Portfolio Management

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Characteristics of the single-index model

- Define the beta on a portfolio β_P as a weighted average of the individual B_is on each stock in the portfolio, where the weights are the fraction of the portfolio invested in each stock. Then:

$$\beta_P = \sum_{i=1}^{N} X_i B_i$$

- Similarly, define the alpha on the portfolio α_P as:

$$\alpha_P = \sum_{i=1}^{N} X_i \alpha_i$$

- Then Equation $\bar{R}_P = \sum_{i=1}^{N} X_i \alpha_i + \sum_{i=1}^{N} X_i \beta_i \bar{R}_m$ can be written as:

$$\bar{R}_P = \alpha_P + \beta_P \bar{R}_m$$

- If the portfolio P is taken to be the market portfolio (all stocks held in the same proportions as they were in constructing R_m), then the expected return on P must be $\bar{R}_m$.

- From the previous equation the only values of β_P and α_P that guarantee $\bar{R}_P = \bar{R}_m$ for any choice of $\bar{R}_m$ are α_P equal to 0 and β_P equal to 1. Thus the beta on the market is 1 and stocks are thought of as being more or less risky than the market, according to whether their beta is larger or smaller than 1.

- The variance of a portfolio of stocks is given by:

$$\sigma_P^2 = \sum_{i=1}^{N} X_i^2 \beta_i^2 \sigma_m^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} X_i X_j \beta_i \beta_j \sigma_m^2 + \sum_{i=1}^{N} X_i^2 \sigma_{ei}^2$$

- In the double summation $i \neq j$, if $i = j$, then the terms would be $X_i\ X_i\ \beta_i^2 \sigma_m^2$. But these are exactly the terms in the first summation. Thus the variance on the portfolio can be written as:

$$\sigma_P^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} X_i X_j \beta_i \beta_j \sigma_m^2 + \sum_{i=1}^{N} X_i^2 \sigma_{ei}^2$$

- Or by rearranging terms,

$$\sigma_P^2 = \left(\sum_{i=1}^{N} X_i\beta_i\right)\left(\sum_{j=1}^{N} X_j\beta_j\right)\sigma_m^2 + \sum_{i=1}^{N} X_i^2\sigma_{ei}^2$$

- Thus the risk of the investor's portfolio could be represented as:

$$\sigma_P^2 = \beta_P^2\sigma_m^2 + \sum_{i=1}^{N} X_i^2\sigma_{ei}^2$$

- As the number of stocks in the portfolio increases, the importance of the residual risk diminishes drastically.

- The risk that is not eliminated as we hold larger and larger portfolios is the risk associated with the term β_P. If we assume that residual risk approaches zero, the risk of the portfolio approaches:

$$\sigma_P = (\beta_P^2 \sigma_m^2)^{1/2} = \beta_P \sigma_m = \sigma_m \left[\sum_{i=1}^{N} X_i B_i\right]$$

Estimating historical betas

- We represented the return on a stock as:

$$R_i = \alpha_i + \beta_i R_m + e_i$$

- This equation is expected to hold at each moment in time, although the value of α_i, β_i, or σ_{ei}^2 might differ over time. When looking at historical data, one cannot directly observe α_i, β_i, or σ_{ei}^2.

- Rather, one observes the past returns on the security and the market. If α_i, β_i, and σ_{ei}^2 are assumed to be constant through time, then the same equation is expected to hold at each point in time. In this case, a straightforward procedure exists for estimating α_i, β_i, and σ_{ei}^2.

- To estimate the beta for a firm for the period from $t = 1$ to $t = 60$ via regression analysis, use:

$$\beta_i = \frac{\sigma_{im}}{\sigma_m^2} = \frac{\sum_{t=1}^{60}[(R_{it} - \bar{R}_{it})(R_{mt} - \bar{R}_{mt})]}{\sum_{t=1}^{60}(R_{mt} - \bar{R}_{mt})^2}$$

- And to estimate alpha, use:

$$\alpha_i = \bar{R}_{it} - \beta_i \bar{R}_{mt}$$

- To learn how this works on a simple example, let us return to Table 6.1. We used the data in Table 6.1 to show how beta interacted with returns. But now assume that all you observed were columns 1 and 2 or the return on the stock and the return on the market.
- Table 6.1 Decomposition of Returns for the Single-Index Model:

	1	2	3	4	5	6
Month	Return on Stock	Return on Market	R_i	$= \alpha_i$	$+ \beta_i R_m$	$+ e_i$
1	10	4	10	2	6	2
2	3	2	3	2	3	2
3	15	8	15	2	12	1
4	9	6	9	2	9	2
5	3	0	3	2	0	1
Total	40	20	40	10	30	0
Average	8	4	8	2	6	0
Variance	20.8	8	20.8	0	18	2.8

- To estimate beta, we need to estimate the covariance between the stock and the market. The average return on the stock was $40/5 = 8$, whereas on the market it was $20/5 = 4$. The beta value for the stock is the covariance of the stock with the market divided by the variance of the market, or:

$$\beta_i = \frac{\left[\sum_{t=1}^{5}[(R_i - \bar{R}_i)(R_m - \bar{R}_m)]\right]/5}{\left[\sum_{t=1}^{5}(R_m - \bar{R}_m)^2\right]/5}$$

- The covariance is found as follows:

Month	Stock Return Minus Mean		Market Return Minus Mean		Value
1	(10 - 8)	×	(4 - 4)	=	0
2	(3 - 8)	×	(2 - 4)	=	10
3	(15 - 8)	×	(8 - 4)	=	28
4	(9 - 8)	×	(6 - 4)	=	2
5	(3 - 8)	×	(0 - 4)	=	20
			Total		60

- The covariance is $60/5 = 12$. The variance of the market return is the average of the sum of squared deviation from the mean:

$$\sigma_m^2 = [(4-4)^2 + (2-4)^2 + (8-4)^2 + (6-4)^2 + (0-4)^2] / 5 = 8$$

- Thus beta $= 12/8 = 1.5$

- Alpha can be computed by taking the difference between the average security return and beta times the average return on the market:

$$\alpha_i = 8 - (1.5)4 = 2$$