2 Stress

2.2.3 Mohr's Circle

Transformation relations

$$\begin{cases} \sigma_\xi = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\varphi + \tau_{xy}\sin 2\varphi\,, \\ \sigma_\eta = \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\varphi - \tau_{xy}\sin 2\varphi\,, \\ \tau_{\xi\eta} = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\varphi + \tau_{xy}\cos 2\varphi\,. \end{cases}$$

Allow a simple and "useful" geometric representation of the stress state . For this purpose, the 1st and the 3d are rewritten as:

$$\sigma_\xi - \frac{1}{2}(\sigma_x + \sigma_y) = \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\varphi + \tau_{xy}\sin 2\varphi\,,$$

$$\tau_{\xi\eta} = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\varphi + \tau_{xy}\cos 2\varphi\,.$$

By squaring and adding, the angle φ can be eliminated: $\left[\sigma_\xi - \frac{1}{2}(\sigma_x + \sigma_y)\right]^2 + \tau_{\xi\eta}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$

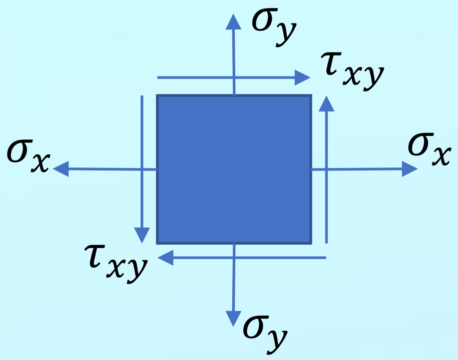
From the transformation relations, if the second and the third are used, same result with σ_ξ replaced by σ_η will be obtained. So, In what follows the subscripts ξ and η will therefore be omitted, and by labeling the right side of the last equation r^2, it becomes clear that the normal stress σ and the shear stress τ related to the angle φ, verify the next equation:

$$(\sigma - \sigma_M)^2 + \tau^2 = r^2 \qquad \text{with} \qquad \sigma_M = \frac{\sigma_x + \sigma_y}{2} \qquad \text{and} \qquad r^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$

For given stresses σ_x, σ_y and τ_{xy} the two parameters σ_M and r, are constant. Then σ & τ verify an equation of a circle .

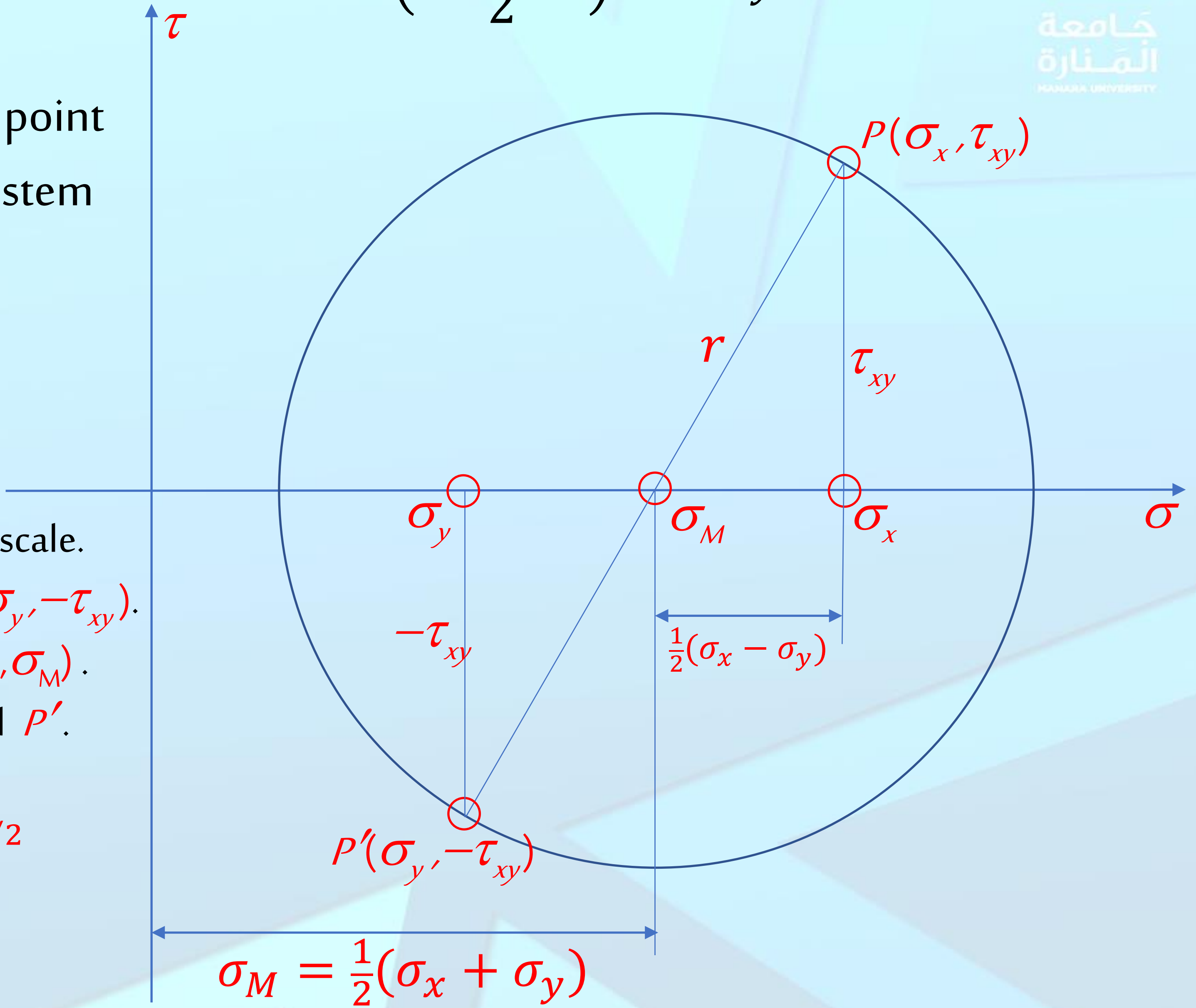
This circle is called Mohr's Circle. If the stresses σ_x, σ_y and τ_{xy}, are knowns, it can be used to represent σ & τ as follows.

$$(\sigma - \sigma_M)^2 + \tau^2 = r^2 \qquad \sigma_M = \frac{\sigma_x + \sigma_y}{2} \qquad r^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$



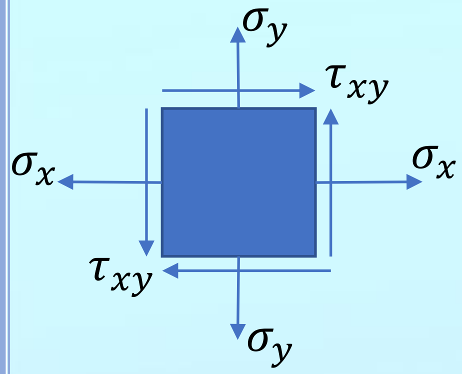
The state of stress at a point is known in the x, y system

Drawing the Mohr's Circle

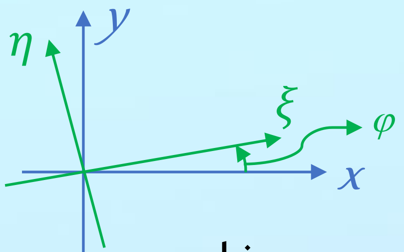


1. Draw two perpendicular axes σ & τ, with the same scale.
2. In the system σ, τ; locate points $P(\sigma_x, \tau_{xy})$ & $P'(\sigma_y, -\tau_{xy})$.
3. Draw line PP', to intersect the axis σ at the point $(0, \sigma_M)$.
4. Draw a circle with $(0, \sigma_M)$ and passing through P and P'.
5. From the figure observe that: $\sigma_M = \frac{1}{2}(\sigma_x + \sigma_y)$.
6. And the radius is: $r = \left[\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2\right]^{1/2}$

$$(\sigma - \sigma_M)^2 + \tau^2 = r^2 \qquad \sigma_M = \frac{\sigma_x + \sigma_y}{2} \qquad r^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$



The state of stress at a point is known in the x, y system

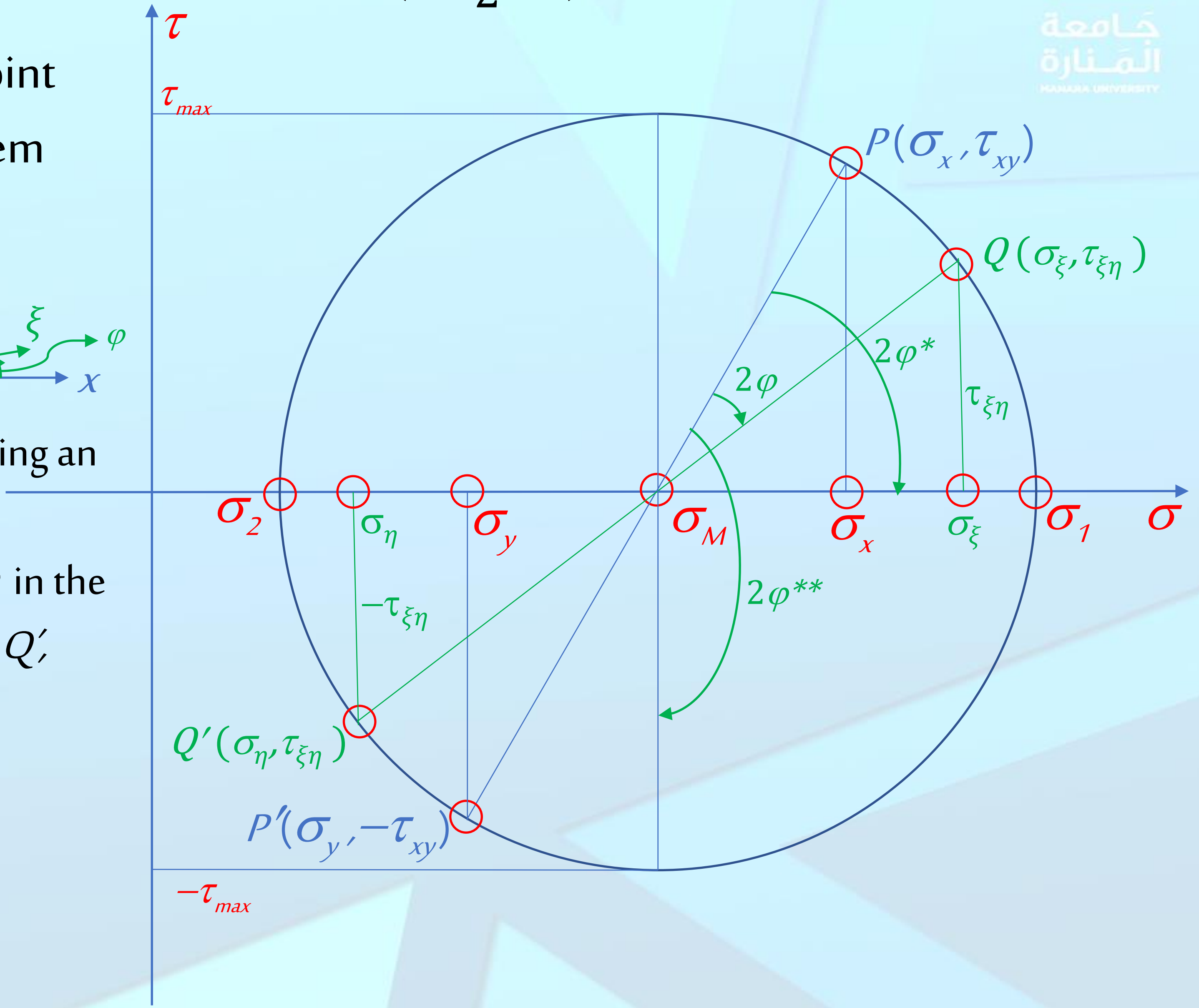


Using Mohr's Circle

1. To determine the stresses in the ξ , η system making an angle φ with x, y system.
At the circle rotate the diameter PP' by an angle 2 φ in the opposite direction of φ, to QQ'. The two points Q & Q', represent the ξ , η system.

2. Principal stresses and principal directions.

3. Maximum shear stress and its directions.



Three special cases

Uniaxial tension

$\sigma_x = \sigma_0 > 0,\ \sigma_y = 0,\ \tau_{xy} = 0$

$\tau_{max} = \sigma_0/2$

at $\varphi = 45°$

pure shear

$\sigma_x = 0,\ \sigma_y = 0\ \&\ \tau_{xy} = \tau_0$

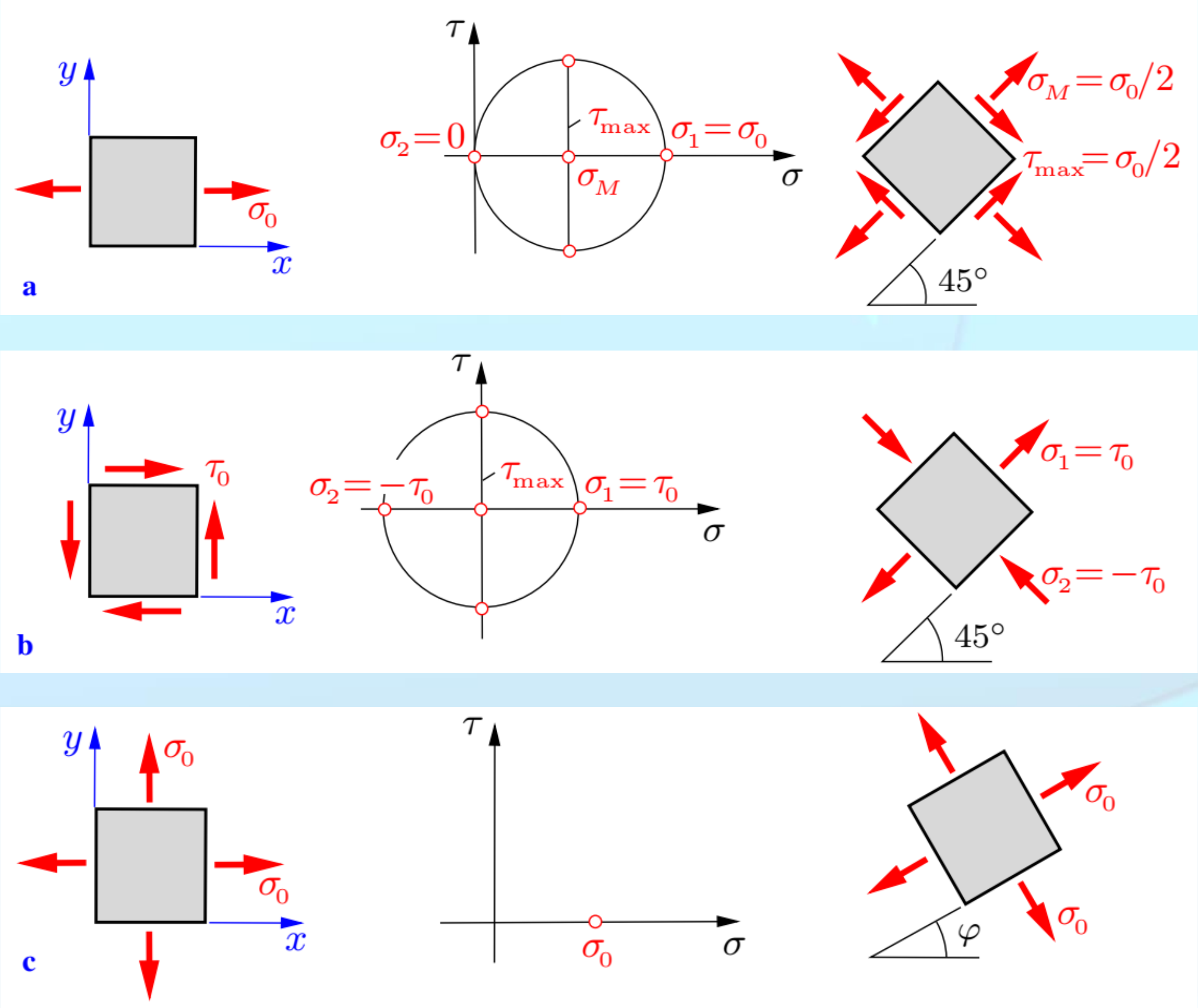
$\sigma_1 = \tau_0\ \&\ \sigma_2 = -\tau_0$

at $\varphi = 45°$

hydrostatic stress state

$\sigma_x = \sigma_y = \sigma_0\ \&\ \tau_{xy} = 0$

$\sigma_\xi = \sigma_\eta = \sigma_0$



2.2.4 The Thin-Walled Pressure Vessel

An important application of plane stress is the *thin walled* cylindrical vessel with radius r & wall thickness $t \ll r$ (Fig.a), subjected to an internal pressure p that causes stresses in its wall which need to be determined (Fig. b).

Far from the end caps of the vessel, the stress state is independent of the location (homogeneous stress state).

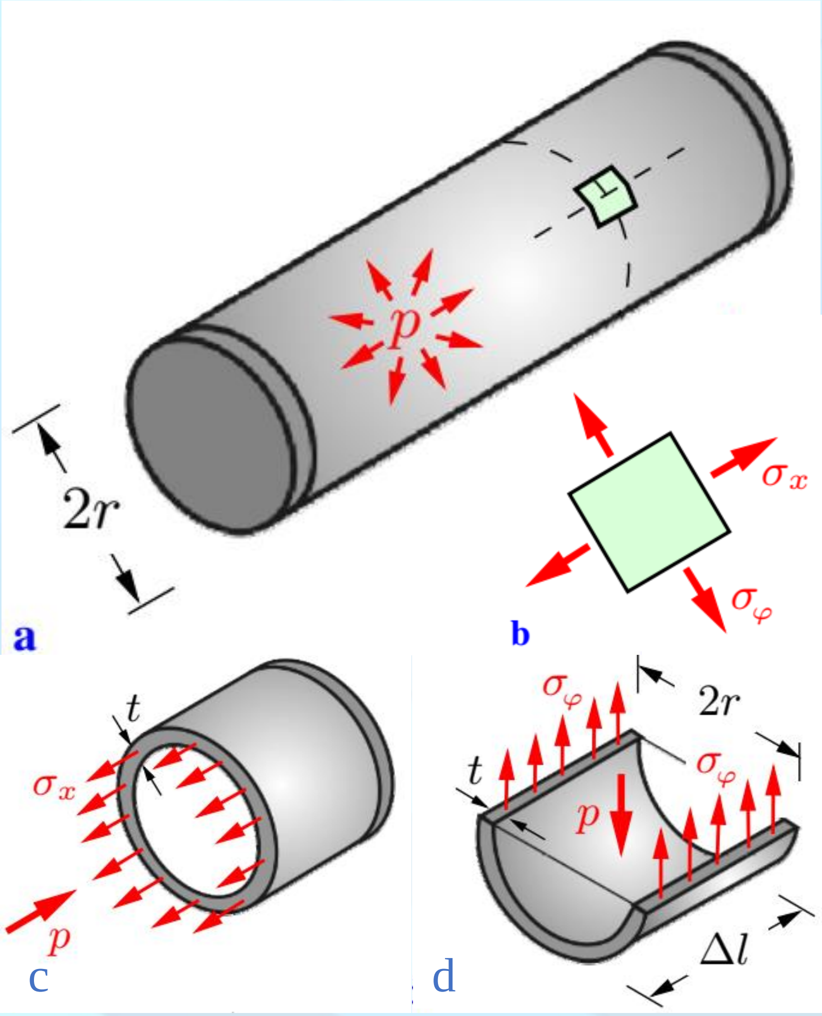
Given that $t \ll r$, the stresses in radial directions can be neglected.

Thus, within a good approximation a plane stress state acts locally in the wall of the vessel (note: although the element in Fig.b is curved, it is replaced by a plane element in the tangent plane).

The stress state can be described by the stresses in two sections perpendicular to each other. First, the vessel is cut perpendicularly to its longitudinal axis (Fig.c).

Since the gas or fluid pressure is independent of the location, the pressure on the section area πr^2 (of the gas or fluid) has the constant value p. Assuming that the *longitudinal stress* σ_x is constant across the wall thickness because of $t \ll r$, the equilibrium condition yields (Fig.c)

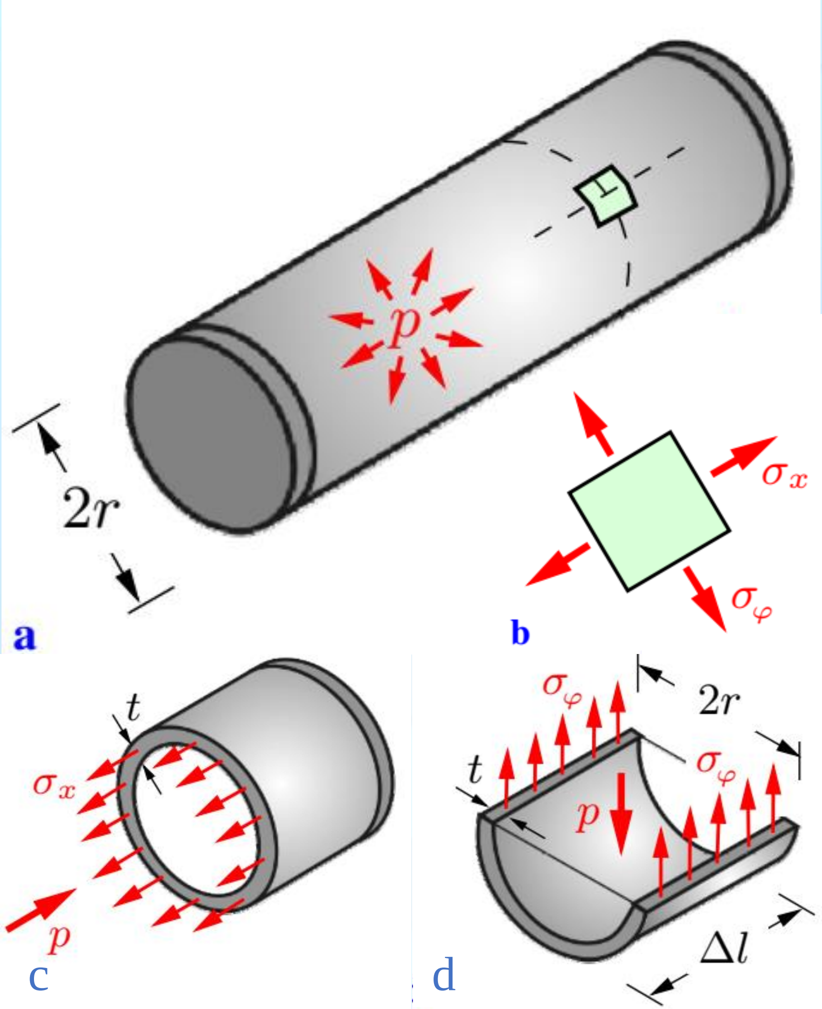
$$\sigma_x(2\pi r t) - p(\pi r^2) = 0 \;\Rightarrow\; \sigma_x = \frac{1}{2}\frac{pr}{t}.$$



As illustrated in Fig.d a half-circular part of length Δl is separated from the vessel. The horizontal sections of the wall are subjected to the *circumferential stress* σ_φ, also called *hoop stress*, which again is constant across the thickness. These stresses will counteract the force $p(2\,r\Delta l)$, exerted from the gas onto the halfcircular part of the vessel. Equilibrium in the vertical direction yields

$$\sigma_\varphi(2t\Delta l) - p(2r\Delta l) = 0 \;\Rightarrow \sigma_\varphi = \frac{pr}{t} \qquad \text{notice that} \qquad \sigma_\varphi = 2\sigma_x$$

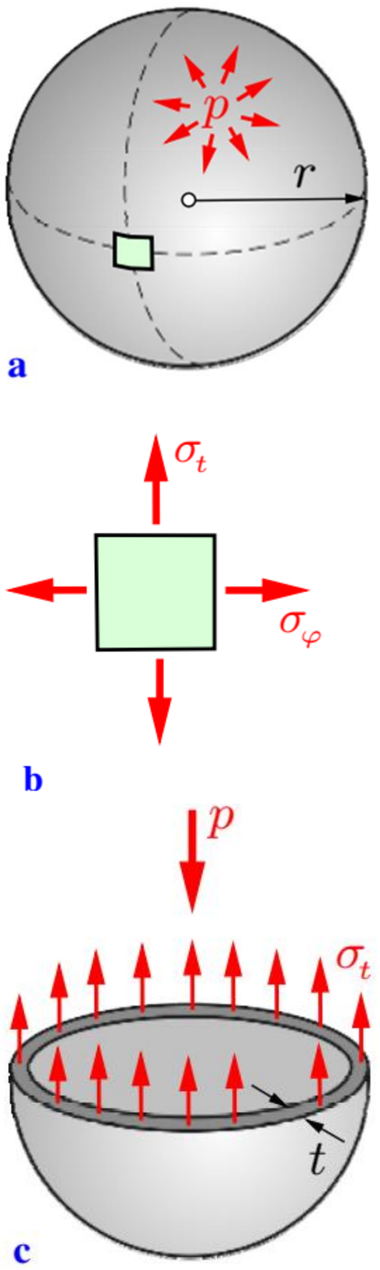
The two equations for σ_x and σ_φ sometimes are called *vessel formulas*. Because of $t \ll r$ it can be seen that $\sigma_x, \sigma_\varphi \gg p$ Therefore, the initially made assumption that the stresses σ_r in radial direction may be neglected is justified ($|\sigma_r| \leq p$). Generally, a vessel is called *thin-walled* when it fulfills the condition $r > 5\,t$. The vessel formulas are also applicable to a vessel subjected to external pressure. In this case only the sign of p has to be changed, i.e. the wall is then under a compressive stress state.



Since no shear stresses are present in both sections, the stresses σ_x and σ_φ are principal stresses: $\sigma_1 = \sigma_\varphi = pr/t\,, \sigma_2 = pr/(2t)$.

The maximum shear stress is given by: $\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_2) = pr/(4t)$.

To note that near the caps more complex stress states are present which cannot be determined with an elementary theory.



Now a thin-walled spherical vessel of radius r, subjected to a gage pressure p (Fig. a) is considered. Here, the stresses σ_t and σ_φ act in the wall (Fig. b). When the vessel is cut into half (Fig. c), σ_t is obtained from the equilibrium condition:

$$\sigma_t(2\pi r t) - p\pi r^2 = 0 \;\Rightarrow\; \sigma_t = \frac{pr}{2t}$$

A cut, perpendicular to the first one, similarly leads to

$$\sigma_\varphi(2\pi r t) - p\pi r^2 = 0 \;\Rightarrow\; \sigma_\varphi = \frac{pr}{2t}$$

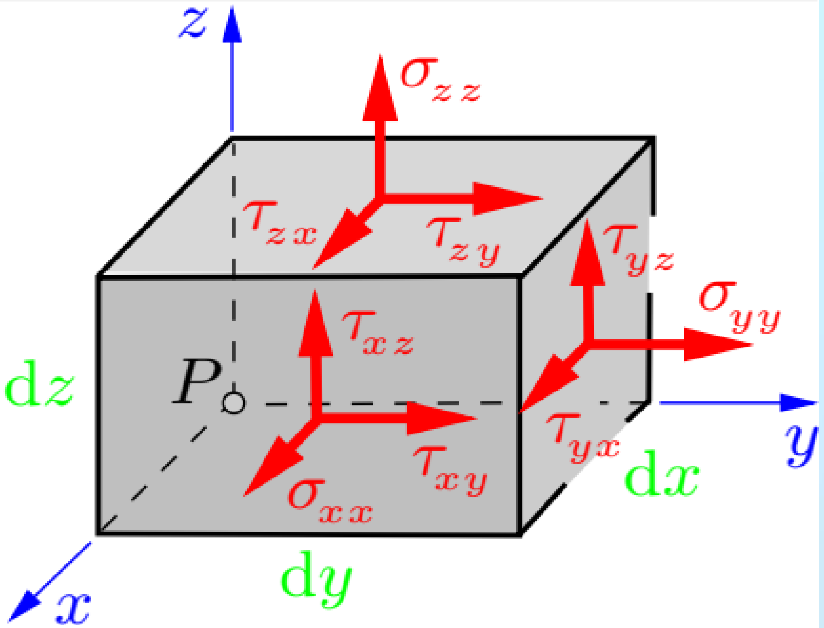
Thus, $\sigma_t = \sigma_\varphi = \dfrac{pr}{2t}$

Therefore, the stress in the wall of a thin-walled spherical vessel has the value $p\,r/(2\,t)$ in any arbitrary direction. The state of stress is a hydrostatic one.
As in the foregoing case, this formula is also valid for an external pressure in which case p is negative.

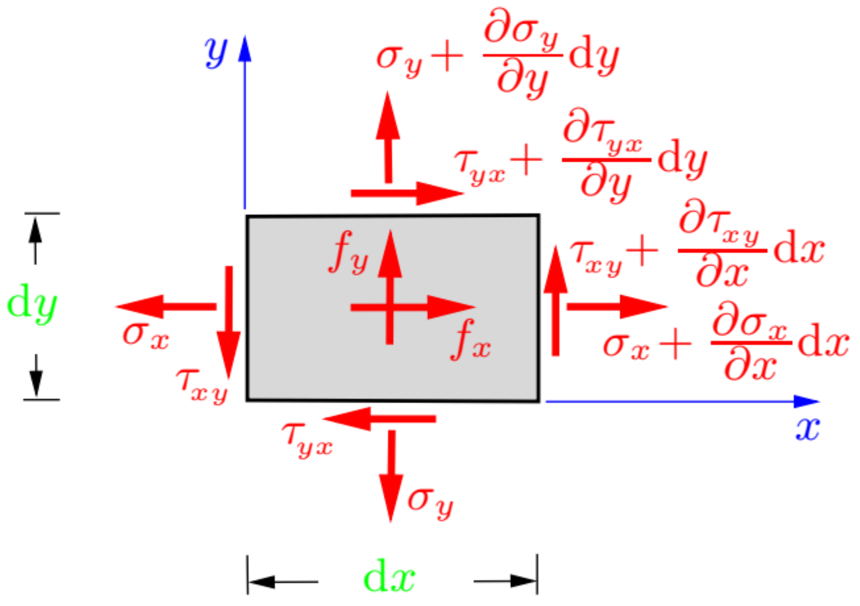
2.3 Equilibrium Conditions

The stress state at a material point of a body is determined by the stress tensor; its components are shown in left Fig. In general, these components vary from point to point and these variations are not independent of each other: they are connected via the *equilibrium conditions*.

Three Dimensional case	Two Dimensional case	One Dimensional case

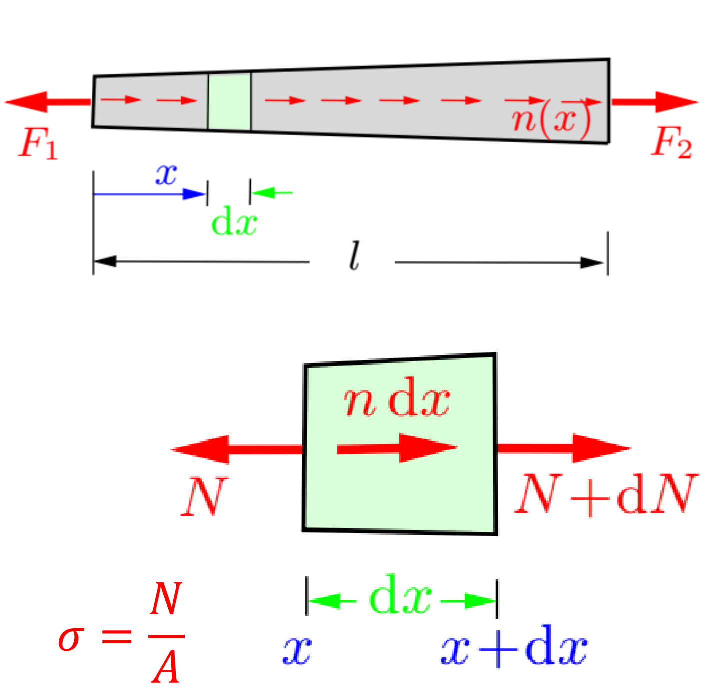


$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$



$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix}$$

$\sigma_x(x,y), \sigma_y(x,y)$ & $\tau_{xy}(x,y)$??



$$\sigma = \frac{N}{A}$$

$$\frac{\boldsymbol{dN}}{\boldsymbol{dx}} + \boldsymbol{n} = \boldsymbol{0} \Rightarrow \sigma(x) = \frac{N(x)}{A(x)}$$

To derive the equilibrium conditions we first consider the stresses acting on an infinitesimal element under plane stress which is cut out from a disk of thickness t.

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix}$$

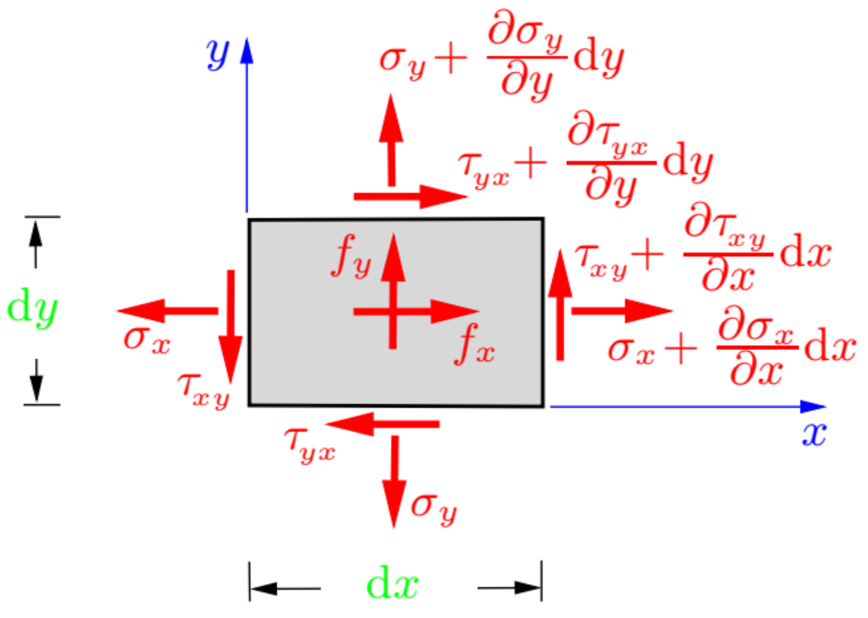
$\sigma_x(x,y), \sigma_y(x,y)$ & $\tau_{xy}(x,y)$??

Since the stresses in general depend on x and y, they are not the same at the opposite sections: they differ by infinitesimal increments.

For example, the left face is subjected to the normal stress σ_x, whereas the stress $\sigma_x + \frac{\partial \sigma_x}{\partial x} dx$ acts on the right face.

The symbol $\partial/\partial x$ denotes the partial derivative with respect to x.

Furthermore, the element may be loaded by the volume force $\vec{f}$ with the components f_x and f_y.



The equilibrium condition in $x-$direction yields

$$-\sigma_x dyt - \tau_{yx} dxt + \left(\sigma_x + \frac{\partial \sigma_x}{\partial x} dx\right) dyt + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy\right) dxt + f_x dxdy = 0$$

By Canceling and dividing by $dxdy$:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + f_x = 0$$

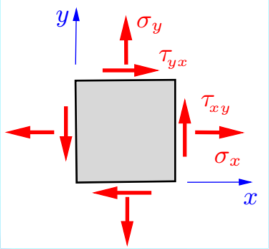
Similarly the equilibrium condition in $y-$direction yields

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + f_y = 0$$

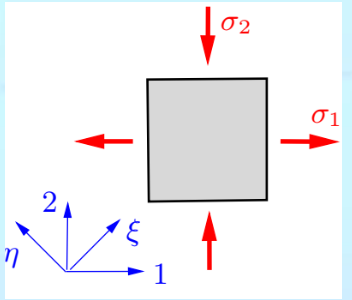
Moment equilibrium about z has given that: $\tau_{xy} = \tau_{yx}$

In plane stress case, two equilibrium equations with three unknowns functions?

2.4 Supplementary Examples



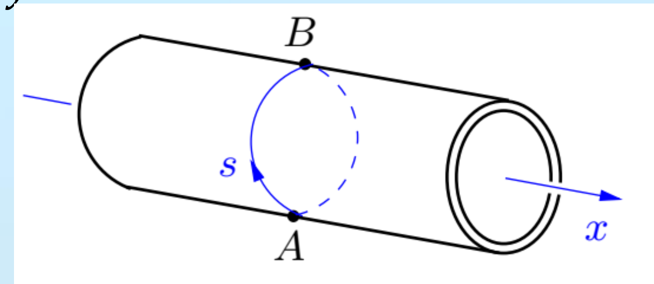
Example 2.2 The stresses $\sigma_x = 20$ MPa, $\sigma_y = 30$ MPa and $\tau_{xy} = 10$ MPa in a metal sheet are known, see Fig. Determine the principal stresses and their directions.



Example 2.3 A plane stress state is given by the principal stresses $\sigma_1 = 30$ MPa and $\sigma_2 = -10$ Mpa, seeFig.

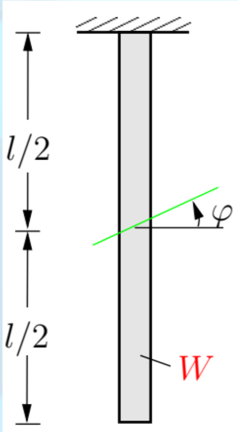
a) Determine the stress components in a ξ, η-coordinate system which is inclined by 45° with respect to the principal axes.

b) Using Mohr's circle, determine the rotation angle α of an x, y-coordinate system where $\sigma_y = 0$ and $\tau_{xy} < 0$. Calculate σ_x and τ_{xy}.



Example 2.4 A thin-walled tube is subjected to bending and torsion such that the following stresses act at points *A* and *B*: $\sigma_x^{A,B} = \pm 25$ MPa, $\sigma_s^{A,B} = 50$ MPa, $\tau_{xs}^{A,B} = 50$ MPa .

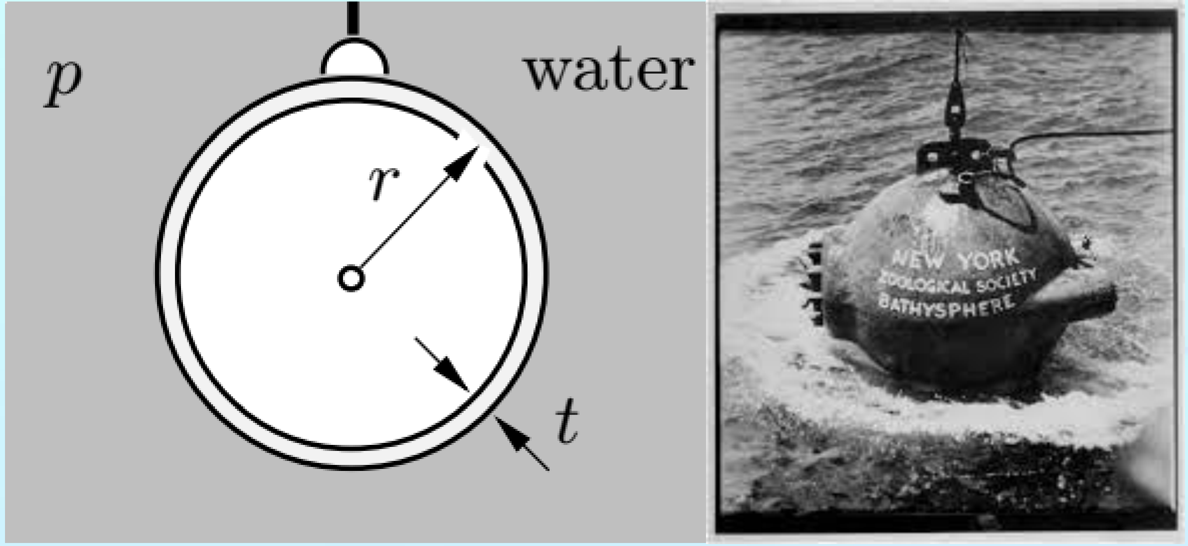
Determine the principal stresses and their directions at *A* and *B*.



Example 2.5 A slender bar (weight *W*, cross sectional area *A*) is suspended from the ceiling.

Given: $\rho = 10^4$ kg/m^3, $g = 10$ m/s^2, $l = 1$ m. Using Mohr's circle, determine the normal stress and the shear stress acting in the section characterized by the angle $\varphi = 20°$ as shown in the figure.

Example 2.6 A thin-walled bathysphere (radius $r = 500$ mm, wall thickness $t = 12.5$ mm) is lowered to a depth of 500m under the water surface (pressure $p = 5$ MPa). Determine the stresses in the wall.



Example 2.9 A thin-walled cylindrical vessel has the radius $r = 1$ m and wall-thickness $t = 10$ mm. Determine the maximum internal pressure p_{max} so that the maximum stress in the wall does not exceed the allowable stress $\sigma_{allow} = 150$ MPa.