Portfolio Management

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Single-index models: An overview (2)

- The key assumption of the single-index model is that e_i is independent of e_j for all values of i and j, or, more formally, $E(e_i e_j) = 0$. This implies that the only reason stocks vary together, systematically, is because of a common co-movement with the market. There are no effects beyond the market (e.g., industry effects) that account for co-movement among securities.
- It is a simplifying assumption that represents an approximation to reality. How well this model performs will depend, in part, on how good (or bad) this approximation is. Let us summarize the single-index model:

- Basic equation:

$R_i = \alpha_i + \beta_i R_m + e_i$ for all stocks $i = 1, \dots N$

- By construction:

Mean of $e_i = E(e_i) = 0$ for all stocks $i = 1, \dots N$

- By assumption:

1. Index unrelated to unique return:

$E[e_i (R_m - \bar{R}_m)] = 0$ for all stocks $i = 1, \dots N$

1. Securities related only through common response to market:

$E(e_i e_j) = 0$ for all pairs of stocks $i = 1, \dots N$ and $j = 1, \dots N$ but $i \neq j$

- By definition:

1. Variance of $e_i = E(e_i)^2 = \sigma_{ei}^2$ for all stocks $i = 1, \dots N$
2. Variance of $R_m = E(R_m - \bar{R}_m)^2 = \sigma_m^2$

- We derive the expected return, standard deviation, and covariance when the single-index model is used to represent the joint movement of securities. The results are:

1. the mean return, $\bar{R}_i = \alpha_i + \beta_i \bar{R}_m$
2. the variance of a security's return, $\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{ei}^2$
3. the covariance of returns between securities i and j, $\sigma_{ij} = \beta_i \beta_j \sigma_m^2$

- Note that the expected return has two components: a unique part α_i and a market-related part $\beta_i\ \bar{R}_m$. Likewise, a security's variance has the same two parts, unique risk σ_{ei}^2 and market-related risk $\beta_i^2\ \sigma_m^2$. In contrast, the covariance depends only on market risk. This is what we mean when we say that the single-index model implied that the only reason securities move together is a common response to market movements.

- These results can be illustrated with a simple example. Consider the returns on a stock and a market index shown in the first two columns of Table 6.1. These returns are what an investor might have observed over the prior five months.
- Now consider the values for the single-index model shown in the remaining columns of the table. Column 3 just reproduced column 1 and is the return on the security. For now assume that $\beta_i = 1.5$. Then, from result 1, $\alpha_i = 8 - 6 = 2$. Because the single-index model must hold as an equality, e_i (column 6) is just defined in each period as the value that makes the equality hold, or

$$e_i = R_i - (\alpha_i + \beta_i R_m)$$

- Table 6.1 Decomposition of Returns for the Single-Index Model

	1	2	3	4	5	6
Month	Return on Stock	Return on Market	$R_i =$	$\alpha_i +$	$\beta_i R_m +$	e_i
1	10	4	10	2	6	2
2	3	2	3	2	3	2
3	15	8	15	2	12	1
4	9	6	9	2	9	2
5	3	0	3	2	0	1
Total	40	20	40	10	30	0
Average	8	4	8	2	6	0
Variance	20.8	8	20.8	0	18	2.8

- For example, in the first period, the sum of α_i and $\beta_i R_m$ is 8. Because the return on the security in the first period is 10, e_i is +2.
- The mean return on the security is:

$$\bar{R}_i = 40/5 = 8$$

- Using the formula from the single-index model,

$$\bar{R}_i = \alpha_i + \beta_i \bar{R}_m = 2 + 1.5(4) = 8$$

- The variance of security i is calculated from the formula derived for the single-index model:

$$\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{ei}^2 = (1.5)^2(8) + 2.8 = 20.8$$

- Having explained the simple example, we can turn to the calculation of the expected return and variance of any portfolio if the single-index model holds. The expected return on any portfolio is given by:

$$\bar{R}_P = \sum_{i=1}^{N} X_i \bar{R}_i$$

- Substituting for $\bar{R}_i$, we obtain:

$$\bar{R}_P = \sum_{i=1}^{N} X_i \alpha_i + \sum_{i=1}^{N} X_i \beta_i \bar{R}_m$$

- We know that the variance of a portfolio of stocks is given by:

$$\sigma_P^2 = \sum_{i=1}^{N} X_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} X_i X_j \sigma_{ij}$$

- Substituting in the results for σ_i^2 and σ_{ij}, we obtain:

$$\sigma_P^2 = \sum_{i=1}^{N} X_i^2 \beta_i^2 \sigma_m^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} X_i X_j \beta_i \beta_j \sigma_m^2 + \sum_{i=1}^{N} X_i^2 \sigma_{ei}^2$$

Derivative Instruments

- Derivative instruments are securities whose value derives from the value of an underlying security or basket of securities. The instruments are also known as contingent claims because their values are contingent on the performance of underlying assets. The most common contingent claims are options and futures.
- An option on a security gives the holder the *right* to either buy (a call option) or sell (a put option) a particular asset or bundle of assets at a future date or during a particular period of time for a specified price. The buyer pays a price for this option but is free not to exercise this option if prices move in the wrong direction. A future is the *obligation* to buy a particular security or bundle of securities at a particular time for a stated price. A future is simply a delayed purchase of a security.

Indirect Investing

- While an investor can purchase any financial instrument, the investor can instead choose to invest indirectly by purchasing the shares of investment companies (mutual funds). A mutual fund holds a portfolio of securities, usually in line with a stated policy and objective.
- Mutual funds exist that hold only a small set of securities (e.g., short-term tax-free securities or stocks in a particular industry or sector) or broad classes of securities (such as stocks from major stock exchanges around the world or a broad representation of American stocks and bonds).

The Return Characteristics Of Alternative Security Types

- One of the basic tenets is that investors like high return but don't like high risk. It is useful to become familiar with the risk and return characteristics of some of the securities.
- We will in most instances use *return* to indicate the return on an investment over a particular span of time called *holding period return*. Return will be measured by the sum of the change in the market price of a security plus any income received over a holding period divided by the price of a security at the beginning of the holding period. Thus, if a stock started the year at $100, paid $5 in dividends at the end of the year, and had a price of $105 at the end of the year, the return would be 10%.

- In describing securities, we mentioned several factors that should affect risk. These included:

1. the maturity of an instrument (in general, the longer the maturity, the riskier it is).
2. the risk characteristic and creditworthiness of the issuer or guarantor of the investment.
3. the nature and priority of the claims the investment has on income and assets.
4. the liquidity of the instrument and the type of market in which it is traded.

If risk is related to these elements, then measures of risk such as the variability of returns should be related to these same factors.