

Exercise 7: Fourier Analysis for Continuous Time Signals and Systems

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CECC507: Signals and Systems

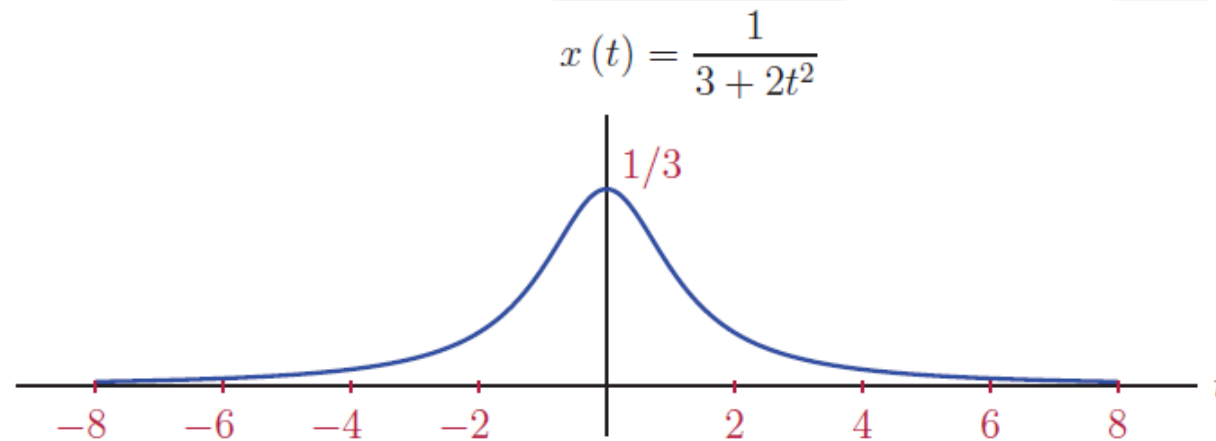
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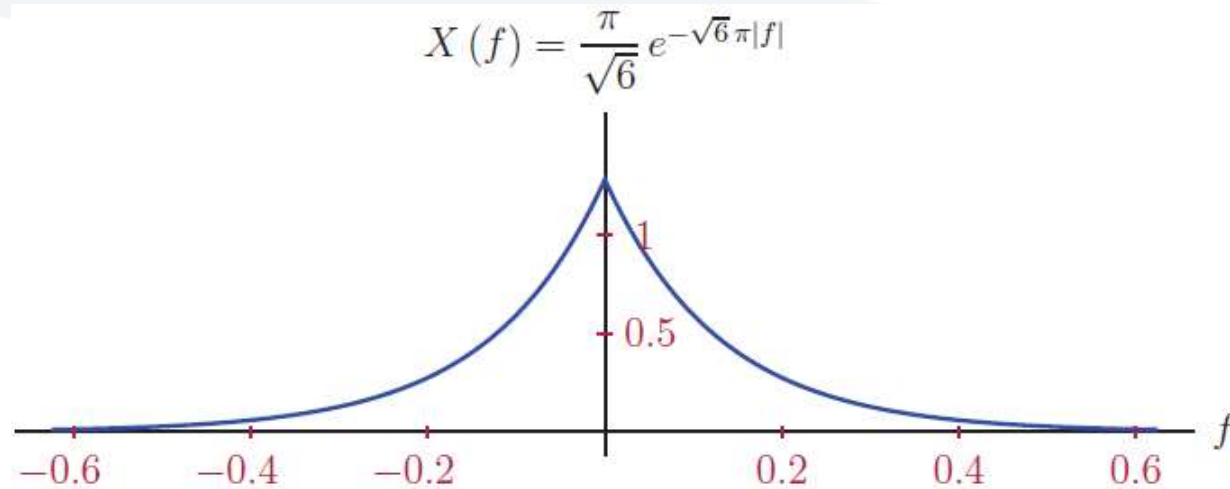
5. The transform pair

$$e^{-a|t|} \xleftrightarrow{F} \frac{2a}{a^2 + \omega^2}$$

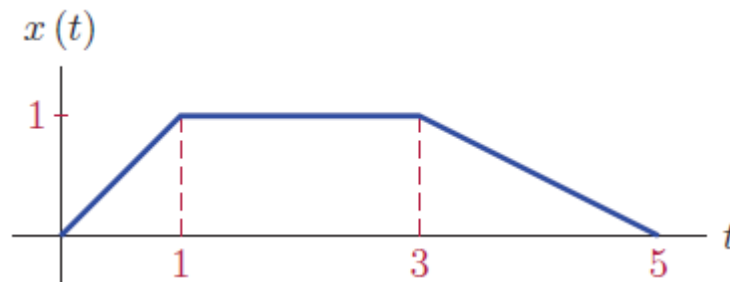
Using this pair along with the duality property, find the Fourier transform of the signal



$$\frac{1}{3 + 2t^2} \xleftrightarrow{F} \frac{\pi}{\sqrt{6}} e^{-\sqrt{3/2}|\omega|}$$



6. Using the differentiation-in-time property of the Fourier transform, determine the transform of the signal shown.



Let $w(t)$ be the derivative of the signal $x(t)$, that is

$$w(t) = \frac{dx(t)}{dt} = 2\Pi(t - 0.5) - 0.5\Pi\left(\frac{t - 4}{2}\right)$$

$$W(f) = \text{sinc}(f)e^{j\pi f} - \text{sinc}(2f)e^{j8\pi f}$$

$$W(f) = j2\pi X(f) \Rightarrow X(f) = \frac{1}{j2\pi} W(f) = \frac{1}{j2\pi} \left[\text{sinc}(f)e^{j\pi f} - \text{sinc}(2f)e^{j8\pi f} \right]$$

7. Determine the Fourier transform of the signal

$$x(t) = \sin(\pi t)\Pi\left(t - \frac{1}{2}\right) = \begin{cases} \sin(\pi t), & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases}$$

a. Using the modulation property of the Fourier transform

b. Using the multiplication property of the Fourier transform

$$a. F\{p(t)\sin(2\pi f_0 t)\} = \frac{1}{2}\left[P(f - f_0)e^{-j\pi/2} + P(f + f_0)e^{j\pi/2}\right]$$

$$p(t) = \Pi\left(t - \frac{1}{2}\right) \Rightarrow P(f) = \text{sinc}(f)e^{-j\pi f}$$

$$f_0 = 0.5 \text{ Hz}$$

$$\begin{aligned}
 X(f) &= \frac{1}{2} \left[P(f - 0.5) e^{-j\pi/2} + P(f + 0.5) e^{j\pi/2} \right] \\
 &= \frac{1}{2} \left[\text{sinc}(f - 0.5) e^{-j\pi(f-0.5)} e^{-j\pi/2} + \text{sinc}(f + 0.5) e^{-j\pi(f+0.5)} e^{j\pi/2} \right] \\
 &= \frac{1}{2} \left[\text{sinc}(f - 0.5) + \text{sinc}(f + 0.5) \right] e^{-j\pi f}
 \end{aligned}$$

b. $F \{ p(t)q(t) \} = P(f) * Q(f)$

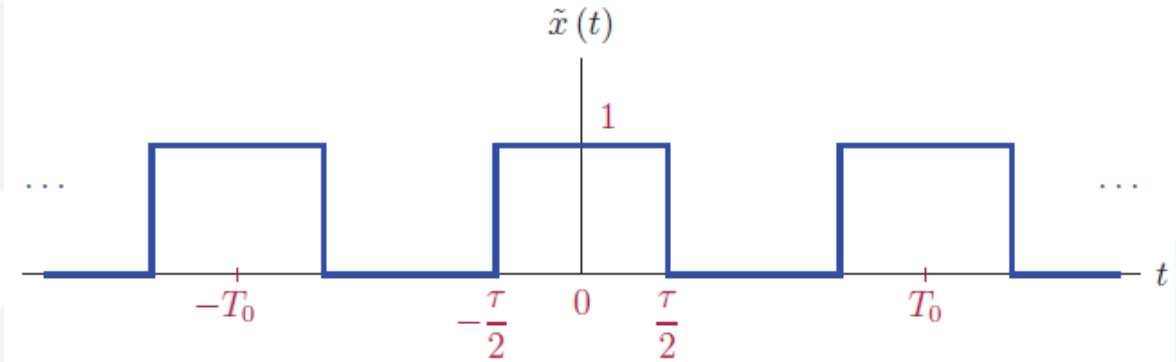
$$p(t) = \Pi \left(t - \frac{1}{2} \right) \Rightarrow P(f) = \text{sinc}(f) e^{-j\pi f}$$

$$q(t) = \sin(\pi t) \Rightarrow Q(f) = -j \frac{1}{2} \delta(f - 0.5) + j \frac{1}{2} \delta(f + 0.5)$$

$$X(f) = Q(f) * P(f) = \frac{1}{2} \left[\text{sinc}(f - 0.5) + \text{sinc}(f + 0.5) \right] e^{-j\pi f}$$

8. Consider the pulse train with duty cycle d shown. Its EFS $c_k = d \operatorname{sinc}(kd)$.

a. Working in the time domain, compute the power of the pulse train as a function of the duty cycle d .



b. Sketch the power spectral density based on the EFS coefficients.

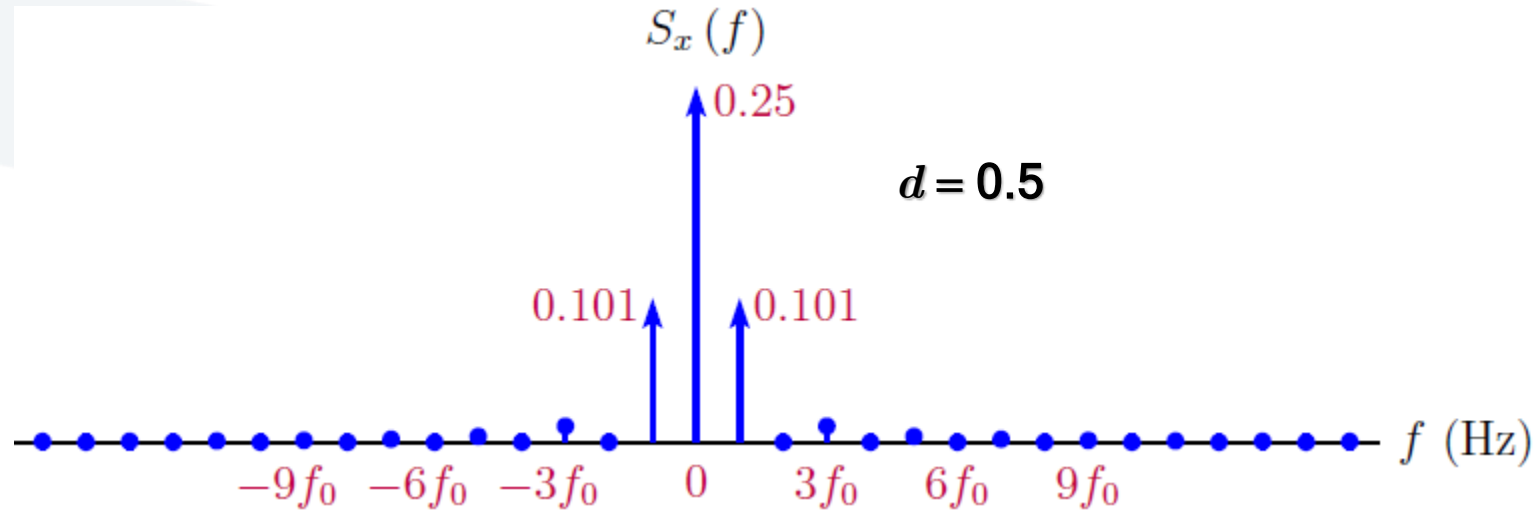
c. Let $d = 0.5$. Suppose this signal is processed through a lowpass system that only retains the first m harmonics and eliminates the others. How many harmonics should be retained if we want to preserve at least 90 percent of the signal power?

- d. How many harmonics should be retained to preserve at least 95 percent of the signal power?
- e. How many harmonics should be retained to preserve at least 99 percent of the signal power?

$$a. P_x = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x^2(t) dt = \frac{1}{T_0} \int_{-\tau/2}^{\tau/2} dt = d \quad (\tau = dT_0)$$

$$b. c_k = dsinc(kd) \quad k = -\infty, \dots, \infty$$

$$S_x(f) = \sum_{k=-\infty}^{\infty} |c_k|^2 \delta(k - f_0) = \sum_{k=-\infty}^{\infty} d^2 \text{sinc}^2(kd) \delta(k - f_0)$$



c. If EFS terms up to and including the M -th harmonic are retained, the normalized average power of the signal would be

$$P_x^{(M)} = \sum_{k=-M}^M |c_k|^2 = \sum_{k=-M}^M d^2 \text{sinc}^2(kd)$$

and the percentage of this to the total average power in the signal $x(t)$ is

$$\eta = \frac{P_x^{(M)}}{P_x} = \frac{P_x^{(M)}}{d} = \sum_{k=-M}^M d \operatorname{sinc}^2(kd)$$

It can be shown that, with $d = 0.5$ and $M = 1$ we get $\sum_{k=-1}^1 0.5 \operatorname{sinc}^2(0.5d) = 0.9053$

d. Frequencies up to the third harmonic are needed to retain 95 percent of the signal power since

$$\sum_{k=-3}^3 0.5 \operatorname{sinc}^2(0.5d) = 0.9503$$

e. Frequencies up to the 21-st harmonic are needed to retain 95 percent of the signal power since

$$\sum_{k=-21}^{21} 0.5 \operatorname{sinc}^2(0.5d) = 0.9908$$

9. Repeat parts (c)-(e) of Problem 8 using $d = 0.2$. Does it take fewer or more harmonics to preserve the same percentage of power when the duty cycle is reduced?

c. If EFS terms up to and including the M -th harmonic are retained, the percentage of this to the total average power in the signal $x(t)$ is

$$\eta = \frac{P_x^{(M)}}{P_x} = \frac{P_x^{(M)}}{d} = \sum_{k=-M}^M d \operatorname{sinc}^2(kd)$$

It can be shown that, with $d = 0.2$ and $M = 4$ we get

$$\sum_{k=-4}^4 0.2 \operatorname{sinc}^2(0.2kd) = 0.9029$$

d. Frequencies up to the 11-*st* harmonic are needed to retain 95 percent of the signal power since

$$\sum_{k=-11}^{11} 0.2 \operatorname{sinc}^2(0.2d) = 0.9528$$

e. Frequencies up to the 51-*st* harmonic are needed to retain 95 percent of the signal power since

$$\sum_{k=-51}^{51} 0.2 \operatorname{sinc}^2(0.2d) = 0.9900$$

10. Determine and sketch the power spectral density of the following signals:

a. $x(t) = 3\cos(20\pi t)$

b. $x(t) = 2\cos(20\pi t) + 3\cos(30\pi t)$

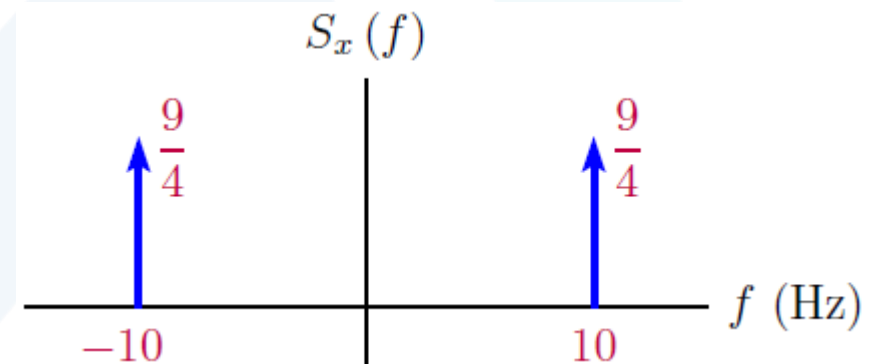
c. $x(t) = 5\cos(200\pi t) + 5\cos(200\pi t) \cos(30\pi t)$

a. For the signal $x(t)$ the fundamental frequency is $f_0 = 10$ Hz, and the EFS coefficients are

$$c_k = \begin{cases} \frac{3}{2}, & k = \pm 1 \\ 0, & \text{otherwise} \end{cases}$$

The power spectral density is

$$S_x(f) = \frac{9}{4} \delta(f + 10) + \frac{9}{4} \delta(f - 10)$$

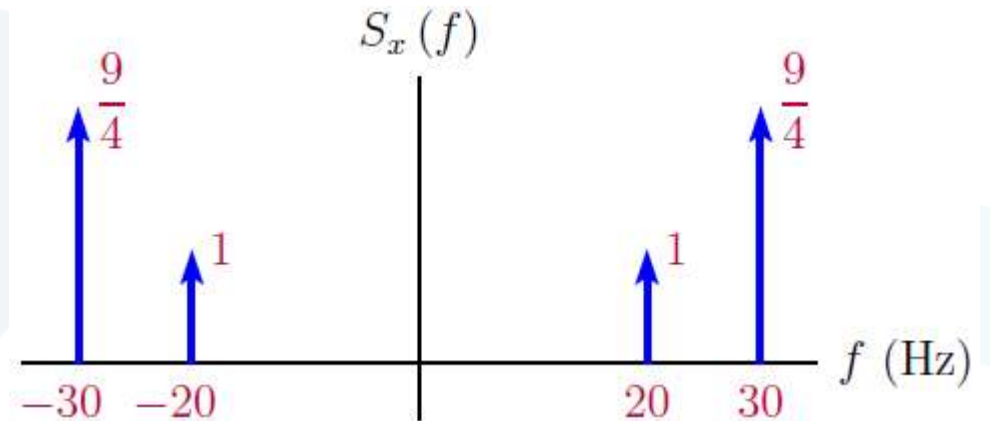


b. For the signal $x(t)$ the fundamental frequency is $f_0 = 5$ Hz, and the EFS coefficients are

$$c_k = \begin{cases} 1, & k = \pm 2 \\ \frac{3}{2}, & k = \pm 3 \\ 0, & \text{otherwise} \end{cases}$$

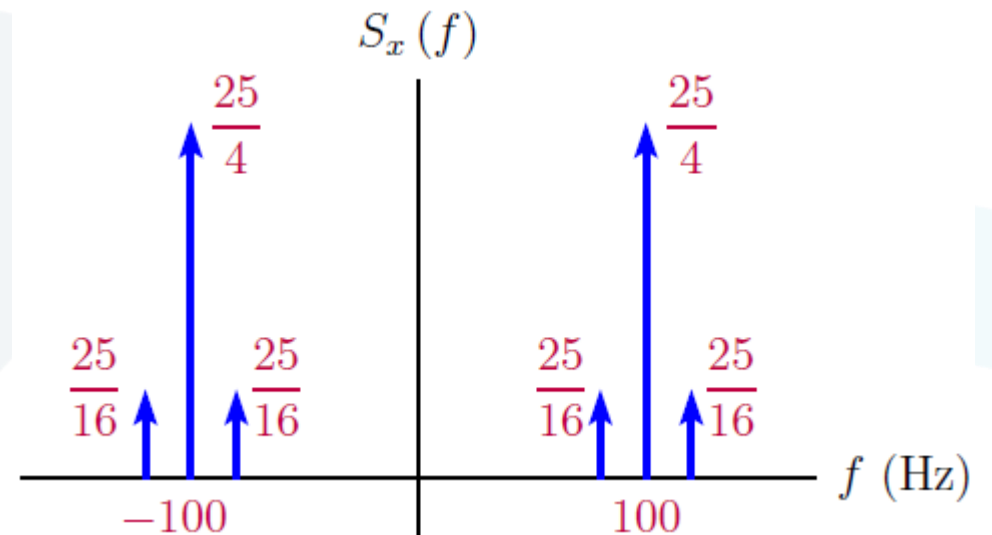
The power spectral density is

$$S_x(f) = \frac{9}{4} \delta(f + 30) + \delta(f + 20) + \delta(f - 20) + \frac{9}{4} \delta(f - 30)$$



c. For the signal $x(t)$ the fundamental frequency is $f_0 = 5$ Hz, and the EFS coefficients are

$$c_k = \begin{cases} \frac{5}{4}, & k = \pm 17 \\ \frac{5}{2}, & k = \pm 20 \\ \frac{5}{4}, & k = \pm 23 \\ 0, & \text{otherwise} \end{cases}$$



The power spectral density is

$$S_x(f) = \frac{25}{16} \delta(f + 230) + \frac{25}{4} \delta(f + 200) + \frac{25}{16} \delta(f + 170) \\ + \frac{25}{16} \delta(f - 170) + \frac{25}{4} \delta(f - 200) + \frac{25}{16} \delta(f - 230)$$