Calculus 2

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Calculus 2

Lecture 2

Exercices 2
Improper Integrals

The integral $\int_{-1}^{1} \frac{dx}{x^3}$ diverges.

Trying to compute that integral by the Fundamental Theorem of Calculus, one gets

$$\int_{-1}^{1} \frac{dx}{x^3} = \frac{1}{-2} x^{-2} \Big|_{-1}^{1} = -\frac{1}{2} - \left(-\frac{1}{2}\right) = 0.$$

This is an incorrect computation.

For what values of p is the integral $\int_1^{+\infty} \frac{dx}{x^p}$ convergent?

Solution

$$\int_1^{\infty} \frac{dx}{x^P} \quad P > 0$$

(P is a constant.)

$$\int_1^{\infty} x^{-P} \, dx$$

$$\lim_{b\to\infty} \int_1^{b} x^{-P} \, dx$$

$$\lim_{b\to\infty} \frac{1}{-P+1} x^{-P+1} \Big|_1^b$$

$$\lim_{b\to\infty} \frac{b^{-P+1}}{-P+1} - \frac{1^{-P+1}}{-P+1}$$

What happens here?

If $P < 1$ then b^{-P+1} gets bigger and bigger as , $b \to \infty$ therefore the integral <u>diverges</u>.

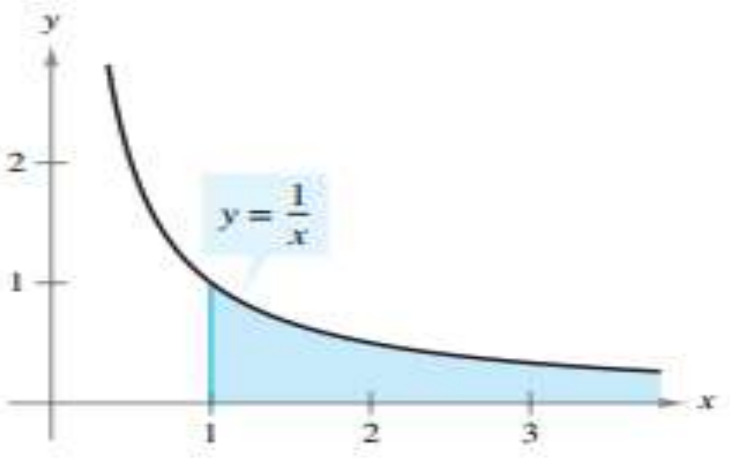
If $P > 1$ then *b* has a negative exponent and , $b^{-P+1} \to 0$ therefore the integral <u>converges</u>.

When $p \neq 1,$

$$\int_1^{+\infty} \frac{dx}{x^p} = \lim_{b \to \infty} \frac{b^{-P+1}}{-P+1} - \frac{1^{-P+1}}{-P+1} = \begin{cases} +\infty & p < 1 \\ \dfrac{1}{p-1} & p > 1 \end{cases}.$$

When $p = 1,$

$$\int_1^{+\infty} \frac{dx}{x} = \lim_{b \to \infty} \ln x \Big|_1^b = \lim_{b \to \infty} (\ln b - \ln 1) = +\infty$$



Diverges (infinite area)

Thus, the integral diverges.

Evaluate $\lim_{b\to 1^-}\int_0^b \sqrt{\frac{1+x}{1-x}}\,dx$

Solution $\int \sqrt{\frac{1+x}{1-x}}\,\frac{\sqrt{1+x}}{\sqrt{1+x}}\,dx$

Rationalize the numerator.

$$\int \frac{1+x}{\sqrt{1-x^2}}\,dx$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx+\int \frac{x}{\sqrt{1-x^2}}\,dx$$

$$\sin^{-1}x-\frac{1}{2}\int u^{-\frac{1}{2}}du$$

$$u=1-x^2$$

$$du=-2x\,dx$$

$$-\frac{1}{2}du=x\,dx$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx + \int \frac{x}{\sqrt{1-x^2}}\,dx$$

$$u = 1 - x^2$$

$$du = -2x\,dx$$

$$-\frac{1}{2}du = x\,dx$$

$$\sin^{-1} x - \frac{1}{2}\int u^{-\frac{1}{2}} du$$

$$\sin^{-1} x - u^{\frac{1}{2}}$$

This integral converges because it approaches a solution.

$$\lim_{b\to 1^-} \sin^{-1} x - \sqrt{1-x^2}\Big|_0^b$$

$$\lim_{b\to 1^-} \left(\sin^{-1} b - \sqrt{1-b^2}\right) - \left(\sin^{-1} 0 - \sqrt{1}\right) = \frac{\pi}{2} + 1$$

($\sin^{-1} b \to \frac{\pi}{2}$, $\sqrt{1-b^2} \to 0$, $\sin^{-1} 0 \to 0$)

Evaluating an integral on $(-\infty, \infty)$

$$\int_{-\infty}^{\infty} e^{-|x|}\, dx$$

Solution

$$\int_{-\infty}^{\infty} e^{-|x|}\,dx = \int_{-\infty}^{0} e^{x}\,dx + \int_{0}^{\infty} e^{-x}\,dx$$

$$= \lim_{a\to-\infty} \int_{a}^{0} e^{x}\,dx + \lim_{a\to\infty} \int_{0}^{a} e^{-x}\,dx$$

$$= \lim_{a\to-\infty} \left(1 - e^{a}\right) + \lim_{a\to\infty} \left(-e^{-a} + 1\right)$$

$$= 1 + 1 = 2$$

evaluate the integral or state that it diverges.

$$\int_{0}^{2} \frac{dx}{1-x^2}$$

Solution

$$\int_0^2 \frac{dx}{1-x^2} \text{ has an infinite discontinuity at } x = 1$$

$$= \int_0^1 \frac{dx}{1-x^2} + \int_1^2 \frac{dx}{1-x^2}$$

$$= \lim_{c \to 1-} \int_0^c \frac{dx}{1-x^2} + \lim_{c \to 1+} \int_c^2 \frac{dx}{1-x^2}$$

$$= \lim_{c \to 1-} \left(-\frac{1}{2} \ln \left(\frac{x - 1}{x + 1} \right) \right)_0^c + \lim_{c \to 1+} \left(-\frac{1}{2} \ln \left(\frac{x - 1}{x + 1} \right) \right)_c^2$$

$$= \infty \text{ , the integral diverges}$$

Use the comparison test to determine if the following integral are convergent or divergent

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{x}} dx \, .$$

(c) $\int_0^{\pi/2} \frac{\cos x}{\sqrt{x}}\,dx$ converges because

$$0 \le \frac{\cos x}{\sqrt{x}} \le \frac{1}{\sqrt{x}} \quad \text{on} \quad \left[0, \frac{\pi}{2}\right].$$

$0 \le \cos x \le 1$ on $\left[0, \frac{\pi}{2}\right]$

and

$$\int_0^{\pi/2} \frac{dx}{\sqrt{x}} = \lim_{a \to 0^+} \int_a^{\pi/2} \frac{dx}{\sqrt{x}}$$

$$= \lim_{a \to 0^+} \sqrt{4x}\,\Big]_a^{\pi/2}$$

$2\sqrt{x} = \sqrt{4x}$

$$= \lim_{a \to 0^+} \left(\sqrt{2\pi} - \sqrt{4a}\right) = \sqrt{2\pi}$$

converges.

Evaluate the improper integral or state that it diverges.

$$\int_{1}^{\infty} \frac{dx}{x^{\frac{3}{2}}}$$

$$\int_{1}^{\infty} \frac{dx}{e^{x} + e^{-x}}$$

$$\int_{1}^{\infty} \frac{dx}{x^{4}}$$

Evaluate the improper integral or state that it diverges.

$$\int_1^\infty \frac{1}{x^3}\,dx$$

$$\int_1^\infty \frac{3}{\sqrt[3]{x}}\,dx$$

$$\int_{-\infty}^0 xe^{-4x}\,dx$$

$$\int_0^\infty x^2e^{-x}\,dx$$

$$\int_4^\infty \frac{1}{x(\ln x)^3}\,dx$$

$$\int_0^\infty \frac{e^x}{1+e^x}\,dx$$

$$\int_0^\infty \sin\frac{x}{2}\,dx$$

$$\int_0^\infty e^{-x}\cos x\,dx$$

TABLE 8.1 **Basic integration formulas**

1. $\int k\,dx = kx + C \qquad \text{(any number } k\text{)}$
2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$
3. $\int \frac{dx}{x} = \ln|x| + C$
4. $\int e^x\,dx = e^x + C$
5. $\int a^x\,dx = \frac{a^x}{\ln a} + C \qquad (a > 0,\ a \neq 1)$
6. $\int \sin x\,dx = -\cos x + C$
7. $\int \cos x\,dx = \sin x + C$
8. $\int \sec^2 x\,dx = \tan x + C$
8. $\int \csc^2 x\,dx = -\cot x + C$
10. $\int \sec x \tan x\,dx = \sec x + C$
11. $\int \csc x \cot x\,dx = -\csc x + C$
12. $\int \tan x\,dx = \ln|\sec x| + C$
13. $\int \cot x\,dx = \ln|\sin x| + C$
14. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$
15. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$
16. $\int \sinh x\,dx = \cosh x + C$
17. $\int \cosh x\,dx = \sinh x + C$
18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$
19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
20. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$
21. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C \qquad (a > 0)$
22. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C \qquad (x > a > 0)$

Thank you for your attention