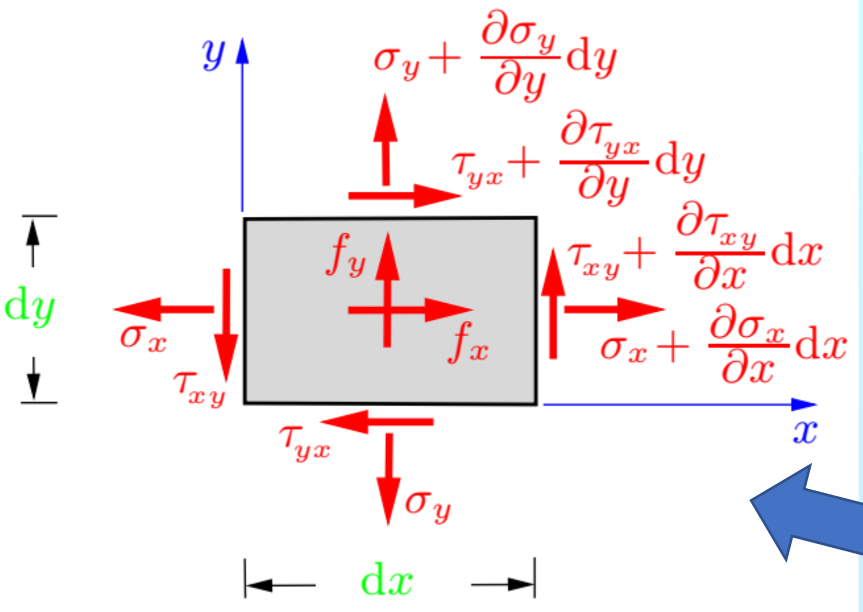
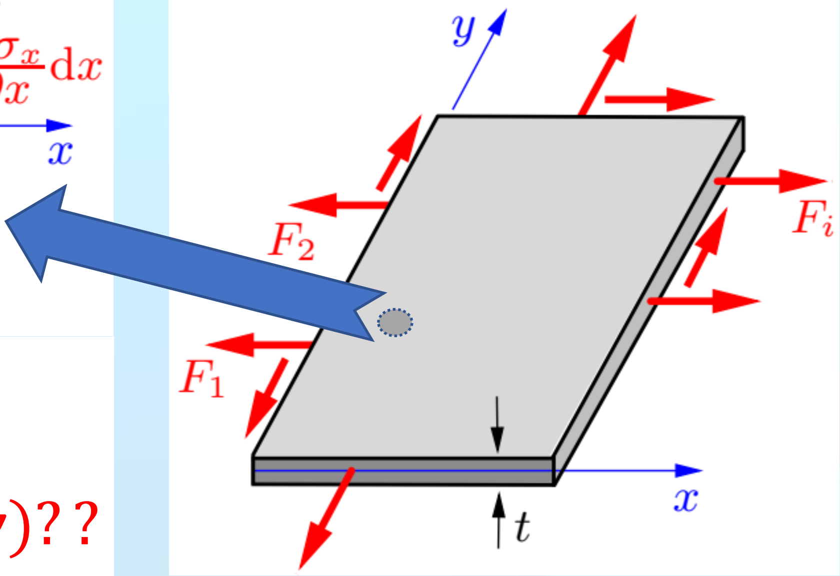
3 Strain, Hooke's Law

3.1 State of Strain

3.2 Hooke's Law

Two Dimensional case





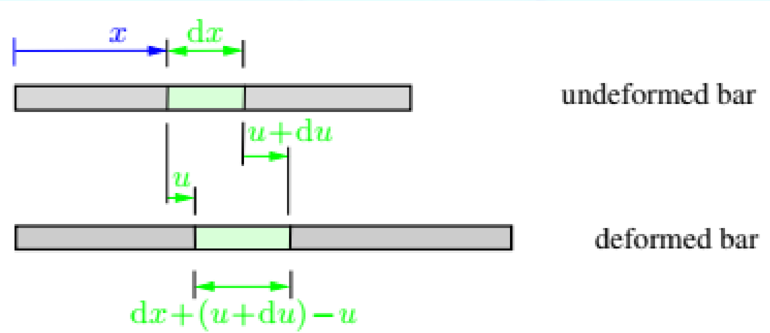
$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix}$$

$\sigma_x(x,y), \sigma_y(x,y)$ & $\tau_{xy}(x,y)$??

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + f_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + f_y = 0$$

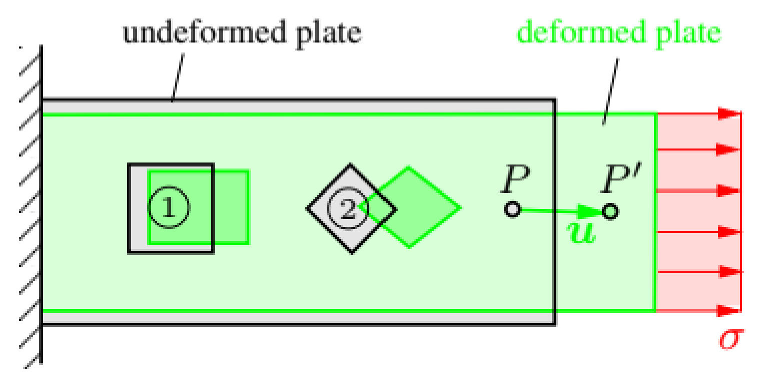
3.1 State of Strain



The uniaxial deformation of a tension bar, was described in first lecture by two kinematic quantities: displacement u and strain $\varepsilon = du/dx$.

The length of the elongated element is: $dx + (u + du) - u = dx + du$.

The longitudinal strain is: $\varepsilon(x) = \dfrac{dx + du - du}{dx} = \dfrac{du}{dx}$



Here the deformation of a plane structure will be described

To this end a panel or a sheet is considered where two squares ① and ② are marked such that they are tilted against each other.

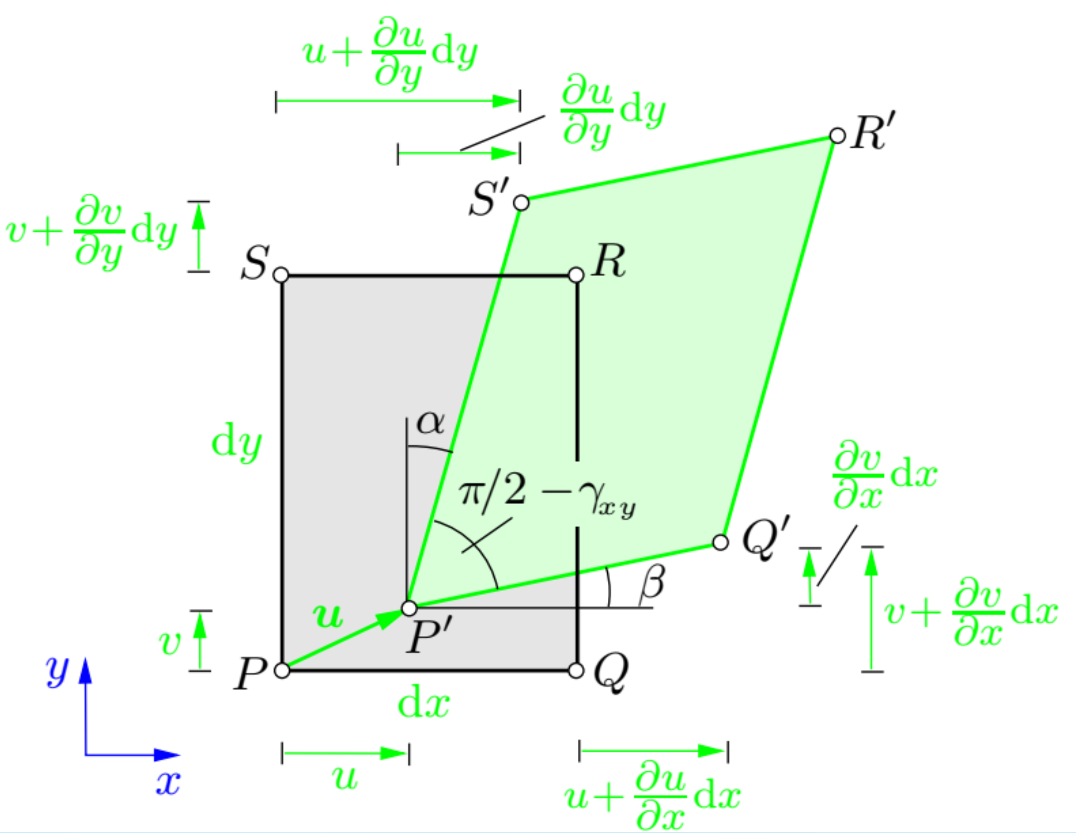
When the panel is loaded by a normal stress σ, point P experiences a displacement $\boldsymbol{u} \equiv \vec{u}$ from its initial position to the new position P' .

Since the *displacement vector* $\vec{u}$ depends on the location, the side lengths (square ①) and the side lengths and angles (square ②), respectively, are changed during the deformation.

In the discussions that follow the changes of side lengths and angles are considered under the assumption of *small* deformations: Changes in lengths and angles are very small.

$$\vec{u}(x,y) = u(x,y)\vec{\imath} + v(x,y)\vec{\jmath},\ \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x} \text{ and } \frac{\partial v}{\partial y}$$ are all much smaller than 1.

Next figure shows an infinitesimal rectangle $PQRS$ of side lengths dx and dy in the undeformed state. During the deformation it is transformed into the new position $P'Q'R'S'$.



Displacement $\vec{u}(x,y)$ of $P(x,y)$ has components: $u(x,y)$ in x direction and $v(x,y)$ in y direction.

displacements of points, adjacent to P, can be described with the help of Taylor expansions. For the functions u and v, depending on x and y, as:

$$u(x+dx,y+dy)=u(x,y)+\frac{\partial u(x,y)}{\partial x}dx+\frac{\partial u(x,y)}{\partial y}dy+\cdots$$
$$v(x+dx,y+dy)=v(x,y)+\frac{\partial v(x,y)}{\partial x}dx+\frac{\partial v(x,y)}{\partial y}dy+\cdots$$

Simplified for $Q(dy=0)$ & $S(dx=0)$.

For point $Q(dy=0)$: $u(x+dx,y)=u(x,y)+\frac{\partial u(x,y)}{\partial x}dx$ & $v(x+dx,y)=v(x,y)+\frac{\partial v(x,y)}{\partial x}dx$

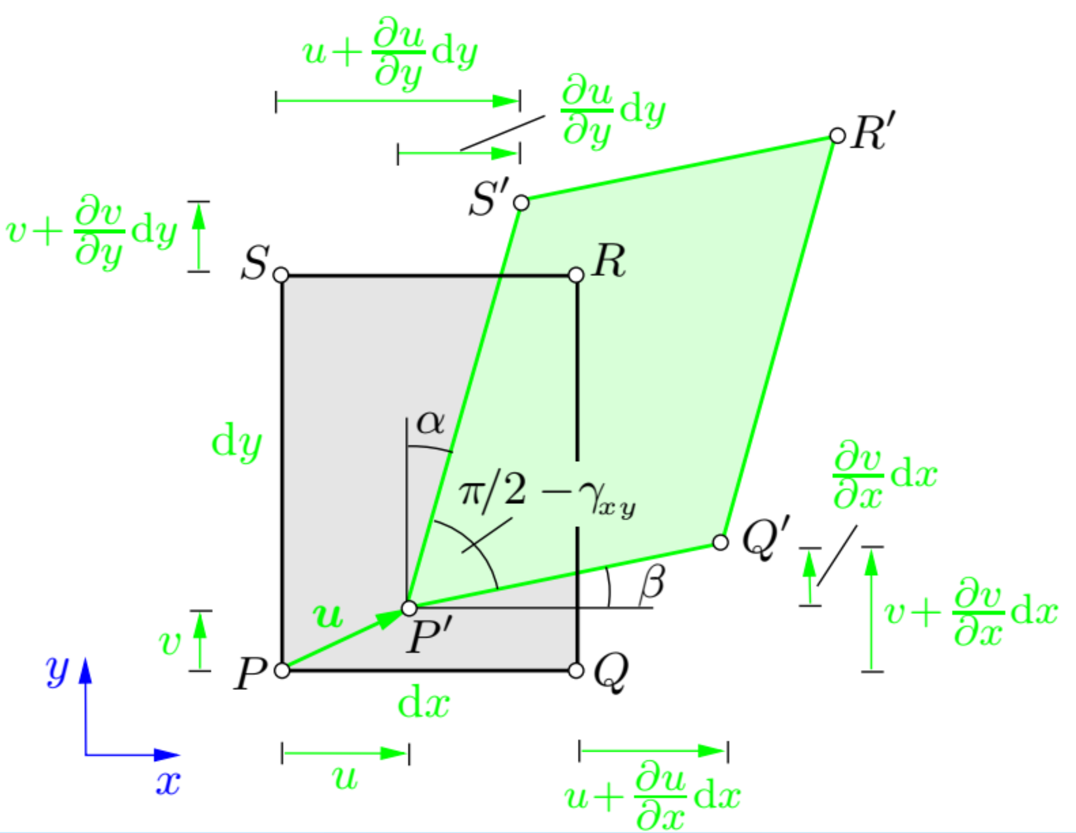
For point $S(dx=0)$: $u(x,y+dy)=u(x,y)+\frac{\partial u(x,y)}{\partial y}dy$ & $v(x,y+dy)=v(x,y)+\frac{\partial v(x,y)}{\partial y}dy$

For point Q: $u(x+dx,y) = u(x,y) + \frac{\partial u(x,y)}{\partial x}dx$

$$v(x+dx,y) = v(x,y) + \frac{\partial v(x,y)}{\partial x}dx$$

For point S: $u(x,y+dy) = u(x,y) + \frac{\partial u(x,y)}{\partial y}dy$

$$v(x,y+dy) = v(x,y) + \frac{\partial v(x,y)}{\partial y}dy$$



During deformation, line PQ is transformed into line $P'Q'$. Since small deformations are assumed ($\beta \ll 1, \cos\beta \approx 1$), the lengths of $P'Q'$ and of its projection on x are approximately the same. So

$$\left|\overrightarrow{P'Q'}\right| \approx \left(x + dx + u + \frac{\partial u}{\partial x}dx\right) - (x+u) = dx + \frac{\partial u}{\partial x}dx$$

Then the *normal strain* ε_x in the *x*-direction defined as the ratio of length increment to initial length, is

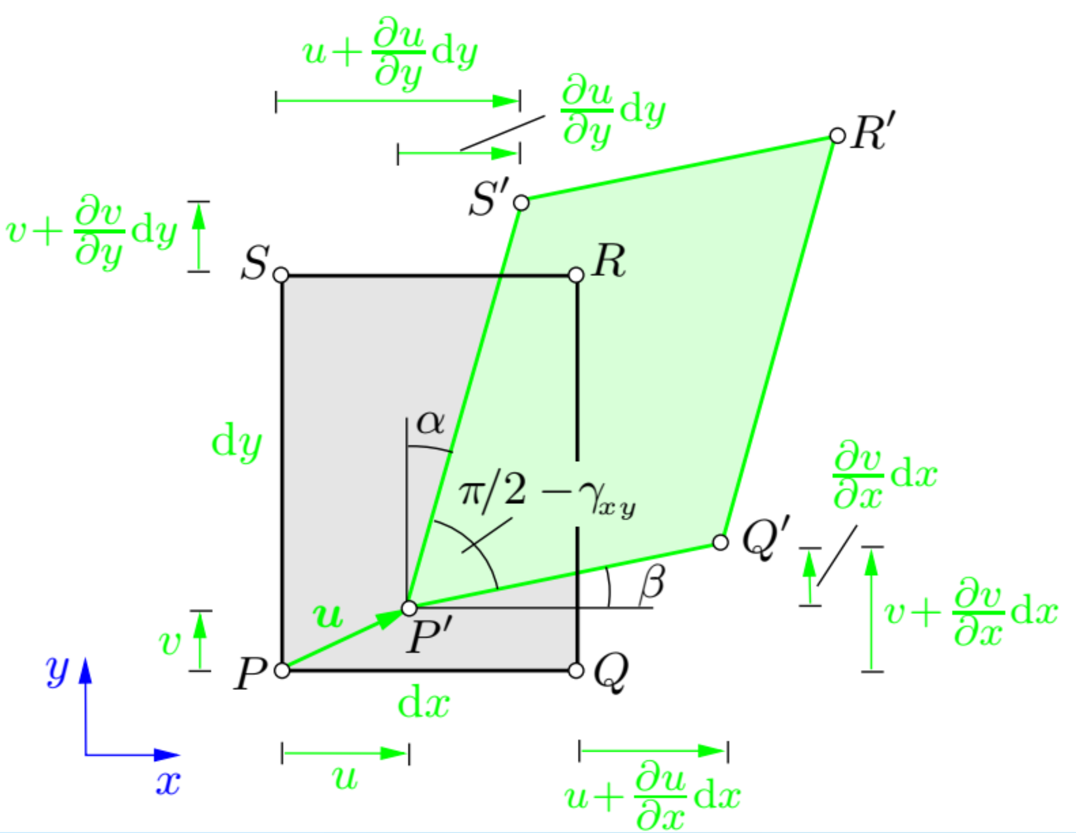
$$\varepsilon_x = \frac{\left|\overrightarrow{P'Q'}\right| - \left|\overrightarrow{PQ}\right|}{\left|\overrightarrow{PQ}\right|} = \frac{dx + \frac{\partial u}{\partial x}dx - dx}{dx} \quad \Rightarrow \quad \varepsilon_x = \frac{\partial u}{\partial x}$$

For point Q: $u(x+dx,y) = u(x,y) + \frac{\partial u(x,y)}{\partial x}dx$

$$v(x+dx,y) = v(x,y) + \frac{\partial v(x,y)}{\partial x}dx$$

For point S: $u(x,y+dy) = u(x,y) + \frac{\partial u(x,y)}{\partial y}dy$

$$v(x,y+dy) = v(x,y) + \frac{\partial v(x,y)}{\partial y}dy$$



During deformation, line PS is transformed into line $P'S'$. Since small deformations are assumed $(\alpha \ll 1, \cos\alpha \approx 1)$, the lengths of $P'S'$ and of its projection on y are approximately the same. So

$$\left|\overrightarrow{P'S'}\right| \approx \left(y + dy + v + \frac{\partial v}{\partial y}dy\right) - (y+v) = dy + \frac{\partial v}{\partial y}dy$$

Then the *normal strain* ε_y in the y-direction defined as the ratio of length increment to initial length, is

$$\varepsilon_y = \frac{\left|\overrightarrow{P'S'}\right| - \left|\overrightarrow{PS}\right|}{\left|\overrightarrow{PS}\right|} = \frac{dy + \frac{\partial v}{\partial y}dy - dy}{dy} \quad \Rightarrow \quad \varepsilon_y = \frac{\partial v}{\partial y}$$

For point Q: $u(x+dx,y) = u(x,y) + \frac{\partial u(x,y)}{\partial x} dx$

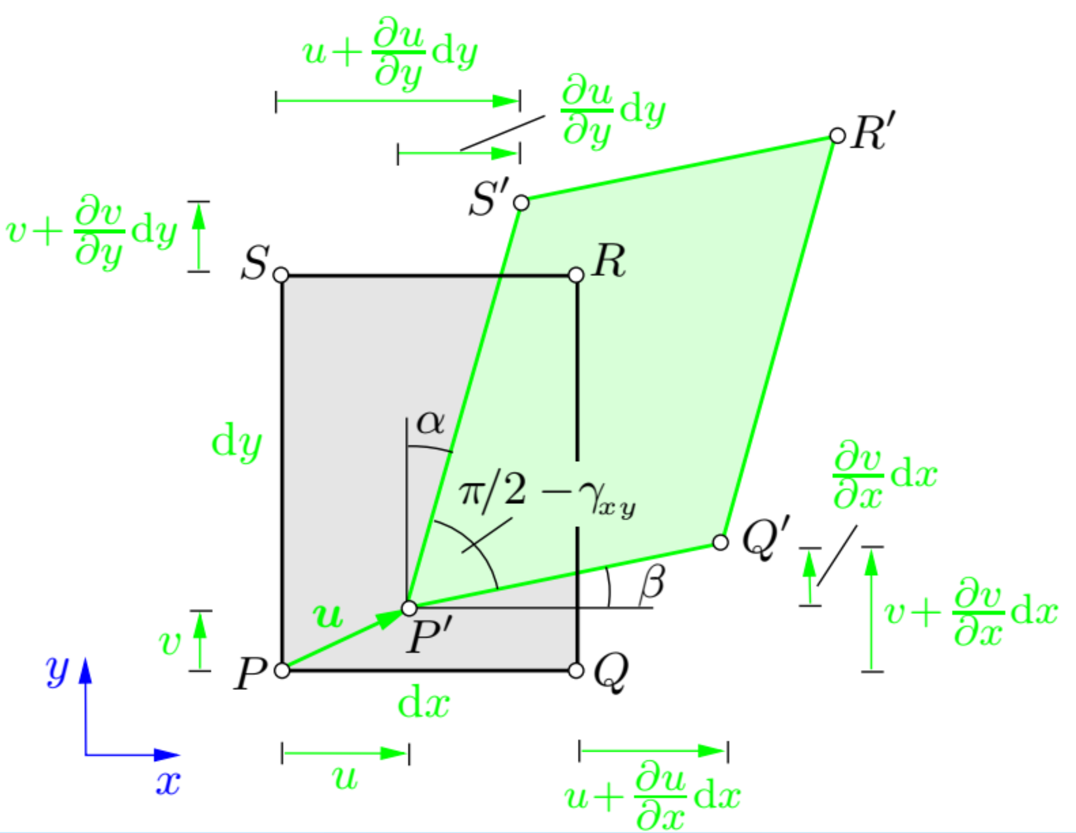
$$v(x+dx,y) = v(x,y) + \frac{\partial v(x,y)}{\partial x} dx$$

For point S: $u(x,y+dy) = u(x,y) + \frac{\partial u(x,y)}{\partial y} dy$

$$v(x,y+dy) = v(x,y) + \frac{\partial v(x,y)}{\partial y} dy$$

During deformation, initially right angle $\widehat{QPS} = \frac{\pi}{2}$ is transformed into angle $\widehat{Q'P'S'} = \frac{\pi}{2} - \gamma_{xy}$.

For assumed small deformations $(\alpha, \beta \ll 1)$, the *shear strain* (*angular strain*) is given as:



$$\gamma_{xy} = \alpha + \beta \approx \tan\alpha + \tan\beta = \frac{u + \frac{\partial u}{\partial y} dy - u}{dy + \varepsilon_y dy} + \frac{v + \frac{\partial v}{\partial x} dx - v}{dx + \varepsilon_x dx} = \frac{\frac{\partial u}{\partial y}}{1 + \varepsilon_y} + \frac{\frac{\partial v}{\partial x}}{1 + \varepsilon_x}$$

Again for small deformation assumption $(\varepsilon_x,\ \varepsilon_y \ll 1)$, So the last expression is approximated to:

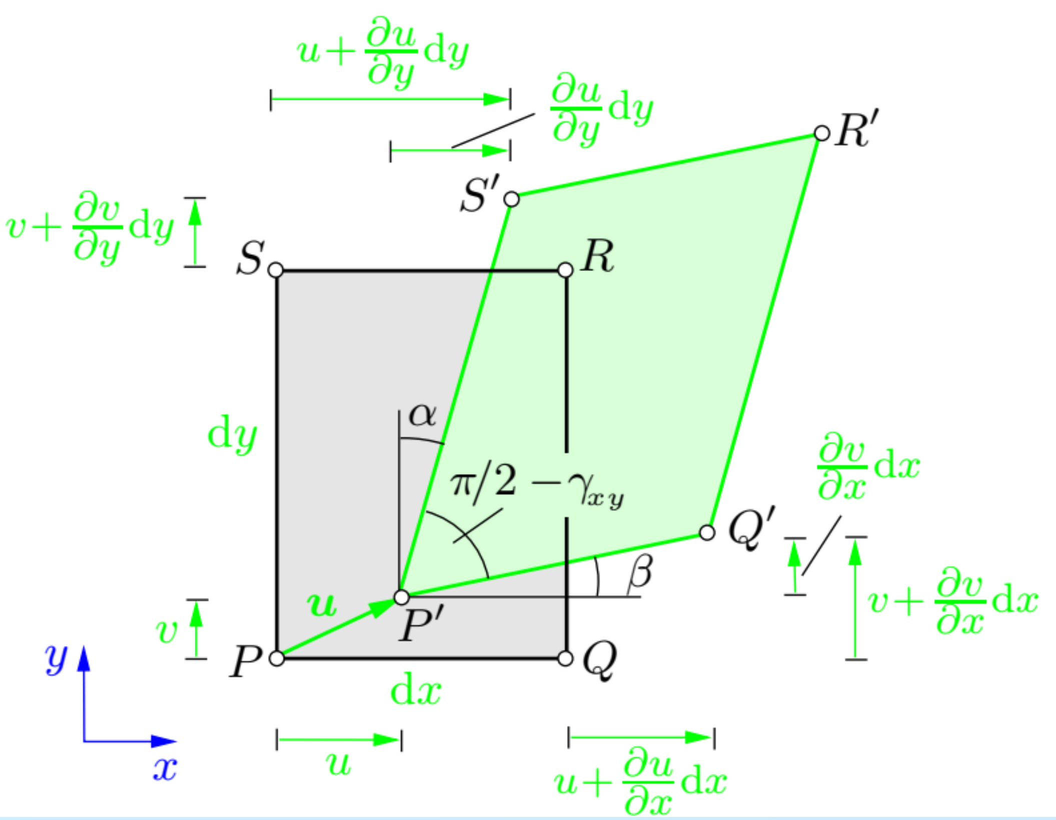
$$\gamma_{xy} \approx \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Conclusion

Displacement of $P(x, y)$ has two components:

$u(x, y)$ in x direction and $v(x, y)$ in y direction.

These two displacements vary with point P, so they are functions of x and y.



Consequently they have the following four 1st order partial derivatives with respect to variables x & y:

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x} \;\&\; \frac{\partial v}{\partial y}$$

Which are generally functions of the variables x & y.

They define the state of strain (deformation) at any point in the planar body by the three relations:

x- normal strain

$$\varepsilon_x = \frac{\partial u}{\partial x}$$

shear strain

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

y- normal strain

$$\varepsilon_y = \frac{\partial v}{\partial y}$$

These are the kinematic equations relating state of displacements to the state of strains.

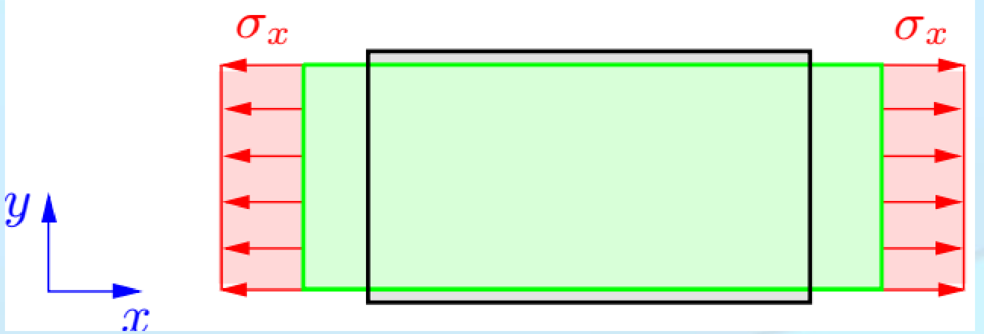
3.2 Hooke's Law

The strains in a structural member depend on the external loading and therefore on the stresses. Stresses and strains are connected by Hooke's law. In the uniaxial case it takes the form $\sigma = E\ \varepsilon$ where E is Young's modulus. Now Hooke's law will be formulated for the planar states of stress and strain.

The formulation will be restricted to materials which are *homogeneous* and *isotropic*.

A *homogeneous* material has the same properties at each material point. For an *isotropic* material these properties are independent of the direction. An example of an anisotropic material is wood.

To derive Hooke's law for planar case, consider a rectangular domain which is cut out from a planar body and which is loaded only by the normal stress σ_x. Then, according to Hooke's law in the uniaxial case: $\sigma_x = E\ \varepsilon_x$ or $\varepsilon_x = \dfrac{\sigma_x}{E}$.



Experiments show that the tensile stress σ_x causes not only an increase of length but also a reduction of width of the rectangle. Thus, also a strain ε_y in y-direction appears. This phenomenon is called *lateral contraction* or *Poisson effect* (Simeon Denis Poisson, 1781–1840). The lateral strain ε_y is proportional to the axial strain ε_x and can be written as : $\varepsilon_y = -\nu\varepsilon_x$, ν is a material property called Poisson ratio.

Poisson's ratio is a material constant which is determined from experiments. Most metallic materials exhibit values about $\nu \approx 0.3$. Generally it can be shown that Poisson's ratio must be in the range of $0 \le \nu \le 1/2$.

As just discussed, the stress σ_x induces the strains $\varepsilon_x = \frac{\sigma_x}{E}$ and $\varepsilon_y = -\nu\varepsilon_x = \frac{-\nu\sigma_x}{E}$.

Analogously, a stress σ_y induces the strains $\varepsilon_y = \frac{\sigma_y}{E}$ and $\varepsilon_x = -\nu\varepsilon_y = \frac{-\nu\sigma_y}{E}$.

When both stresses, σ_x as well as σ_y, are acting, the total strains are obtained by superposition:

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \text{ and } \varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$$

If the domain which cut out from a planar body is loaded solely by shear stresses τ_{xy} (pure shear), a linear relationship between the angle change γ_{xy} and τ_{xy} is experimentally observed:

$$\tau_{xy} = G\gamma_{xy}$$

The constant G is called *shear modulus*. It is a material parameter which can be experimentally determined in a shear test or a torsion test. The shear modulus G has the same dimension as Young's modulus E, i.e. force/area, and it is usually expressed in MPa.

It can be shown that there exist only *two independent* material constants for isotropic, linear elastic materials. The following relationship holds between the three constants E, G and ν:

$$G = \frac{E}{2(1+\nu)}$$

Conclusion If the state of stress in a planar homogeneous, isotropic body is described by

$$\sigma_x(x,y), \sigma_y(x,y)\ and\ \tau_{xy}(x,y)$$

Then the state of strain described by

$$\varepsilon_x(x,y), \varepsilon_y(x,y)\ and\ \gamma_{xy}(x,y)$$

Is related at every point to the state of stress via the materials properties $E, G, \nu,$ by Hook's law:

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y)$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy} = \frac{2(1+\nu)}{E}\tau_{xy}$$

Hook's law can also be written in the inverse form

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y)$$

$$\sigma_y = \frac{E}{1-\nu^2}(\nu\varepsilon_x + \varepsilon_y)$$

$$\tau_{xy} = G\gamma_{xy} = \frac{E}{2(1+\nu)E}\gamma_{xy}$$