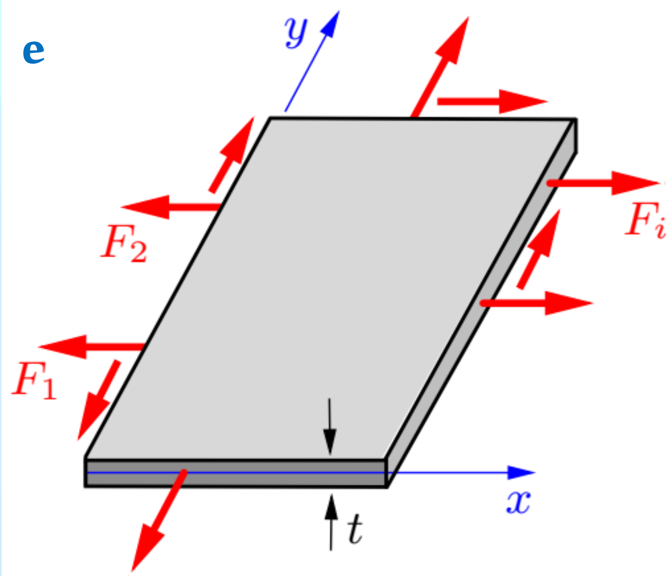
2 Stress

2.2 Plane Stress

We will now examine the state of stress in a *disk*. This plane structural element has a thickness *t* much smaller than its inplane dimensions and it is loaded solely *in* its plane by in-plane forces (Fig.e).

The upper and the lower face of the disk are load-free. Since no external forces in the *z*-direction exist, we can assume with sufficient accuracy that also no stresses will appear in this direction: $\tau_{xz} = \tau_{yz} = \sigma_z = 0$.



Because of the small thickness we furthermore can assume that the stresses σ_x, σ_y and $\tau_{xy} = \tau_{yx}$ are constant across the thickness of the disk. Such a stress distribution is called a ***state of plane stress***.

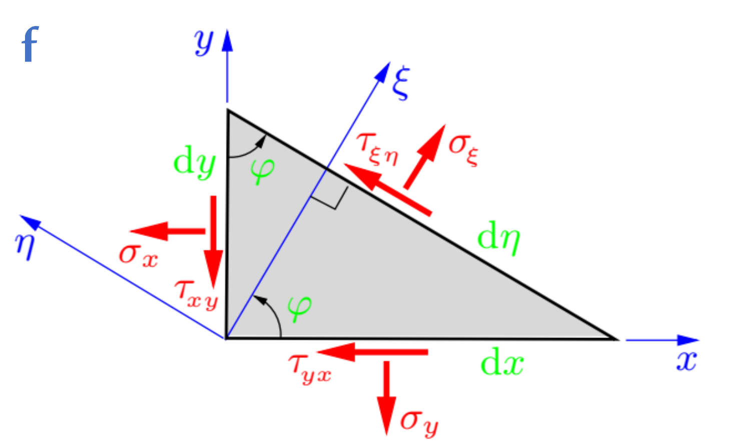
In this case, the third row and the third column of the stress matrix vanish and we get

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix}$$

In general, the stresses depend on the location, i.e. on the coordinates *x* and *y*.

In the special case when the stresses are independent of the location, the stress state is called *homogeneous*.

2.2.1 Coordinate Transformation



Up to now only stresses in sections parallel to the coordinate axes have been considered. Now we will show how from these stresses, the stresses in an arbitrary section perpendicular to the disk can be determined.

For this purpose we consider an infinitesimal wedge-shaped element of thickness t cut out from the disk (Fig. f).

The directions of the sections are characterized by the x , y - coordinate system and the angle φ.

We introduce a ξ , η -system which is rotated with respect to the x, y-system by the angle φ and whose ξ - axis is normal to the inclined section. Here φ is counted positive *counterclockwise*.

According to the coordinate directions, the stresses in the inclined section are denoted as σ_ξ and $\tau_{\xi\eta}$. The corresponding cross section is given by $dA = d\eta\ t$. The other two cross sections perpendicular to the y- and x- axis, respectively, are $dA \sin\varphi$ and $dA \cos\varphi$. The equilibrium conditions for the forces in ξ- and in η-direction are

$$\nearrow: \sigma_\xi dA - (\sigma_x dA \cos\varphi)\cos\varphi - (\tau_{xy} dA \cos\varphi)\sin\varphi - (\sigma_y dA \sin\varphi)\sin\varphi - (\tau_{yx} dA \sin\varphi)\cos\varphi = 0$$

$$\nwarrow: \tau_{\xi\eta} dA + (\sigma_x dA \cos\varphi)\sin\varphi - (\tau_{xy} dA \cos\varphi)\cos\varphi - (\sigma_y dA \sin\varphi)\cos\varphi + (\tau_{yx} dA \sin\varphi)\sin\varphi = 0$$

Taking into account $\tau_{yx} = \tau_{xy}$, we get

$$\begin{cases} \sigma_\xi = \sigma_x \cos^2\varphi + \sigma_y \sin^2\varphi + 2\tau_{xy} \sin\varphi\cos\varphi \\ \tau_{\xi\eta} = -(\sigma_x - \sigma_y)\sin\varphi\cos\varphi + \tau_{xy}(\cos^2\varphi - \sin^2\varphi) \end{cases}$$

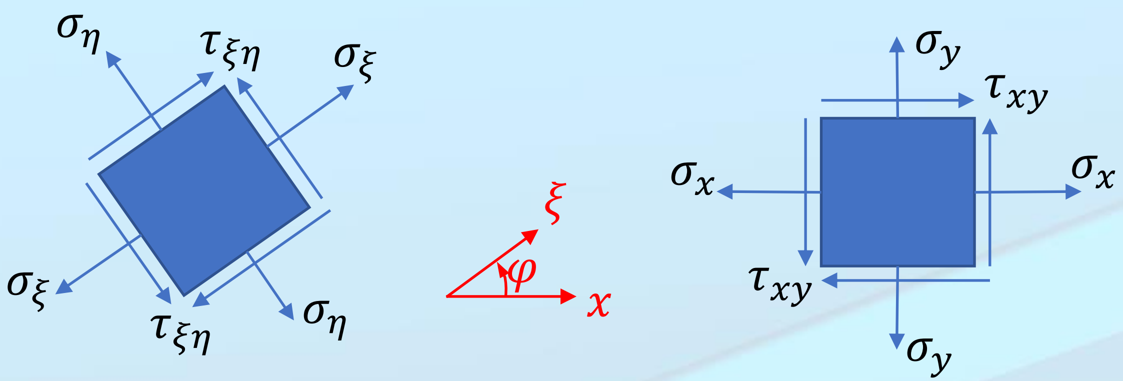
Additionally, we will now determine the normal stress σ_η which acts in a section with normal pointing in η-direction. The cutting angle of this section is given by $\varphi + \pi/2$. Therefore, σ_η is obtained by replacing in the equation of σ_η, the angle φ by $\varphi + \pi/2$. Recalling that $\cos(\varphi + \pi/2) = -\sin\varphi$ and $\sin(\varphi + \pi/2) = \cos\varphi$,

We obtain: $$\sigma_\eta = \sigma_x \sin^2\varphi + \sigma_y \cos^2\varphi - 2\tau_{xy}\cos\varphi\sin\varphi$$

Usually the last three equations are written in a different form. Using the standard trigonometric relations:

$\cos^2\varphi - \sin^2\varphi = \cos 2\varphi,\quad \cos^2\varphi = \frac{1}{2}(1+\cos 2\varphi),\quad \sin^2\varphi = \frac{1}{2}(1-\cos 2\varphi),\quad 2\sin\varphi\cos\varphi = \sin 2\phi$ We get:

$$\sigma_\xi = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\varphi + \tau_{xy}\sin 2\varphi,$$
$$\sigma_\eta = \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\varphi - \tau_{xy}\sin 2\varphi,$$
$$\tau_{\xi\eta} = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\varphi + \tau_{xy}\cos 2\varphi.$$



These are called transformation relations for components of stress from the system x, y to the system ξ, η.

It is important to emphasize that either groups of the stress components represent the same state of stress at the studied point of the disk.

We finally consider the special case of equal normal stresses $(\sigma_x = \sigma_y)$ and vanishing shear stress $(\tau_{xy} = 0)$ in the x, y system. The equations

$$\left.\begin{aligned}
\sigma_\xi &= \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\varphi + \tau_{xy}\sin 2\varphi\,, \\
\sigma_\eta &= \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\varphi - \tau_{xy}\sin 2\varphi\,, \\
\tau_{\xi\eta} &= \qquad\qquad\quad -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\varphi + \tau_{xy}\cos 2\varphi\,.
\end{aligned}\right\}$$

Show that: $\sigma_\xi = \sigma_\eta = \sigma_x = \sigma_y$, and $\tau_{\xi\eta} = 0$

Accordingly, the normal stress for all directions of the sections are the same (independent of φ) whereas the shear stresses always vanish. Such a state is called Hydrostatic because it corresponds to the pressure in a fluid at rest.

A quantity whose components have two subscripts and which are transformed from one coordinate system to a rotated coordinate system, by similar rules to the here seen, is called a second rank tensor.

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix}$$

Adding the first two equations of these equations, we obtain $\sigma_\xi + \sigma_\eta = \sigma_x + \sigma_y$

Thus the sum of the normal stresses has the same value in each coordinate system. For this reason this sum is called an invariant of the stress tensor.

It can also be verified that the determinant $\sigma_x\sigma_y - \tau_{xy}^2$ of the stress tensor is further invariant, that is

$$\sigma_x\sigma_y - \tau_{xy}^2 = \sigma_\xi\sigma_\eta - \tau_{\xi\eta}^2$$

2.2.2 Principal Directions and Principal Stresses

$$\left.\begin{aligned}\sigma_\xi &= \frac{1}{2}(\sigma_x+\sigma_y)+\frac{1}{2}(\sigma_x-\sigma_y)\cos 2\varphi+\tau_{xy}\sin 2\varphi\,,\\ \sigma_\eta &= \frac{1}{2}(\sigma_x+\sigma_y)-\frac{1}{2}(\sigma_x-\sigma_y)\cos 2\varphi-\tau_{xy}\sin 2\varphi\,,\\ \tau_{\xi\eta} &= \qquad\qquad -\frac{1}{2}(\sigma_x-\sigma_y)\sin 2\varphi+\tau_{xy}\cos 2\varphi\,.\end{aligned}\right\}$$

Then the stresses σ_ξ, σ_η & $\tau_{\xi\eta}$ depend on φ (direction of the section).
We now determine the angle for which these stresses have maximum and minimum values and we calculate these extreme values.

1. Normal stresses reach extreme values when $d\sigma\xi/d\varphi=0$ or $d\sigma\eta/d\varphi=0$, respectively. Both conditions lead to:

$-(\sigma_x-\sigma_y)\sin 2\varphi+2\tau_{xy}\cos 2\varphi=0.$ Hence, the angle $\varphi=\varphi^*$ that leads to a maximum or a minimum is given by

$$\tan 2\varphi^*=\frac{2\tau_{xy}}{\sigma_x-\sigma_y}$$

The tangent function is π-periodic: $\tan 2\varphi^*=\tan(2\varphi^*+\pi)$. Therefore, there exist two directions of the sections, φ^* & $\varphi^*+\pi/2$, perpendicular to each other, for which the normal stresses are maximum or minimum.

These directions of the sections are called ***principal directions.*** and the two normal stresses corresponding to these directions acting on theses sections are called ***principal stresses***.

To distinguish normal principal stresses corresponding to principal directions φ^* & $\varphi^*+\pi/2$, they are labeled: σ_1 & σ_2.
Using this notation and the above equations they can be determined as follows:

2.2.2 Principal Directions and Principal Stresses (Cont.)

$$\tan 2\varphi^* = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$$\sigma_\xi = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\varphi + \tau_{xy}\sin 2\varphi\,,$$

$$\sigma_\eta = \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\varphi - \tau_{xy}\sin 2\varphi\,,$$

$$\tau_{\xi\eta} = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\varphi + \tau_{xy}\cos 2\varphi\,.$$

Using the following trigonometric relations

$$\cos 2\varphi^* = \frac{1}{\sqrt{1+\tan^2 2\varphi^*}} = \frac{\sigma_x-\sigma_y}{\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}},$$

$$\sin 2\varphi^* = \frac{2\tan 2\varphi^*}{\sqrt{1+\tan^2 2\varphi^*}} = \frac{2\tau_{xy}^2}{\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}},$$

$$\sigma_{1,2} = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{\frac{1}{2}(\sigma_x - \sigma_y)^2}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}} \pm \frac{2\tau_{xy}^2}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}}$$

Simplifying the two fractions to get:

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \qquad \sigma_1 > \sigma_2$$

If the angles φ^* & $\varphi^* + \pi/2$, respectively, are introduced into the third equation of $\tau_{\xi\eta}$ we find $\tau_{\xi\eta} = 0$.

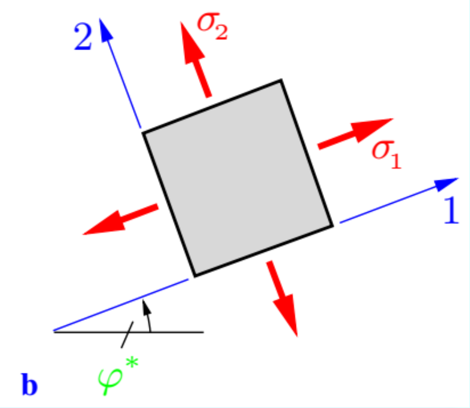
Thus, the shear stresses vanish in sections where the normal stresses take on their extreme (principal) values.

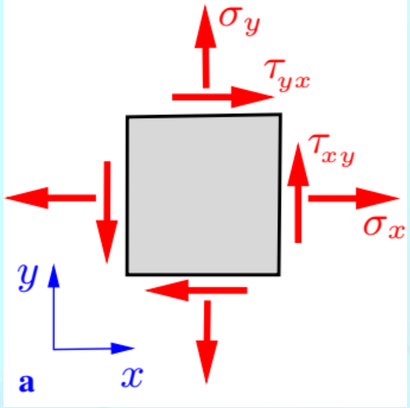
Inversely, when the shear stress in a section is zero, the normal stress in this section is a principal stress.

A coordinate system with its axes pointing in the principal directions is called *principal coordinate system*.

Its two axes are denoted by 1* & 2*, they are corresponding respectively to σ_1 & σ_2 (first & second principal stress: $\sigma_1 > \sigma_2$).

In next figures the stresses at an element in the x, y - system and in the principal coordinate system 1 & 2 are displayed.





After determining the extreme values of the normal stresses & the associated directions, the extreme values of the shear stresses:
$\tau_{\xi\eta} = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\varphi + \tau_{xy}\cos 2\varphi$, will be investigated

2. <u>Shear stresses</u> reach extreme values when $d\tau_{\xi\eta}/d\varphi = 0 \Rightarrow -(\sigma_x - \sigma_y)\cos 2\varphi - 2\tau_{xy}\sin 2\varphi = 0$

Hence the angle $\varphi = \varphi^{**}$ for an extreme value of the shear stresses, is obtained: $\tan 2\varphi^{**} = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$

Again this defines two perpendicular angles φ^{**} & $\varphi^{**} + \pi/2$ where the shear stress reaches maximum or minimum.

By comparing: $\tan 2\varphi^*$ with $\tan 2\varphi^{**}$, we find that: $\tan 2\varphi^{**} = -1/\tan 2\varphi^*$. $\Rightarrow 2\varphi^{**} = 2\varphi^* + \pi/2 \Rightarrow \varphi^{**} = \varphi^* + \pi/4$

S0, the direction of the extreme shear stress is rotated by 45^0 with respect to the direction of the extreme normal stress.

The extreme shear stresses are obtained by introducing $\tan 2\varphi^{**}$, into $\tau_{\xi\eta}$ and using the same trigonometric relations:

$$\tau_{max} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Since they differ only in the sign (i.e. in the sense of direction) both stresses are commonly called *maximum shear stresses*.

Summarizing the results giving the principal directions and principal values and of

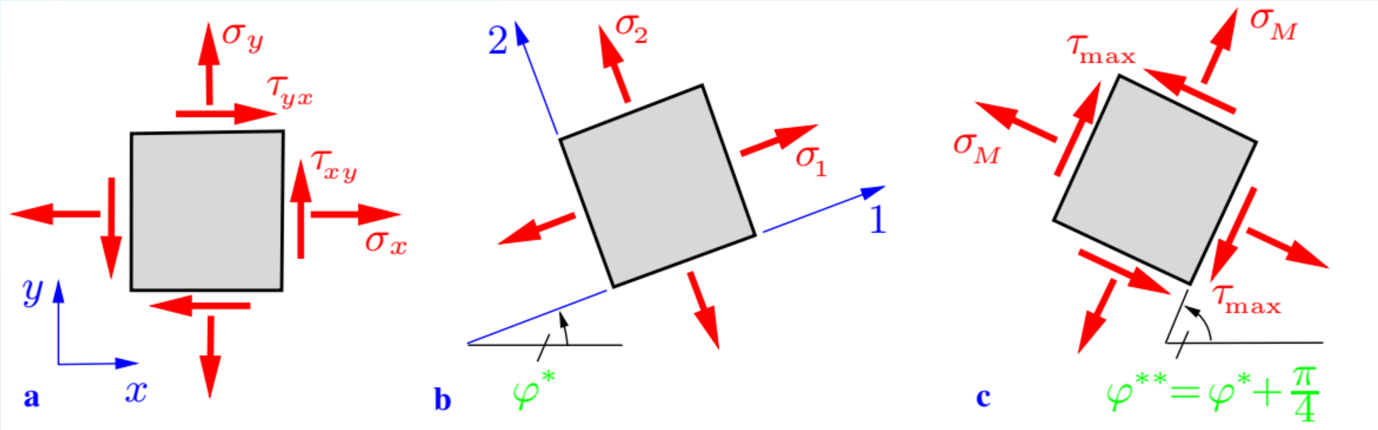
normal stresses: $\tan 2\varphi^* = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$ $\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$

Shear stresses: $\tan 2\varphi^{**} = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$ $\tau_{max} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \pm\frac{1}{2}(\sigma_1 - \sigma_2)$

Introducing φ^{**} into the equations giving σ_ξ & σ_η , leads to a normal stress in the sections where the shear stress is maximum. We denote this stress as σ_M; it is given by

$$\sigma_M = \frac{1}{2}(\sigma_x + \sigma_y) = \frac{1}{2}(\sigma_1 + \sigma_2)$$

Therefore, the normal stresses generally do not vanish in the sections with extreme shear stresses. Next figure shows stresses in the x, y - system , 1,2 system and principal shear directions.



Example 2.1

The state of plane stress in a metal sheet is given by: $\sigma_x = -64$ MPa, $\sigma_y = 32$ MPa and $\tau_{xy} = -20$ MPa.

a) Display these stresses acting on an x-y system element. Then determine

b) The stresses in a section which is defined by $\varphi = -30^0$,

c) The principal stresses and principal directions,

d) The maximum shear stress and the associated directions.

Display the stresses at an element for each case.

$$\sigma_\xi = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\varphi + \tau_{xy}\sin 2\varphi\,,$$

$$\sigma_\eta = \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\varphi - \tau_{xy}\sin 2\varphi\,,$$

$$\tau_{\xi\eta} = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\varphi + \tau_{xy}\cos 2\varphi\,.$$