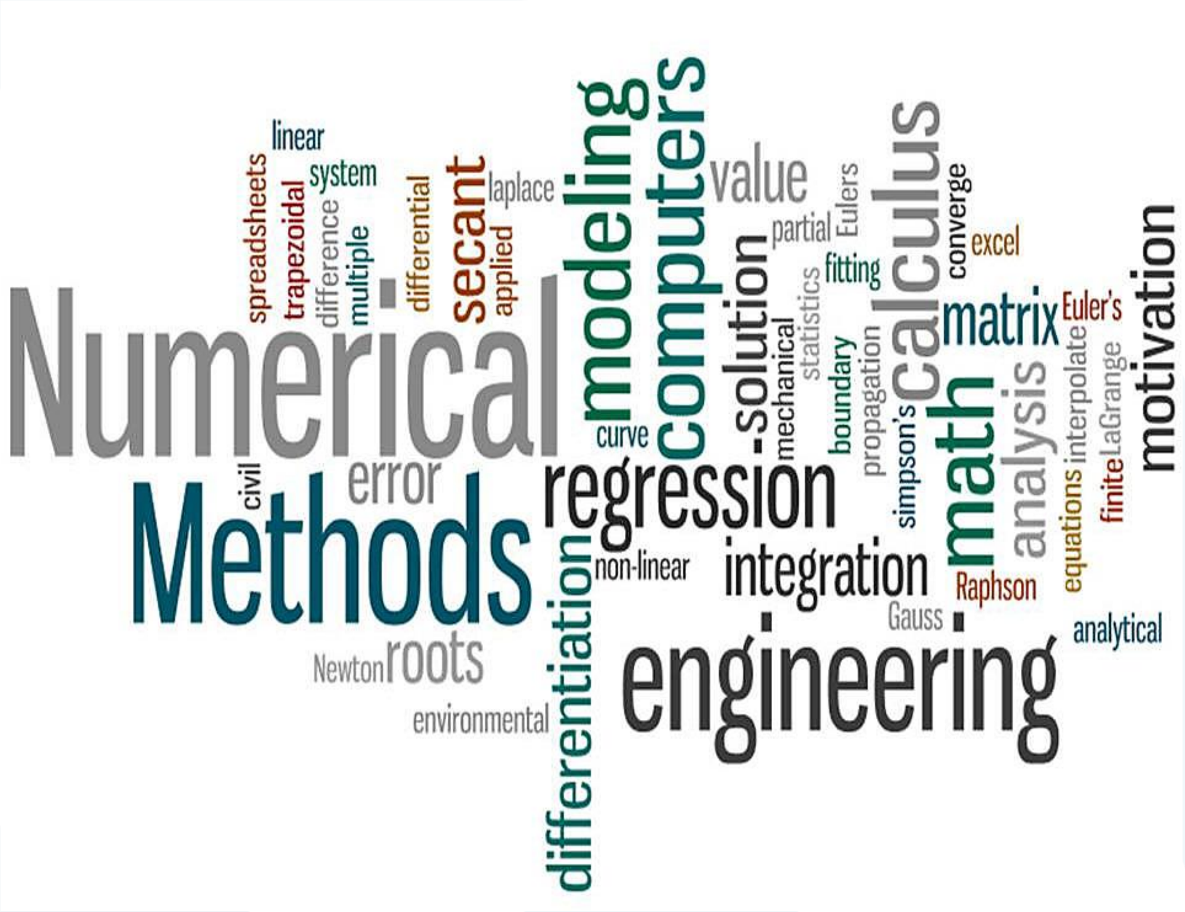
Programming and Numerical Analysis



07/11/2022
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Truncation Errors and Taylor Series

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What is The Error ?
How it is happened ?

Error Definition

Error Definitions

True Value = Approximation + Error

Et = True value – Approximation (+/-)

True Error

$$\text{Relative error} = \frac{\text{true error}}{\text{true value}}$$

$$\text{Percent relative error, } \varepsilon_t = \frac{\text{true error}}{\text{true value}} \times 100\%$$

Error Definition

Example

Problem Statement: Suppose that you have the task of measuring the lengths of a bridge and a rivet and come up with 9999 and 9 cm, respectively. If the true values are 10,000 and 10 cm, respectively, compute (a) the true error and (b) the true percent relative error for each case.

Solution:

(a) The error for measuring the bridge is (True Value = Approximation + Error) : Et = 10,000 − 9999 = 1 cm

and for the rivet it is: Et = 10 − 9 = 1 cm

(b) The percent relative error for the bridge is $\varepsilon_t = \frac{\text{true error}}{\text{true value}} \times 100\%$: $\varepsilon_t = \frac{1}{10{,}000} \times 100\% = 0.01\%$

and for the rivet it is $\varepsilon_t = \frac{1}{10} \times 100\% = 10\%$

Thus, although both measurements have an error of 1 cm, the relative error for the rivet is much greater. We would conclude that we have done an adequate job of measuring the bridge, whereas our estimate for the rivet leaves something to be desired

Error Definition con.

- **For numerical methods, the true value will be known only when we deal with functions that can be solved analytically (simple systems). In real world applications, we usually not know the answer a priori. Then:**

$$\varepsilon_{\mathrm{a}} = \frac{\text{approximate error}}{\text{approximation}} \times 100\%$$

- **Iterative approach, example Newton's method**

$$\underset{(+\,/\,-)}{\varepsilon_{\mathrm{a}}} = \frac{\text{current approximation} - \text{previous approximation}}{\text{current approximation}} \times 100\%$$

Error Definition *con.*

- Use absolute value.
- Computations are repeated until stopping criterion is satisfied.

$$|\varepsilon_a| = \varepsilon_s$$

Pre-specified % tolerance based on the knowledge of your solution

- If the following criterion is met

$$\varepsilon_s = \left(0.5 \times 10^{(2-n)}\right) \times 100\%$$

- you can be sure that the result is correct to at least ***n significant*** figures.

Source of Error

- Round-off error:
  - Caused by the limited number of digits that represent numbers in a computer and
  - The ways numbers are stored and additions and substructions are performed in a computer

- Truncation Error:
  - Caused by approximation used in the mathematical formula of the scheme

Background of Truncation Error

- Numerical solutions are mostly approximations for exact solution
- Most numerical methods are based on approximating function by polynomials
- How accurately the polynomial is approximating the true function ?
- Comparing the polynomial to the exact solution it becomes possible to evaluate the error, called truncation error

Taylor Series

- The most important polynomials used to derive numerical schemes and analyze truncation errors
- With an infinite power series, it is exactly represents a function within a certain radius about a given point

Taylor Series - Taylor's Theorem

- If the function f and its first n + 1 derivatives are continuous on an interval containing a and x, then the value of the function at x is given by:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n$$

- With the remainder Rn is defined as:

$$R_n = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt$$

Taylor Series

- Taylor series is of great value in the study of numerical methods. In essence, the Taylor series provides a means to predict a function value at one point in terms of the function value and its derivatives at another point. In particular, the theorem states that any smooth function can be approximated as a polynomial.
- A useful way to gain insight into the Taylor series is to build it term by term. For example, the first term in the series is:

$$f(x_{i+1}) \cong f(x_i)$$

Taylor Series

$f(x_{i+1}) \cong f(x_i)$ — *Zero-order approximation*

$f(x_{i+1}) \cong f(x_i) + f'(x_i)(x_{i+1} - x_i)$ — *First-order approximation*

$f(x_{i+1}) \cong f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2$ — *Second-order approximation*

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2 + \frac{f^{(3)}(x_i)}{3!}(x_{i+1} - x_i)^3 + \cdots + \frac{f^{(n)}(x_i)}{n!}(x_{i+1} - x_i)^n + R_n$$

Taylor Series

nth order approximation

$$f(x_{i+1}) \cong f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''}{2!}(x_{i+1} - x_i)^2 + \ldots$$

$$+ \frac{f^{(n)}}{n!}(x_{i+1} - x_i)^n + R_n$$

$(x_{i+1}-x_i)= h$ *step size* (define first)

$$R_n = \frac{f^{(n+1)}(\varepsilon)}{(n+1)!} h^{(n+1)}$$

- Reminder term, R_n, accounts for all terms from (n+1) to infinity.

Taylor Series Approximation of a Polynomial

Example

Problem Statement: Use zero- through fourth-order Taylor series expansions to approximate the function:

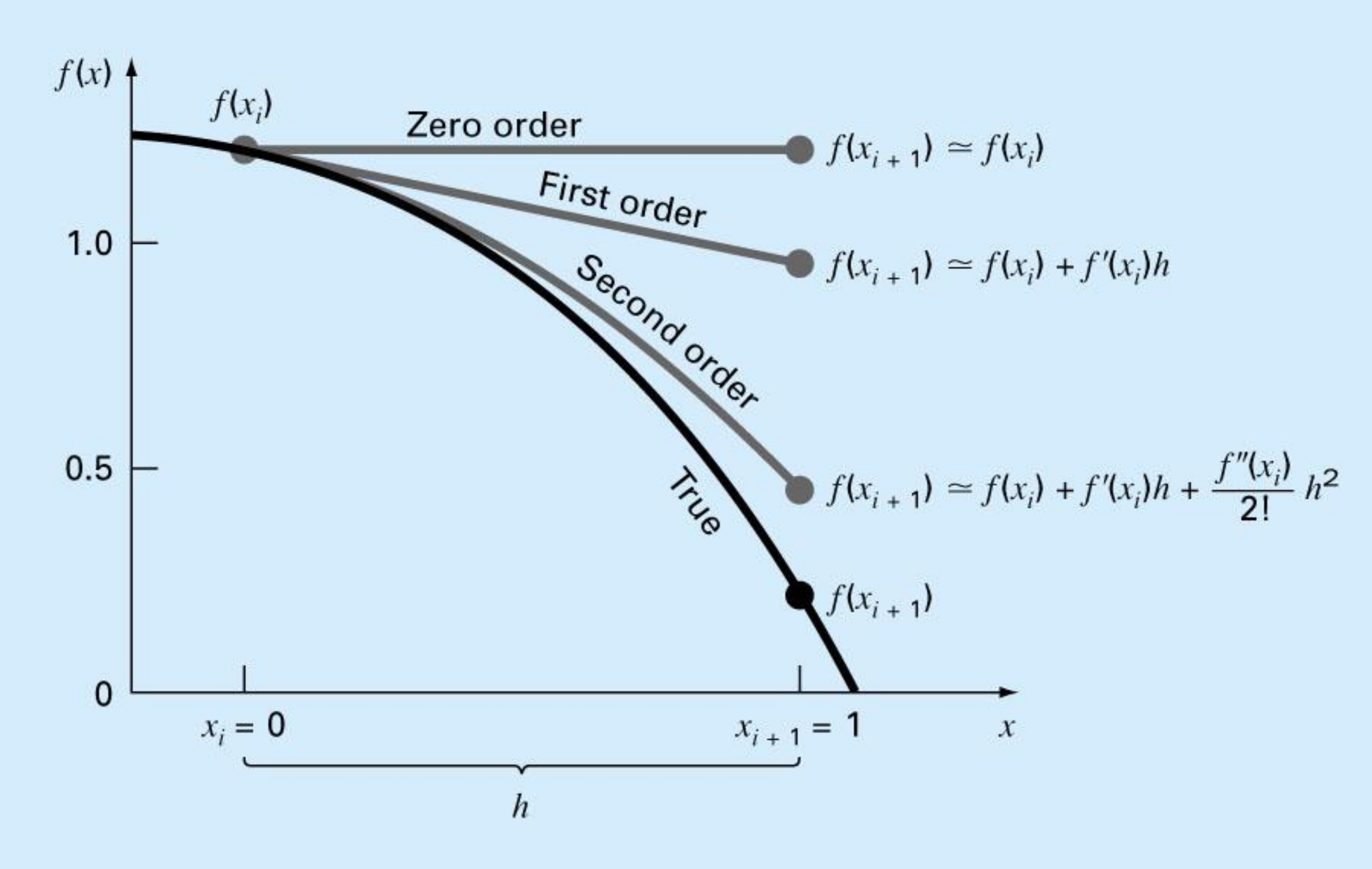
$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

From $x_i = 0$ with h=1, that is, predict the function's value at $x_{i+1} = 1$

Solution:

Because we are dealing with a known function, we can compute values for *f(x)* between 0 and 1.

Taylor Series Approximation of a Polynomial



Homework

Problem Statement: Use the Taylor series expansions to estimate :

$$f(x) = e^{-x} \text{ at } x_{i+1} = 1 \text{ for } x_i = 0$$

Employ the zero-, first-, second-, and third-order versions and compute |εt| for each case

أوجد قيمة التابع التالي $f(x) = e^{-x}$ في النقطة $x_{i+1} = 1$ منطلقاً من نشر التابع في جوار x_i $= 0$ مستخدماً أربعة حدود من سلسلة تايلور. واحسب الخطأ |εt| في كل مرحلة.