a body loaded by single forces F_i & area forces p (Fig.**a**).

The external load generates internal forces.

In an imaginary cut *s-s* through the body the internal area forces (stresses) are distributed over the entire area *A*.



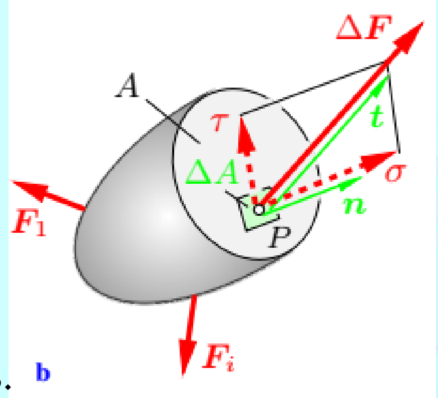
In the bar these stresses are constant over the cross section they now generally vary throughout the section.

Then the stress must be defined at the arbitrary point *P* of the cross section (Fig.**b**). The area element ΔA containing *P* is subjected to the resultant internal force ΔF (note: according to the law of action and reaction the same force acts in the opposite cross section with opposite direction).

The average stress in the area element is defined as the ratio $\Delta \boldsymbol{F}/\Delta A$ (force per area).

Stress vector at point *P* of the section s-s is defined by: $\vec{t} = \lim_{\Delta A \to 0} \frac{\Delta \vec{F}}{\Delta A}$, decomposed into normal stress σ and shear stress τ

In general, the stress vector $\boldsymbol{t}$ depends on the location of point P in the section area A.

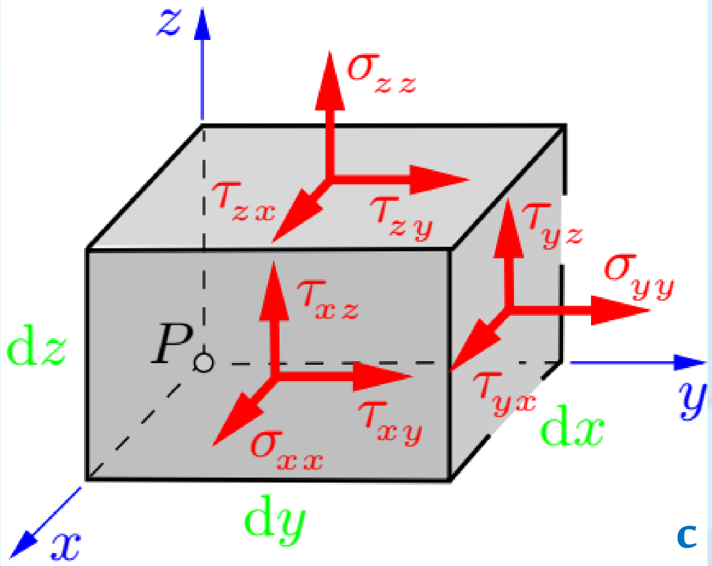


b

The stresses also depend on the orientation of the section which is defined normal vector $\boldsymbol{n}$.

It can be shown that the stress state at point P is uniquely determined by three stress vectors for three sections through P, perpendicular to each other.

Useful to choose the directions of a Cartesian coordinate system for the respective orientations.



c

The three sections can most easily be visualized if we imagine them to be the surfaces of a volume element with edge lengths dx, dy and dz at point P (Fig.c).

A stress vector acts on each of its 6 surfaces. It can be decomposed into its components perpendicular to the section (normal stress) and tangential to the section (shear stress).

The shear stress subsequently can be further decomposed into its components according to the coordinate directions.

To characterize the components double subscripts are used: the first subscript indicates the orientation of the section by the direction of its normal vector whereas the second subscript indicates the direction of the stress component.

For example, τ_{yx} is a shear stress acting in a section whose normal points in y-direction; the stress itself points in x-direction.

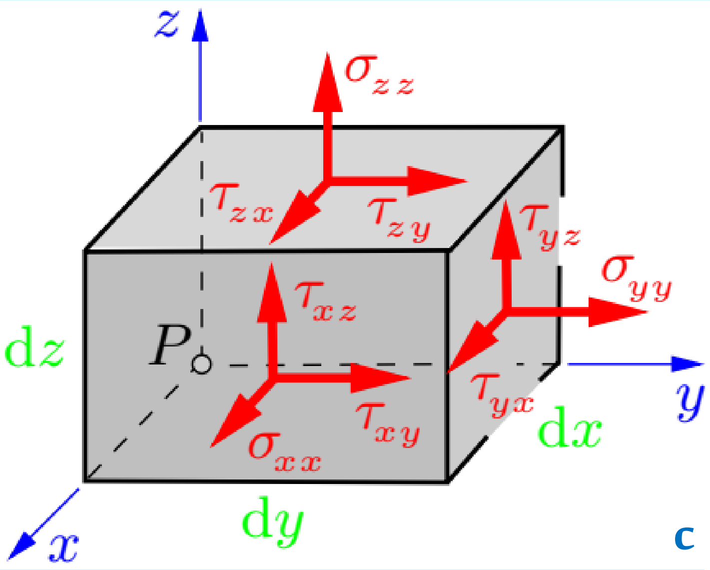
The notation can be simplified for the normal stresses $\sigma_{xx} = \sigma_x$, $\sigma_{yy} = \sigma_y$, $\sigma_{zz} = \sigma_z$

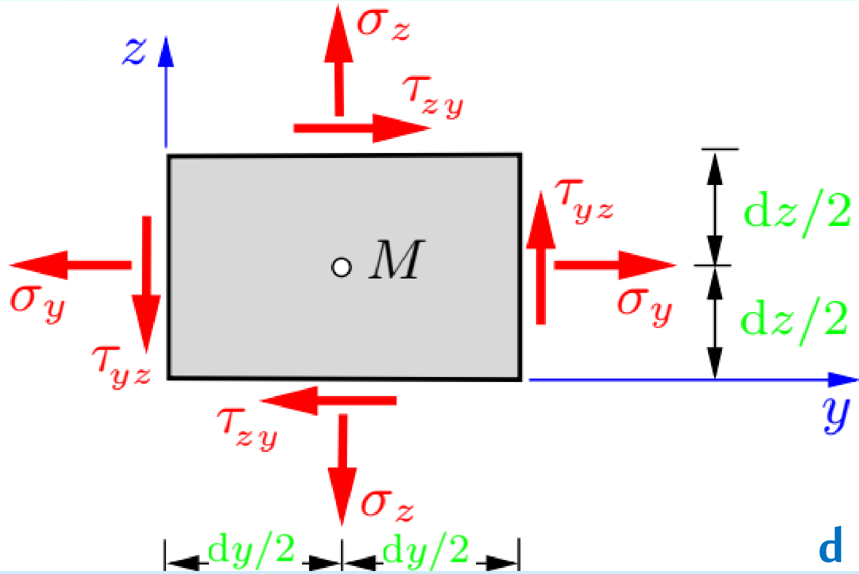
Using the introduced notation, the stress vectors in the sections with the normal vectors pointing in x, y, z direction, can be written as: $\vec{t}_x = \sigma_x \vec{\imath} + \tau_{xy} \vec{\jmath} + \tau_{xz} \vec{k}$ $\quad \vec{t}_y = \tau_{yx} \vec{\imath} + \sigma_y \vec{\jmath} + \tau_{yz} \vec{k}$ $\quad \vec{t}_z = \tau_{zx} \vec{\imath} + \tau_{zy} \vec{\jmath} + \sigma_z \vec{k}$

Positive **stresses at a *positive* face point in *positive* directions of the coordinates.**

Positive **stresses at a *negative* face point in *negative* directions of the coordinates.**

Accordingly, positive (negative) normal stresses cause tension (compression) in the volume element.





By means of the decomposition of the three stress vectors into their components we have obtained three normal stresses $(\sigma_x, \sigma_y, \sigma_z)$ and six shear stresses $(\tau_{xy}, \tau_{xz}, \tau_{yx}, \tau_{yz}, \tau_{zx}, \tau_{zy})$. So nine stress components.

But shear stresses are not independent. This can be shown by formulating the equilibrium condition for the moments about an axis parallel to the x-axis through the center of the volume element (Fig.**d**).

$$\overset{\curvearrowleft}{M}: 2\frac{dy}{2}\left(\tau_{yz} dx dy\right) - 2\frac{dz}{2}\left(\tau_{zy} dx dy\right) = 0 \Rightarrow \tau_{yz} = \tau_{zy} \qquad \text{similarly} \quad \tau_{xy} = \tau_{yx} \ \text{ and } \ \tau_{xz} = \tau_{zx}$$

Shear stresses with the same subscripts in two orthogonal sections are equal. *complementary shear stresses*

As a final result of: There exist only six independent stress components at any point of the three dimensional body. The components of the three stress vectors can be arranged in a matrix:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$

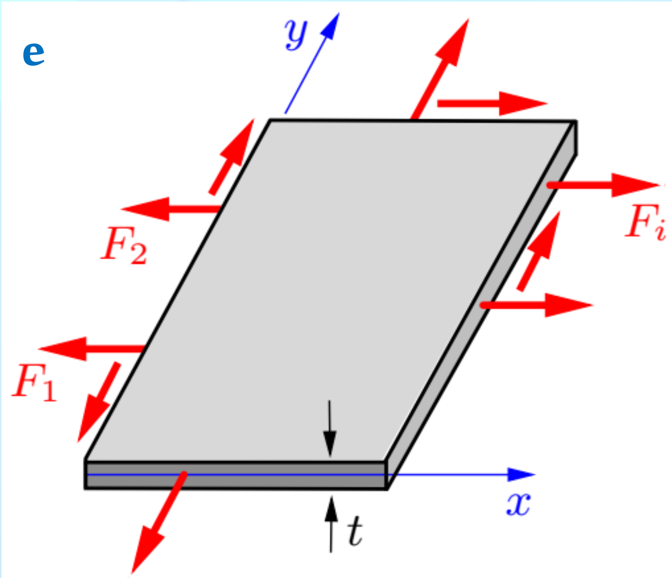
The main diagonal contains the normal stresses; the other elements are the *symmetric* shear stresses.

The quantity $\boldsymbol{\sigma}$ is called *stress tensor* (the concept *tensor* will be explained). The elements of $\boldsymbol{\sigma}$ are the components of the stress tensor. The *stress state* at a material point is uniquely defined by the stress vectors for three sections, orthogonal to each other, and consequently by the stress tensor.

2.2 Plane Stress

We will now examine the state of stress in a *disk*. This plane structural element has a thickness *t* much smaller than its inplane dimensions and it is loaded solely *in* its plane by in-plane forces (Fig.e).

The upper and the lower face of the disk are load-free. Since no external forces in the z-direction exist, we can assume with sufficient accuracy that also no stresses will appear in this direction: $\tau_{xz} = \tau_{yz} = \sigma_z = 0$.



Because of the small thickness we furthermore can assume that the stresses σ_x, σ_y and $\tau_{xy} = \tau_{yx}$ are constant across the thickness of the disk. Such a stress distribution is called a ***state of plane stress***.

In this case, the third row and the third column of the stress matrix vanish and we get

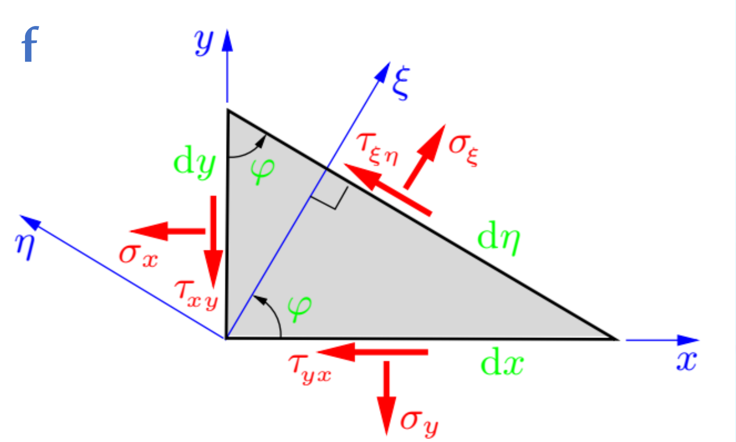
$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix}$$

In general, the stresses depend on the location, i.e. on the coordinates *x* and *y*.

In the special case when the stresses are independent of the location, the stress state is called *homogeneous*.

2.2.1 Coordinate Transformation

Up to now only stresses in sections parallel to the coordinate axes have been considered. Now we will show how from these stresses, the stresses in an arbitrary section perpendicular to the disk can be determined.



For this purpose we consider an infinitesimal wedge-shaped element of thickness t cut out from the disk (Fig. f).

The directions of the sections are characterized by the x , y - coordinate system and the angle φ.

We introduce a ξ , η -system which is rotated with respect to the x, y-system by the angle φ and whose ξ - axis is normal to the inclined section. Here φ is counted positive *counterclockwise*.

According to the coordinate directions, the stresses in the inclined section are denoted as σ_ξ and $\tau_{\xi\eta}$. The corresponding cross section is given by $dA = d\eta\ t$. The other two cross sections perpendicular to the y- and x- axis, respectively, are $dA \sin\varphi$ and $dA\cos\varphi$. The equilibrium conditions for the forces in ξ- and in η-direction are

$$\nearrow: \sigma_\xi dA - (\sigma_x dA\cos\varphi)\cos\varphi - (\tau_{xy} dA\cos\varphi)\sin\varphi - (\sigma_y dA\sin\varphi)\sin\varphi - (\tau_{yx} dA\sin\varphi)\cos\varphi = 0$$

$$\nwarrow: \tau_{\xi\eta} dA + (\sigma_x dA\cos\varphi)\sin\varphi - (\tau_{xy} dA\cos\varphi)\cos\varphi - (\sigma_y dA\sin\varphi)\cos\varphi + (\tau_{yx} dA\sin\varphi)\sin\varphi = 0$$

Taking into account $\tau_{yx} = \tau_{xy}$, we get

$$\begin{cases} \sigma_\xi = \sigma_x\cos^2\varphi + \sigma_y\sin^2\varphi + 2\tau_{xy}\sin\varphi\cos\varphi \\ \tau_{\xi\eta} = -(\sigma_x - \sigma_y)\sin\varphi\cos\varphi + \tau_{xy}(\cos^2\varphi - \sin^2\varphi) \end{cases}$$

Additionally, we will now determine the normal stress σ_η which acts in a section with normal pointing in η-direction. The cutting angle of this section is given by $\varphi + \pi/2$. Therefore, σ_η is obtained by replacing in the equation of σ_η, the angle φ by $\varphi + \pi/2$. Recalling that $\cos(\varphi + \pi/2) = -\sin\varphi$ and $\sin(\varphi + \pi/2) = \cos\varphi$,

We obtain: $\sigma_\eta = \sigma_x \sin^2\varphi + \sigma_y \cos^2\varphi - 2\tau_{xy} \cos\varphi \sin\varphi$

Usually the last three equations are written in a different form. Using the standard trigonometric relations:

$\cos^2\varphi - \sin^2\varphi = \cos 2\varphi,\quad \cos^2\varphi = \frac{1}{2}(1 + \cos 2\varphi),\quad \sin^2\varphi = \frac{1}{2}(1 - \cos 2\varphi),\quad 2\sin\varphi\cos\varphi = \sin 2\phi$ We get:

$$\sigma_\xi = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\varphi + \tau_{xy}\sin 2\varphi\,,$$
$$\sigma_\eta = \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\varphi - \tau_{xy}\sin 2\varphi\,,$$
$$\tau_{\xi\eta} = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\varphi + \tau_{xy}\cos 2\varphi\,.$$

These are called transformation relations for components of stress from the system x, y to the system ξ, η.

It is important to emphasize that either groups of the stress components represent the same state of stress at the studied point of the disk.

A quantity whose components have two subscripts and which are transformed from one coordinate system to a rotated coordinate system, by similar rules to the here seen, is called a second rank tensor.

Adding the first two equations of these equations, we obtain

$$\sigma_\xi + \sigma_\eta = \sigma_x + \sigma_y$$

Thus the sum of the normal stresses has the same value in each coordinate system. For this reason this sum is called an invariant of the stress tensor.

It can also be verified that the determinant $\sigma_x\sigma_y - \tau_{xy}^2$ of the matrix of the stress tensor is further invariant, that is

$$\sigma_x\sigma_y - \tau_{xy}^2 = \sigma_\xi\sigma_\eta - \tau_{\xi\eta}^2$$

We finally consider the special case of equal normal stresses $(\sigma_x = \sigma_y)$ and vanishing shear stress $(\tau_{xy} = 0)$ in the x, y system. The equations

$$\sigma_\xi = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\varphi + \tau_{xy}\sin 2\varphi\,,$$
$$\sigma_\eta = \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\varphi - \tau_{xy}\sin 2\varphi\,,$$
$$\tau_{\xi\eta} = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\varphi + \tau_{xy}\cos 2\varphi\,.$$

Show that:

$$\sigma_\xi = \sigma_\eta = \sigma_x = \sigma_y, \text{ and } \tau_{\xi\eta} = 0$$

Accordingly, the normal stress for all directions of the sections are the same (independent of φ) whereas the shear stresses always vanish. Such a state is called Hydrostatic because it corresponds to the pressure in a fluid at rest where the normal stress is the same in all directions.