Exercise 12: z-Transform for Discrete-Time Signals and Systems

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1. Using the definition of the z-transform, compute $X(z)$ for the signals listed below. Write each transform using non-negative powers of z. In each case determine the poles and the zeros of the transform, and the region of convergence.

$a.\ x[n] = \{\underset{\underset{n=0}{\uparrow}}{1}, 1, 1, 1, 1\}$ $b.\ x[n] = \{1, 1, \underset{\underset{n=0}{\uparrow}}{1}, 1, 1\}$ $c.\ x[n] = \{1, 1, 1, 1, \underset{\underset{n=0}{\uparrow}}{1}\}$

$$a.\ X(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} = \frac{z^4 + z^3 + z^2 + z + 1}{z^4}, \quad \text{ROC: } |z| > 0$$

$$b.\ X(z) = z^2 + z + 1 + z^{-1} + z^{-2} = \frac{z^4 + z^3 + z^2 + z + 1}{z^2}, \quad \text{ROC: } 0 < |z| < \infty$$

$$c.\ X(z) = z^4 + z^3 + z^2 + z + 1, \quad \text{ROC: } |z| < \infty$$

2. For each transform $X(z)$ listed below, determine whether the DTFT of the corresponding signal $x[n]$ exists. If it does, find it.

$$a.\ X(z) = \frac{z(z-1)}{(z+1)(z+2)}, \quad \text{ROC: } |z| > 2$$

$$b.\ X(z) = \frac{z^2}{z^2+5z+6}, \quad \text{ROC: } |z| < 2$$

a. No, since the ROC does not include the unit circle.

b. Yes, since the ROC includes the unit circle.

$$X(\Omega) = X(z)\Big|_{z=e^{j\Omega}} = \frac{e^{j2\Omega}}{e^{j2\Omega}+5e^{j\Omega}+6}$$

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3. **Determine the z-transforms of the signals given below. Indicate the ROC for each.**

$$a.\ x[n] = \begin{cases} n, & n = 0,\cdots,9 \\ 0, & \text{otherwise} \end{cases} \qquad b.\ x[n] = \begin{cases} n, & n = 0,\cdots,9 \\ 10, & n \geq 10 \\ 0, & \text{otherwise} \end{cases}$$

$$a.\ X(z) = \sum_{n=0}^{9} n z^{-n}$$

$$\text{Let } A(z) = \sum_{n=0}^{9} z^{-n} = \frac{1 - z^{-10}}{1 - z^{-1}}, \quad |z| > 0$$

$$\frac{d}{dz}[A(z)] = -\sum_{n=0}^{9} n z^{-n-1} \Rightarrow -z\frac{d}{dz}[A(z)] = \sum_{n=0}^{9} n z^{-n} = X(z)$$

$$X(z)=\frac{d}{dz}\left[A(z)\right]=\frac{-z^{-2}+10z^{-11}-9z^{-12}}{(1-z^{-1})^2},\quad |z|>0$$

b. $X(z)=X_1(z)+X_2(z)$

$$X_1(z)=\frac{-z^{-2}+10z^{-11}-9z^{-12}}{(1-z^{-1})^2},\quad |z|>0$$

$$X_2(z)=\sum_{n=10}^{\infty}10z^{-n}=10\sum_{m=0}^{\infty}z^{-(n+10)}=\frac{10z^{-10}}{1-z^{-1}},\quad |z|>1$$

$$X(z)=\frac{-z^{-2}+10z^{-11}-9z^{-12}}{(1-z^{-1})^2}+\frac{10z^{-10}}{1-z^{-1}}=\frac{z^{-1}-z^{-11}}{(1-z^{-1})^2},\quad |z|>1$$

4. Find the z-transform of the signal.

$$x[n] = \begin{cases} 1, & n \geq 0 \text{ and even} \\ (2/3)^n, & n > 0 \text{ and odd} \\ 0, & n < 0 \end{cases}$$

Also indicate the ROC.

$$X(z) = X_1(z) + X_2(z)$$

$$X_1(z) = \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} z^{-n} = \sum_{m=0}^{\infty} z^{-2m} = \frac{1}{1 - z^{-2}}, \quad \left|z^{-2}\right| < 1 \Rightarrow |z| > 1$$

$$X_2(z) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} (2/3)^n z^{-n} = \sum_{m=0}^{\infty} (2/3)^{2m+1} z^{-(2m+1)} = \left((2/3)z^{-1}\right) \sum_{m=0}^{\infty} \left(\tfrac{4}{9} z^{-2}\right)^m$$

$$= \frac{(2/3)z^{-1}}{1 - \frac{4}{9} z^{-2}} = \frac{6z^{-1}}{9 - 4z^{-2}}, \quad \left|\tfrac{4}{9} z^{-2}\right| < 1 \Rightarrow |z| > \tfrac{2}{3}$$

$$X(z) = \frac{1}{1 - z^{-2}} + \frac{6z^{-1}}{9 - 4z^{-2}} = \frac{9 + 6z^{-1} - 4z^{-2} - 6z^{-3}}{9z^{-4} - 13z^{-2} + 4z^{-4}} = \frac{9z^4 + 6z^3 - 4z^2 - 6z}{9z^4 - 13z^2 + 4}, \quad |z| > 1$$

5. a. Use the correlation property of the z-transform to determine the autocorrelation function for $x[n] = u[n] - u[n-2]$

Hint: Write $X(z)$ in polynomial form.

b. Generalize the result in part (a) to find the autocorrelation function for $x[n] = u[n] - u[n-N]$, $N > 0$ using z-transform techniques.

$$a.\ R_{xx}(z) = X(z)X(z^{-1})$$

$$X(z) = 1 + z^{-1} \Rightarrow R_{xx}(z) = (1 + z^{-1})(1 + z^{1}) = z + 2 + z^{-1}$$

$$r_{xx}[m] = \delta[m+1] + 2\delta[m] + \delta[m-1]$$

$$b.\ X(z) = \frac{1}{1 - z^{-1}}(1 - z^{-N}), \quad |z| > 0$$

$$X(z^{-1}) = \frac{1}{1-z}(1-z^{N}), \quad |z| < \infty$$

$$R_{xx}(z) = \frac{1-z^{-N}}{1-z^{-1}}\frac{1-z^{N}}{1-z} = \frac{z^{N+1} - 2z + z^{-N+1}}{(z-1)^2}, \quad 0 < |z| < \infty$$

$$R_{xx}(z) = \frac{z}{(z-1)^2}[z^{N} - 2 + z^{-N}]$$

$$r_{xx}[m] = (m+N)u[m+N] - 2mu[m] + (m-N)u[m-N]$$

5. Use the correlation property of the z-transform to determine the cross correlation function for the signals

$$x[n] = u[n] - u[n-3] \text{ and } y[n] = u[n] - u[n-5]$$

Hint: Write $X(z)$ and $Y(z)$ in polynomial form.

$$X(z) = 1 + z^{-1} + z^{-2}, \qquad Y(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4}$$

$$R_{xy}(z) = X(z)Y(z^{-1}) = (1 + z^{-1} + z^{-2})(1 + z^{-1} + z^{-2} + z^{-3} + z^{-4})$$

$$= z^{4} + 2z^{3} + 3z^{2} + 3z + 3 + 2z^{-1} + z^{-2}$$

$$r_{xx}[m] = a.\ x[n] = \{1, 2, 3, 3\ \underset{\substack{\uparrow \\ n=0}}{3}\ , 2, 1\}$$

6. Let $x[n] = a^n\, u[n]$. A signal $w[n]$ is defined in terms of $x[n]$ as

$$w[n] = \sum_{k=-\infty}^{n} x[k]$$

a. Determine $X(z)$. Afterwards determine $W(z)$ from $X(z)$ using the summation property of the z-transform.

b. Determine $w[n]$ from the transform $W(z)$ using partial fraction expansion.

c. Determine $w[n]$ directly from the summation relationship using the geometric series formula and compare to the result found in part (b).

$$a.\ X(z) = \frac{z}{z-a}, \quad |z| > |a|$$

$$W(z) = \frac{z}{z-1} X(z) = \frac{z^2}{(z-1)(z-a)}, \quad |z| > \max(1, |a|)$$ summation property

$$b.\ W(z) = \frac{1}{1-a}\frac{z}{z-1} + \frac{a}{a-1}\frac{z}{z-a}$$

$$w[n] = \frac{1}{1-a} u[n] + \frac{a}{a-1} a^n u[n] = \frac{1}{1-a}(1-a^{n+1})u[n]$$

$$c.\ w[n] = \sum_{k=0}^{n} a^k u[k] = \sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}, \quad n \geq 0$$

7. Consider the z-transforms listed below along with their ROCs. For each case determine the inverse transform $x[n]$ using partial fraction expansion.

$$a.\ X(z) = \frac{z}{(z+1)(z+2)}, \quad \text{ROC: } |z| < 1$$

$$b.\ X(z) = \frac{z+1}{(z+1/2)(z+2/3)}, \quad \text{ROC: } |z| > \frac{2}{3}$$

$$c.\ X(z) = \frac{z(z+1)}{(z+3/4)(z-1/2)(z-3/2)}, \quad \text{ROC: } \frac{3}{4} < |z| < \frac{3}{2}$$

a. $$\frac{X(z)}{z} = \frac{1}{(z+1)(z+2)} = \frac{1}{z+1} - \frac{1}{z+2} \Rightarrow X(z) = \frac{z}{z+1} - \frac{z}{z+2}$$

$x[n]$ is an anti-causal signal. Therefore

$$x[n] = -(-1)^n u[-n-1] + (-2)^n u[-n-1]$$

b. $$\frac{X(z)}{z} = \frac{z+1}{(z+1/2)(z+3/2)} = \frac{3}{z} - \frac{6}{z+1/2} + \frac{3}{z+3/2} \Rightarrow X(z) = 3 - \frac{6z}{z+1/2} + \frac{3z}{z+3/2}$$

$x[n]$ is a causal signal. Therefore

$$x[n] = 3\delta[n] - 6(-1/2)^n u[n] + 3(-2/3)^n u[n]$$

c. $$\frac{X(z)}{z} = \frac{z+1}{(z+3/4)(z-1/2)(z-3/2)} = \frac{4/45}{z+3/4} - \frac{6/5}{z-1/2} + \frac{10/9}{z-3/2}$$

$$X(z) = \underbrace{\frac{\frac{4}{45}z}{z+3/4} - \frac{\frac{6}{5}z}{z-1/2}}_{causal} + \underbrace{\frac{\frac{10}{9}z}{z-3/2}}_{anti-causal}$$

$$x[n] = \frac{4}{45}(-3/4)^n u[n] - \frac{6}{5}(1/2)^n u[n] - \frac{10}{9}(3/2)^n u[-n-1]$$

8. The transform $X(z)$ is given by

$$X(z)=\frac{(z+1)(z-2)}{(z+1/2)(z-1)(z+2)}$$

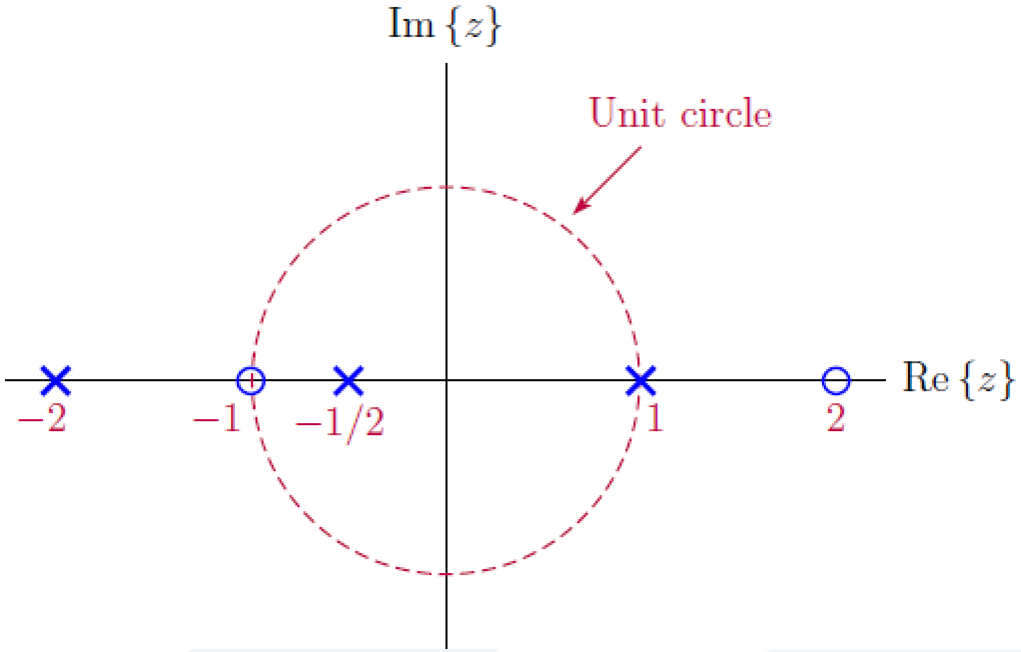
The ROC is not specified.

a. Construct a pole-zero plot for $X(z)$ and identify all possibilities for the ROC.

b. Express $X(z)$ using partial fractions and determine the residues.

c. For each choice of the ROC, determine the inverse transform $x[n]$.

a.



b.
$$\frac{X(z)}{z} = \frac{2}{z} - \frac{10/9}{z+1/2} - \frac{4/9}{z-1} - \frac{4/9}{z+2}$$

$$X(z) = 2 - \frac{\frac{10}{9}z}{z+1/2} - \frac{\frac{4}{9}z}{z-1} - \frac{\frac{4}{9}z}{z+2}$$

c.

Case 1: $|z| < ½$

$$x[n] = 2\delta[n] + \frac{10}{9}(-1/2)^n u[-n-1] + \frac{4}{9}u[-n-1] + \frac{4}{9}(-2)^n u[-n-1]$$

Case 2: $½ < |z| < 1$

$$x[n] = 2\delta[n] - \frac{10}{9}(-1/2)^n u[n] + \frac{4}{9}u[-n-1] + \frac{4}{9}(-2)^n u[-n-1]$$

Case 3: $1 < |z| < 2$

$$x[n] = 2\delta[n] - \frac{10}{9}(-1/2)^n u[n] - \frac{4}{9}u[n] + \frac{4}{9}(-2)^n u[-n-1]$$

Case 4: $|z| > 2$

$$x[n] = 2\delta[n] - \frac{10}{9}(-1/2)^n u[n] - \frac{4}{9}u[n] - \frac{4}{9}(-2)^n u[n]$$

9. The transforms listed below have complex poles. Each is known to be the transform of a causal signal. Determine the inverse transform $x[n]$ using partial fraction expansion.

$$a.\ X(z) = \frac{z^2 + 3z}{z^2 - 1.4z + 0.85}$$

$$b.\ X(z) = \frac{z^2}{z^2 - 1.6z + 1}$$

$$a.\ X(z)=\frac{z(z+3)}{(z-0.7-j0.6)(z-0.7+j0.6)}$$

$$\frac{X(z)}{z}=\frac{z+3}{(z-0.7-j0.6)(z-0.7+j0.6)}=\frac{0.5-j3.0833}{z-0.7-j0.6}+\frac{0.5+j3.0833}{z-0.7+j0.6}$$

$$X(z)=\frac{(0.5-j3.0833)z}{z-0.7-j0.6}+\frac{(0.5+j3.0833)z}{z-0.7+j0.6}$$

$$x[n]=(0.5-j30.833)(0.7+j0.6)^n u[n]+(0.5+j30.833)(0.7-j0.6)^n u[n]$$

$$x[n]=(3.1236e^{-j1.41})(0.922e^{j0.7086})^n u[n]+(3.1236e^{j1.41})(0.922e^{-j0.7086})^n u[n]$$

$$x[n]=6.2472(0.922)^n\cos(0.7086n-1.41)u[n]$$

$$b.\ X(z)=\frac{z^2}{(z-0.8-j0.6)(z-0.8+j0.6)}$$

$$\frac{X(z)}{z}=\frac{z}{(z-0.8-j0.6)(z-0.8+j0.6)}=\frac{\frac{1}{2}-j\,\frac{2}{3}}{z-0.7-j0.6}+\frac{\frac{1}{2}+j\,\frac{2}{3}}{z-0.7+j0.6}$$

$$X(z)=\frac{X(z)}{z}=\frac{\left(\frac{1}{2}-j\,\frac{2}{3}\right)z}{z-0.7-j0.6}+\frac{\left(\frac{1}{2}+j\,\frac{2}{3}\right)z}{z-0.7+j0.6}$$

$$x[n]=(\tfrac{1}{2}-j\,\tfrac{2}{3})(0.8+j0.6)^n u[n]+(\tfrac{1}{2}+j\,\tfrac{2}{3})(0.8-j0.6)^n u[n]$$

$$x[n]=(\tfrac{5}{6}\,e^{-j0.9273})e^{j0.7086n}u[n]+(\tfrac{5}{6}\,e^{j0.9273})e^{-j0.7086n}u[n]$$

$$x[n]=\tfrac{5}{3}\cos(0.9273n-0.7086)u[n]$$

10. The transforms listed below have multiple poles. Express each transform using partial fractions, and determine the residues.

$$X(z) = \frac{z^2 + 3z + 2}{z^2 - 2z + 1}$$

$$X(z) = \frac{(z+1)(z+2)}{(z-1)^2}$$

$$\frac{X(z)}{z} = \frac{(z+1)(z+2)}{z(z-1)^2} = \frac{2}{z} + \frac{6}{z-1} - \frac{1}{(z-1)^2}$$

$$X(z) = \frac{(z+1)(z+2)}{z(z-1)^2} = 2 + \frac{6z}{z-1} - \frac{z}{(z-1)^2}$$