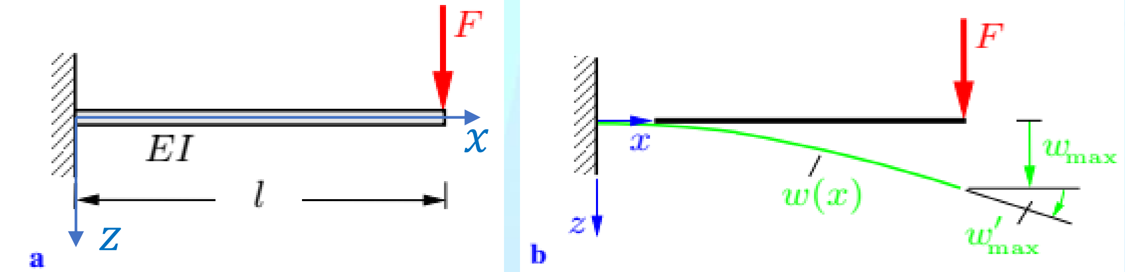
4.4.2 Beams with one Interval of Integration

The following examples show how the differential equations are used to obtain the deflection curve.

This section is concerned with beams where the integration can be performed in *one interval*, i.e., we assume that each of the quantities $q(x), V(x), M(x), w'(x)$ and $w(x)$ is given by *one* function for the entire length of the beam.



Ex.1 A cantilever beam (flexural rigidity EI) subjected to a concentrated force F (Fig.a). Since the system is statically determinate, the bending moment can be calculated from the equilibrium conditions.

With the coordinate system shown in Fig. a, $M = -F(l - x)$. Introducing into $M(x) = -EIw''$ to get

$$EIw'' = F(l - x). \quad \text{integrating twice yields}$$

$$EIw' = F\left(lx - \frac{x^2}{2}\right) + C_1$$

$$EIw = F\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) + C_1 x + C_2$$

The geometrical boundary conditions: $w(0) = 0, w'(0) = 0$

lead to the constants of integration: $C_1 = 0, C_2 = 0$.

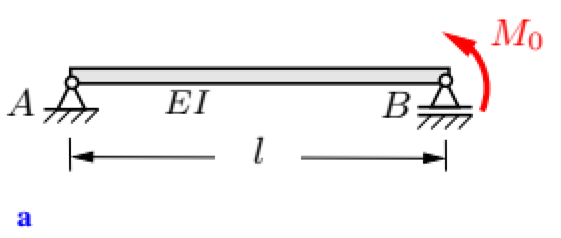
Hence, the slope and the deflection are obtained as

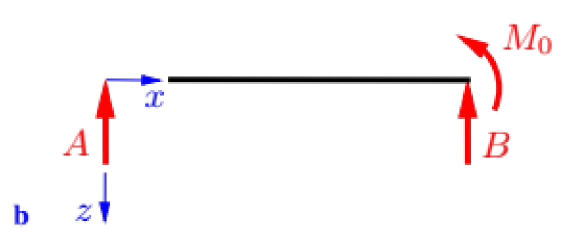
$$w' = \frac{Fl^2}{2EI}\left(\frac{2x}{l} - \frac{x^2}{l^2}\right), \quad w = \frac{Fl^3}{6EI}\left(3\frac{x^2}{l^2} - \frac{x^3}{l^3}\right)$$

maximum slope & maximum deflection (at $x = l$, Fig.b) are

$$w'_{max} = \frac{Fl^2}{2EI} \qquad w_{max} = \frac{Fl^3}{3EI}$$

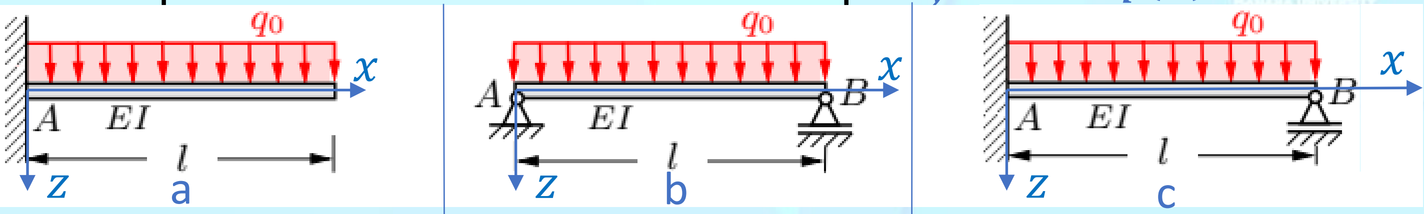
Ex.2 A simply supported beam (bending stiffness EI) is loaded by a moment M_0 (Fig. a). Determine the location and magnitude of the maximum deflection.





Ex.3 Consider three beams (bending stiffness EI) subjected to a constant line load q_0. The supports in the three cases are different; the systems in the (Figs. a & b) are statically determinate, the system in (Fig. c) is statically indeterminate.

Since in (c) the bending moment can not be calculated from Eqm. conditions, the 4th order Diff. Eq. $EI_y w^{IV} = q(x)$ will be used in all three cases. A coordinate system is introduced, integration is done 4 times starting from :



$$EI_y w^{IV} = q_0$$

$$EIw''' = -V = q_0 x + C_1,$$

$$EIw'' = -M = \frac{1}{2} q_0 x^2 + C_1 x + C_2,$$

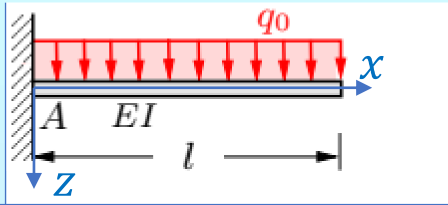
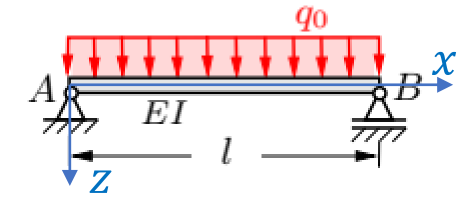
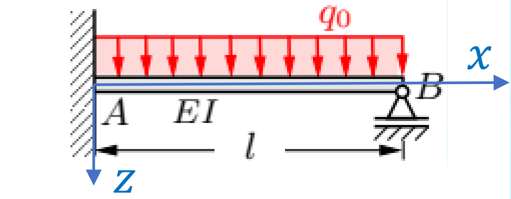
$$EIw' = \frac{1}{6} q_0 x^3 + \frac{1}{2} C_1 x^2 + C_2 x + C_3,$$

$$EIw = \frac{1}{24} q_0 x^4 + \frac{1}{6} C_1 x^3 + \frac{1}{2} C_2 x^2 + C_3 x + C_4,$$

These equations are independent of the supports & therefore are valid for all cases. Different boundary conditions lead to different constants of integration:

a	b	c
$w'(0) = 0 \rightarrow C_3 = 0$	$M(0) = 0 \rightarrow C_2 = 0$	$w'(0) = 0 \rightarrow C_3 = 0$
$w(0) = 0 \rightarrow C_4 = 0$	$w(0) = 0 \rightarrow C_4 = 0$	$w(0) = 0 \rightarrow C_4 = 0$
$V(l) = 0 \rightarrow C_1 = -q_0 l$	$M(l) = 0 \rightarrow C_1 = -\frac{1}{2} q_0 l$	$M(l) = 0 \rightarrow$ $\frac{1}{2} q_0 l^2 + C_1 l + C_2 = 0$
$M(l) = 0 \rightarrow C_2 = \frac{1}{2} q_0 l^2$	$w(l) = 0 \rightarrow C_3 = \frac{1}{24} q_0 l^3$	$w(l) = 0 \rightarrow$ $\frac{1}{24} q_0 l^4 + \frac{1}{6} C_1 l^3 + \frac{1}{2} C_2 l^2 = 0$
		$C_1 = -\frac{5}{8} q_0 l$ & $C_2 = \frac{1}{8} q_0 l^2$

	a	b	c
The deflection function is given by: $w(x) = \frac{q_0 l^4}{24EI} \times$	$\left[\left(\frac{x}{l}\right)^4 - 4\left(\frac{x}{l}\right)^3 + 6\left(\frac{x}{l}\right)^2\right]$	$\left[\left(\frac{x}{l}\right)^4 - 2\left(\frac{x}{l}\right)^3 + \left(\frac{x}{l}\right)\right]$	$\left[\left(\frac{x}{l}\right)^4 - \frac{5}{2}\left(\frac{x}{l}\right)^3 + \frac{3}{2}\left(\frac{x}{l}\right)^2\right]$
Maximum deflection is given by: $w_{max} =$	$\frac{q_0 l^4}{8EI}$	$\frac{5 q_0 l^4}{348EI}$	Indeterminate! no more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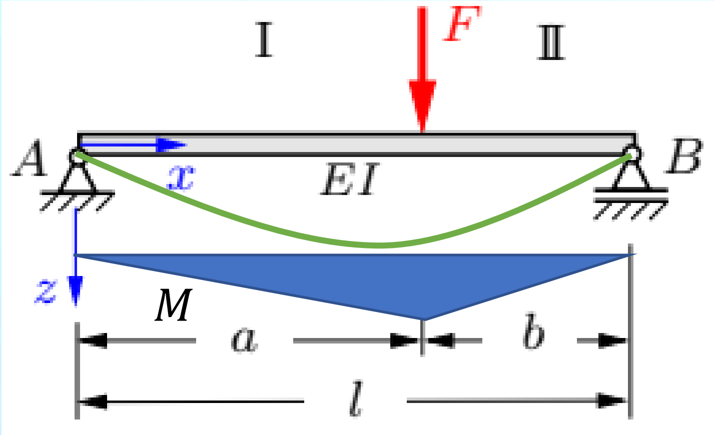
Maximum deflection is given by: $w_{max} =$	$\frac{q_0 l^4}{8EI}$	$\frac{5q_0 l^4}{348EI}$	Indeterminate! no more
$w'(x) = \frac{q_0 l^3}{6EI} \times$	$\left[\left(\frac{x}{l}\right)^3 - 3\left(\frac{x}{l}\right)^2 + 3\left(\frac{x}{l}\right)\right]$	$\left[\left(\frac{x}{l}\right)^3 - \frac{3}{2}\left(\frac{x}{l}\right)^2 + \frac{1}{4}\right]$	$\left[\left(\frac{x}{l}\right)^3 - \frac{15}{8}\left(\frac{x}{l}\right)^2 + \frac{3}{4}\left(\frac{x}{l}\right)\right]$
$M = -EIw'' = -\frac{q_0 l^2}{2} \times$	$\left[\left(\frac{x}{l}\right)^2 - 2\left(\frac{x}{l}\right) + 1\right]$	$\left[\left(\frac{x}{l}\right)^2 - \left(\frac{x}{l}\right)\right]$	$\left[\left(\frac{x}{l}\right)^2 - \frac{5}{4}\left(\frac{x}{l}\right) + \frac{1}{4}\right]$
$V = -EIw''' = -q_0 l \times$	$\left[\left(\frac{x}{l}\right) - 1\right]$	$\left[\left(\frac{x}{l}\right) - \frac{1}{2}\right]$	$\left[\left(\frac{x}{l}\right) - \frac{5}{8}\right]$
			$A_z = V(0) = \frac{5}{8} q_0 l \ (\uparrow)$ $B_z = -V(l) = \frac{3}{8} q_0 l \ (\uparrow)$ $A_M = -M(0) = \frac{q_0 l^2}{8} \ (\curvearrowleft)$

4.4.3 Beams with several Regions of Integration

Frequently, one or several of the quantities $q, V\ , M, w'\ , w$ or the flexural rigidity EI are given through different functions of x in different portions of the beam. In this case the beam must be divided into several regions and the integration has to be performed separately in each of these regions.

The constants of integration can be calculated from both, boundary conditions and *matching conditions*, also called *continuity conditions.* The treatment of such problems will be illustrated by means of the following example.

Ex. 4 A simply supported beam is subjected to a concentrated force F at $x = a$ (Fig.). Determine the deflection w at $x = a$.



$$M(x) = \begin{cases} F\dfrac{b}{l}x: & \text{In Region I} \\ F\dfrac{a}{l}(l-x) & \text{In Region II} \end{cases}$$

Boundary Conditions: $w(0) = 0, w(l) = 0$

Continuity Conditions of Elastic line :

$w'(a_l) = w'(a_r)$ & $w(a_l) = w(a_r)$

In Region I: $0 \le x \le a$ | $EIw'' = -M = -F\dfrac{b}{l}x$ | $EIw' = -F\dfrac{b}{l}\dfrac{x^2}{2} + C_1$ | $EIw = -F\dfrac{b}{l}\dfrac{x^3}{6} + C_1x + C_2$

In Region II: $a \le x \le l$ | $EIw'' = -M = -F\dfrac{a}{l}(l-x)$ | $EIw' = F\dfrac{a}{l}\dfrac{(l-x)^2}{2} + C_3$ | $EIw = -F\dfrac{a}{l}\dfrac{(l-x)^3}{6} - C_3(l-x) + C_4$

Boundary Conditions give: $w(0) = 0 \Rightarrow C_2 = 0$ & $w(l) = 0 \Rightarrow C_4 = 0$

Continuity Conditions give:

$$\begin{cases} w'(a_l) = w'(a_r) \Rightarrow -F\dfrac{b}{l}\dfrac{a^2}{2} + C_1 = F\dfrac{a}{l}\dfrac{b^2}{2} + C_3 \Rightarrow C_1 - C_3 = \dfrac{Fab}{2} \\ w(a_l) = w(a_r) \Rightarrow -F\dfrac{b}{l}\dfrac{a^3}{6} + C_1a = -F\dfrac{a}{l}\dfrac{b^3}{6} - C_3b \Rightarrow aC_1 + bC_3 = \dfrac{Fab}{2}\left(\dfrac{a-b}{3}\right) \end{cases}$$

$$C_1 = \frac{Fab}{6l}(a+2b), \quad C_3 = -\frac{Fab}{6l}(2a+b)$$

In Region I: $w = \dfrac{Fbl^2}{6EI}\left\{\left[1-\left(\dfrac{b}{l}\right)^2\right]\left(\dfrac{x}{l}\right) - \left(\dfrac{x}{l}\right)^3\right\}$ & $w' = \dfrac{Fbl}{6EI}\left\{\left[1-\left(\dfrac{b}{l}\right)^2\right] - 3\left(\dfrac{x}{l}\right)^2\right\}$

In Region II: $w = \dfrac{Fal^2}{6EI}\left\{\left[1-\left(\dfrac{a}{l}\right)^2\right]\left(\dfrac{l-x}{l}\right) - \left(\dfrac{l-x}{l}\right)^3\right\}$ & $w' = \dfrac{Fal}{6EI}\left\{\left[1-\left(\dfrac{a}{l}\right)^2\right] - 3\left(\dfrac{l-x}{l}\right)^2\right\}$

$$w(a) = \frac{Fa^2b^2}{3EIl}$$

$$w'(a) = \frac{Fab(b-a)}{3EIl}$$

a) Determine the deflection function of the cantilever beam shown in the figure

b) Find the deflection at the free end

c) Find the slope at the free end

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