

Roots Of Equations

Root Finding Problems

14/11/2022

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Numerical Analysis

Many problems in Science and Engineering are expressed as:

Given a continuous function $f(x)$,
find the value r such that $f(r) = 0$

These problems are called root finding problems.

Roots of Equations

A number r that satisfies an equation is called a root of the equation.

The equation : $x^4 - 3x^3 - 7x^2 + 15x = -18$

has four roots : $-2, 3, 3,$ and -1 .

i.e., $x^4 - 3x^3 - 7x^2 + 15x + 18 = (x + 2)(x - 3)^2(x + 1)$

*The equation has two simple roots (-1 and -2)
and a repeated root (3) with multiplicity = 2.*

Zeros of a Function

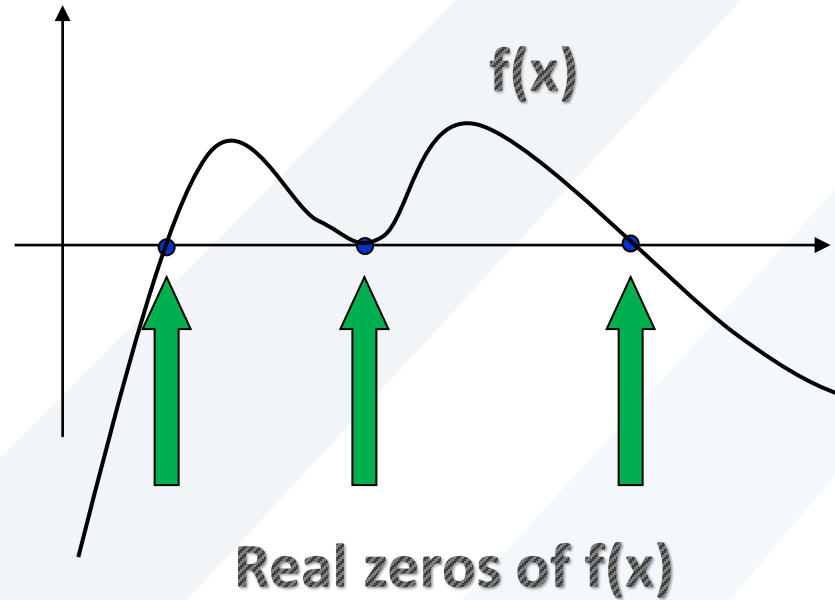
Let $f(x)$ be a real-valued function of a real variable. Any number r for which $f(r)=0$ is called a zero of the function.

Examples:

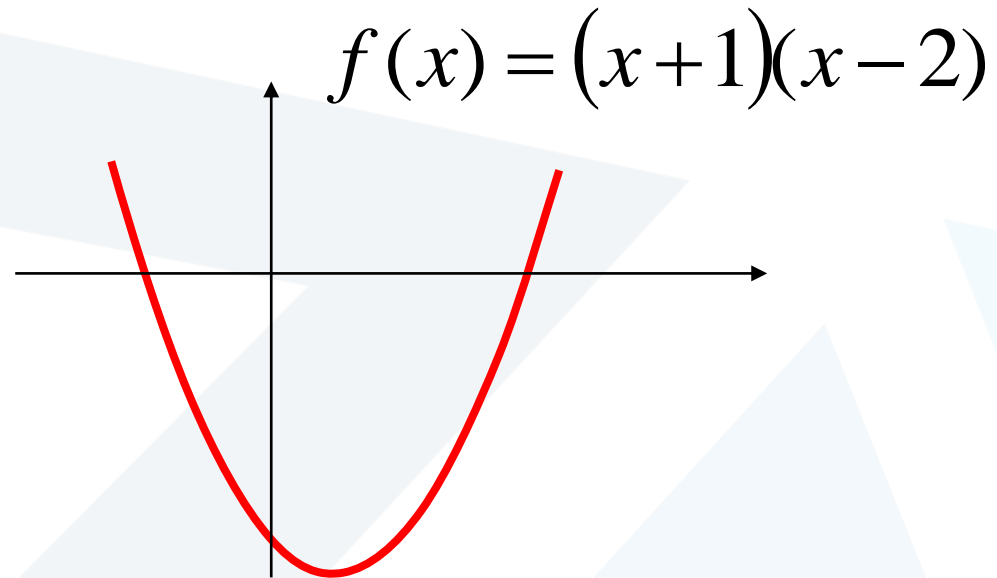
2 and **3** are zeros of the function $f(x) = (x-2)(x-3)$

Graphical Interpretation of Zeros

The real zeros of a function $f(x)$ are the values of x at which the graph of the function crosses (or touches) the x-axis.



Simple Zeros



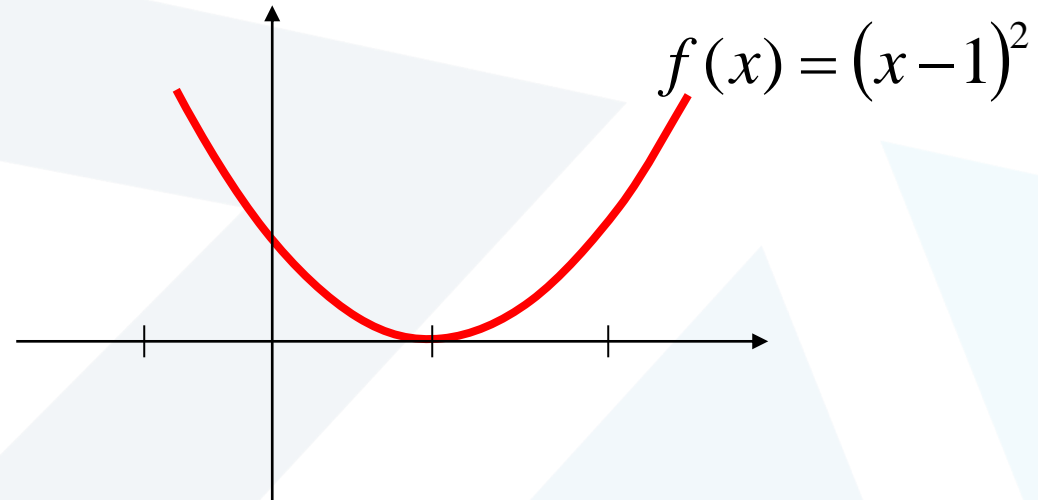
$$f(x) = (x+1)(x-2) = x^2 - x - 2$$

has two simple zeros (one at $x = 2$ and one at $x = -1$)

Multiple Zeros

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$$f(x) = (x-1)^2 = x^2 - 2x + 1$$

has double zeros (zero with multiplicity $y = 2$) at $x = 1$

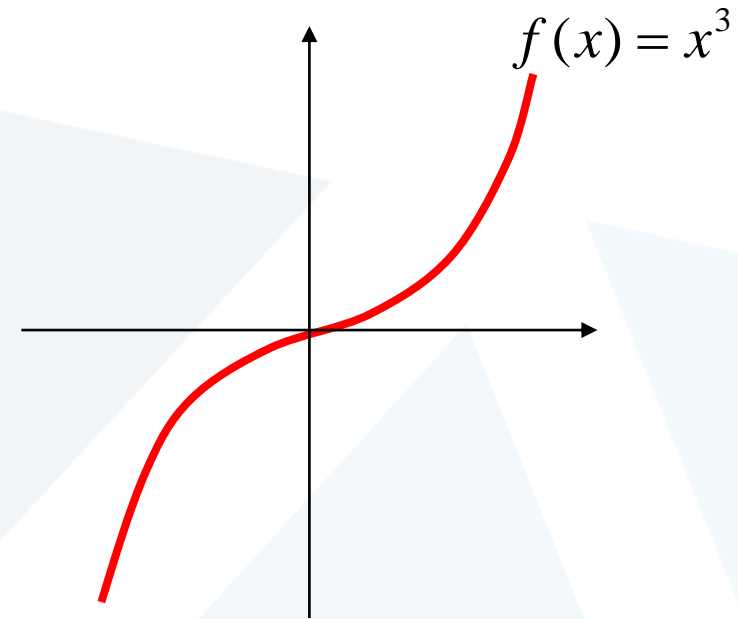
Numerical Analysis

Multiple Zeros

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Numerical Analysis



$$f(x) = x^3$$

has a zero with multiplicity $y = 3$ at $x = 0$

Facts

- Any n^{th} order polynomial has exactly n zeros (counting real and complex zeros with their multiplicities).
- Any polynomial with an odd order has at least one real zero.
- If a function has a zero at $x=r$ with multiplicity m then the function and its first $(m-1)$ derivatives are zero at $x=r$ and the m^{th} derivative at r is not zero.

Roots of Equations & Zeros of Function

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Given the equation :

$$x^4 - 3x^3 - 7x^2 + 15x = -18$$

Move all terms to one side of the equation :

$$x^4 - 3x^3 - 7x^2 + 15x + 18 = 0$$

Define $f(x)$ as :

$$f(x) = x^4 - 3x^3 - 7x^2 + 15x + 18$$

The zeros of $f(x)$ are the same as the roots of the equation $f(x) = 0$
(Which are $-2, 3, 3,$ and -1)

Nonlinear Equation Solvers

Graphical
Solutions

Numerical Solutions

Analytical
Solutions

Bracketing Methods

Open Methods

Bisection
Method

False
Position
Method

Fixed Point
Iteration

Newton
Raphson
Method

Secant
Method

Analytical Methods

- Analytical Solutions are available for special equations only.
- Analytical Solution for: $ax^2 + bx + c = 0$

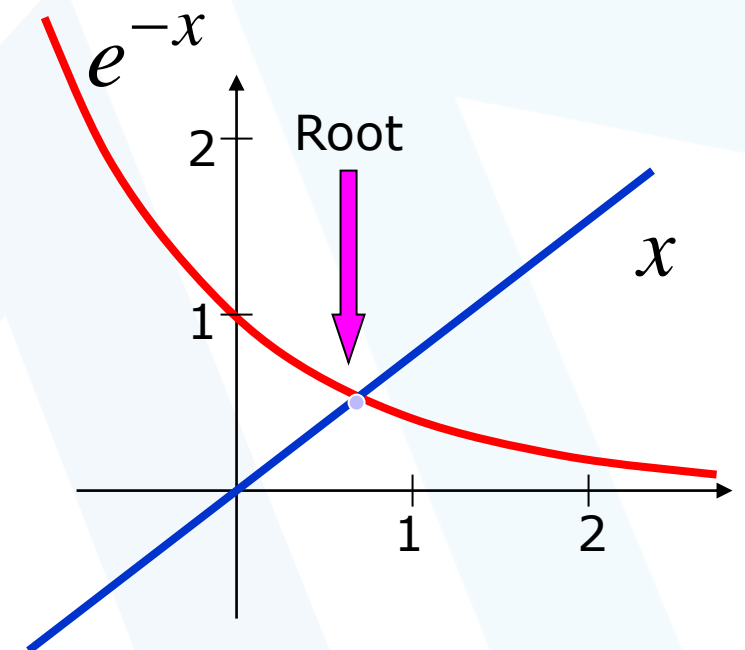
$$\mathit{roots} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

No analytical solution is available for: $x - e^{-x} = 0$

Graphical Methods

- A simple method for obtaining an estimate of the root of the equation $f(x) = 0$ is to make a plot of the function and observe where it crosses the x axis. This point, which represents the x value for which $f(x) = 0$, provides a rough approximation of the root.
- Graphical methods are useful to provide an **initial guess** to be used by other methods.

Solve
 $x = e^{-x}$
The root $\in [0,1]$
root ≈ 0.6



Graphical Methods

- **Example:** Use the graphical approach to determine the drag coefficient c needed for a parachutist of mass $m = 68.1$ kg to have a velocity of 40 m/s after free-falling for time $t = 10$ s. Note: The acceleration due to gravity is 9.81 m/s²

$$v(t) = \frac{gm}{c}(1 - e^{-(c/m)t})$$

- **Solution:** This problem can be solved by determining the root of previous equation using the parameters $t = 10$, $g = 9.81$, $v = 40$, and $m = 68.1$

$$f(c) = \frac{9.81(68.1)}{c}(1 - e^{-(c/68.1)10}) - 40$$

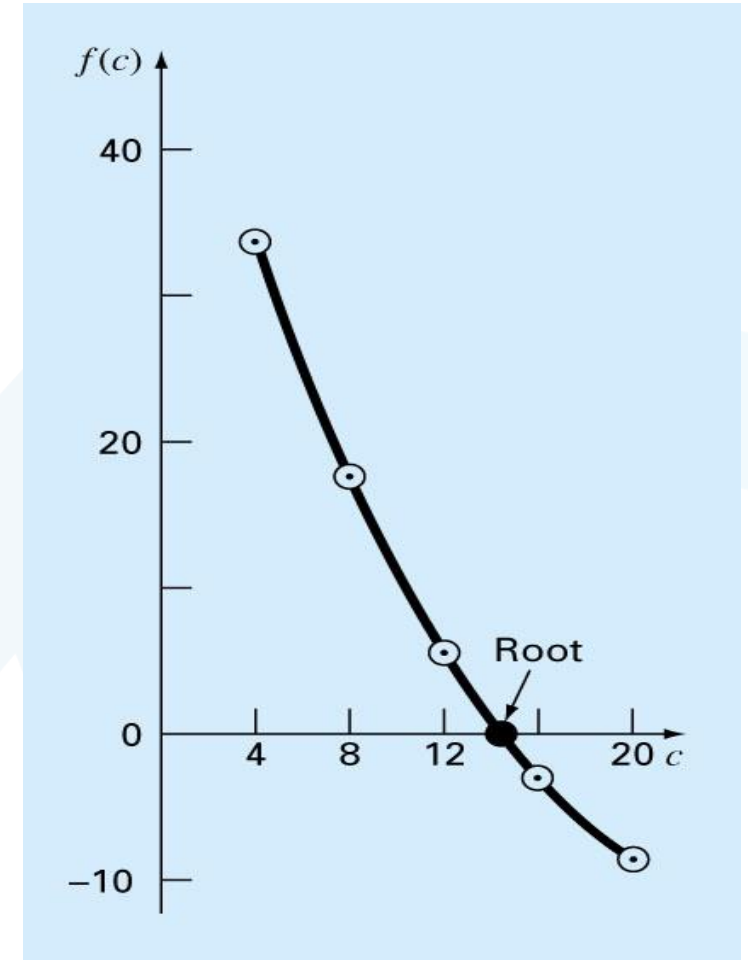
$$f(c) = \frac{668.06}{c}(1 - e^{-0.146843c}) - 40$$

Various values of c can be substituted into this equation to compute the following points:

Graphical Methods

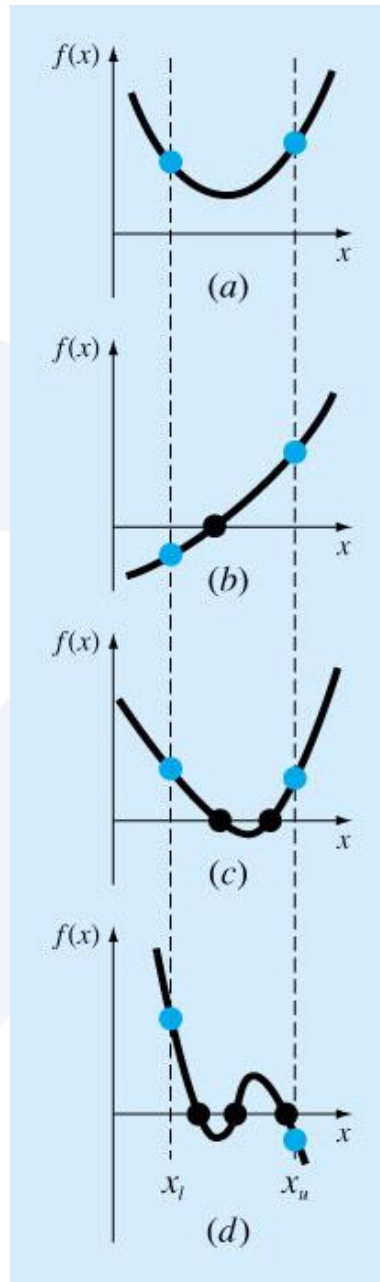
The resulting curve crosses the c axis **between 12 and 16**. Visual inspection of the plot provides a rough estimate of the **root of 14.75**. The validity of the graphical estimate can be checked by substituting it into

c	$f(c)$
4	34.190
8	17.712
12	6.114
16	-2.230
20	-8.368



Graphical Methods

- **Graphical techniques** are of **limited practical value** because they are not precise. However, graphical methods can be utilized to **obtain rough estimates of roots**. These estimates can be employed as **starting guesses for numerical methods** discussed in this and the next lectures. Aside from providing rough estimates of the roots, graphical interpretations are **important tools** for **understanding the properties of the functions** and **anticipating the pitfalls of the numerical methods**. For example, the following shows a number of ways in which roots can occur (or be absent) in an interval prescribed by a lower bound x_l and an upper bound x_u .



No root (same sign)

Single root (change sign)

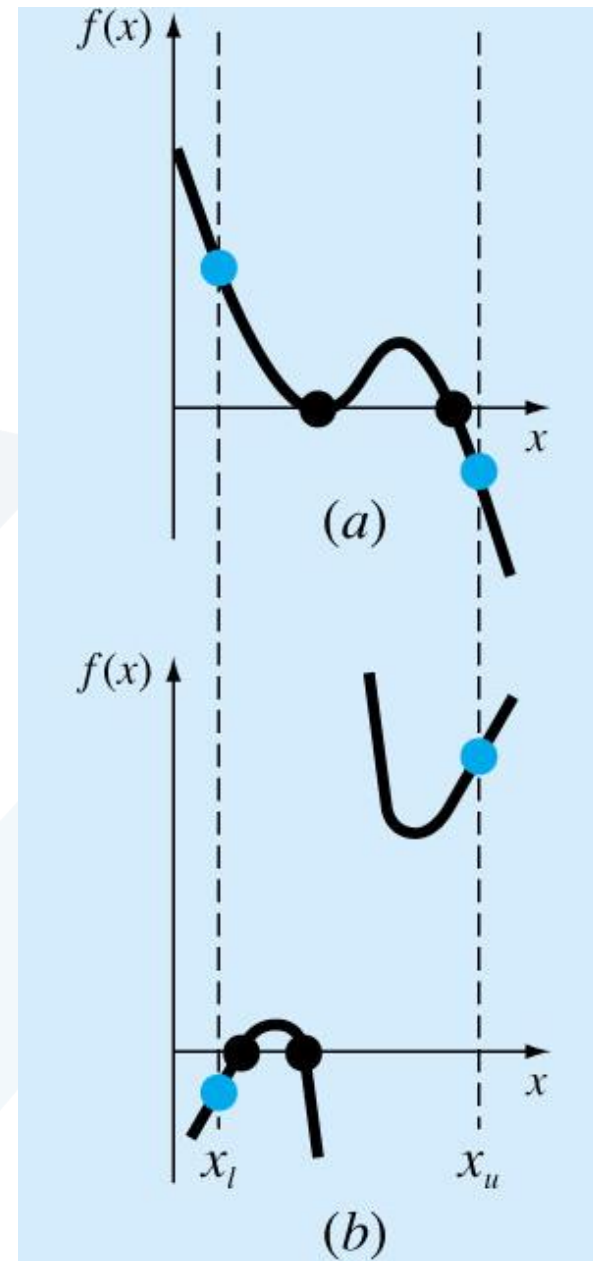
Two roots (same sign)

Three roots (change sign)

Special Cases

Multiple Roots

Discontinuity



Graphical Methods

- **Conclusion:**

Graphical method is useful for getting an idea of what's going on in a problem, but depends on eyeball.

- **Recommendation:**

Use bracketing methods to improve the accuracy

Bracketing Methods

- In bracketing methods (two points method for finding roots), the method starts with an **interval** that contains the root and a procedure is used to obtain a smaller interval containing the root.
- In other words, two initial guesses for the root are required. These guesses must “bracket” or be on either side of the root.
- If one root of a real and continuous function, $f(x)=0$, is bounded by values $x=x_l$, $x=x_u$ then $f(x_l) \cdot f(x_u) < 0$. (The function changes sign on opposite sides of the root)

Bisection Method

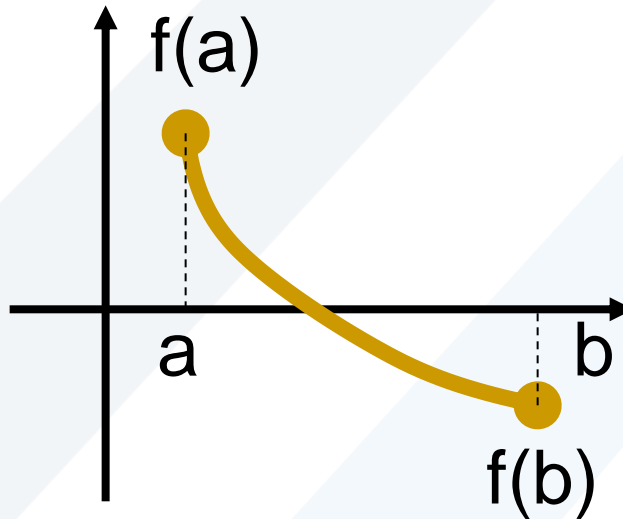
- **Bisection Method:** is one of the simplest methods to find a zero of a nonlinear function.
- It is also called **interval halving** method.
- To use the Bisection method, one needs an **initial interval** that is known to contain a zero of the function.
- The method systematically **reduces** the interval. It does this by dividing the interval into two equal parts, performs a simple test and based on the result of the test, half of the interval is thrown away.
- The procedure is repeated until the desired interval size is obtained.

Bisection Method

- Let $f(x)$ be defined on the interval $[a,b]$.

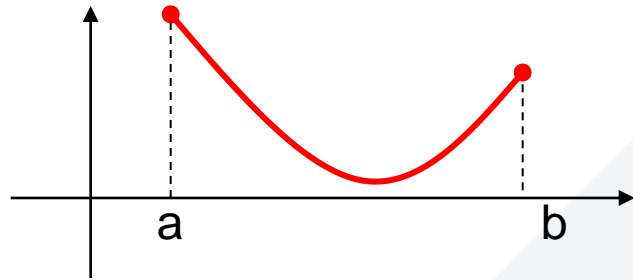
Intermediate value theorem:

if a function is continuous and $f(a)$ and $f(b)$ have different signs then the function has at least one zero in the interval $[a,b]$.

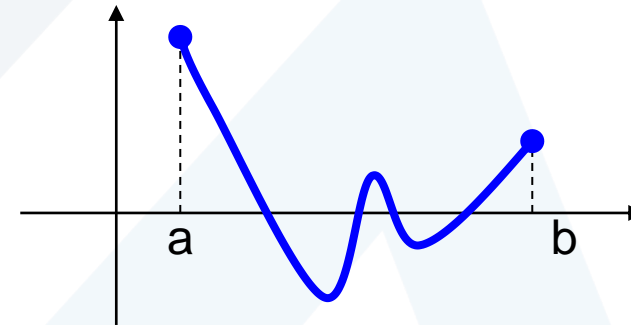


Bisection Method

- If $f(a)$ and $f(b)$ have the same sign, the function may have an even number of real zeros or no real zeros in the interval $[a, b]$.
- Bisection method **can not** be used in these cases.



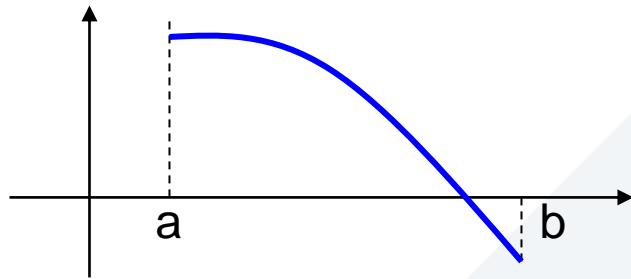
The function has no real zeros



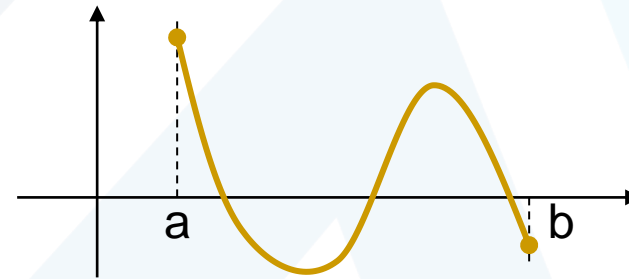
The function has four real zeros

Bisection Method

- If $f(a)$ and $f(b)$ have different signs, the function has at least one real zero.
- Bisection method **can** be used to find one of the zeros.



The function has one real zeros



The function has three real zeros

Bisection Method

- If the function is continuous on $[a,b]$ and $f(a)$ and $f(b)$ have different signs, Bisection method obtains a new interval that is half of the current interval and the sign of the function at the end points of the interval are different.
- This allows us to repeat the Bisection procedure to further reduce the size of the interval.

Bisection Method

Assumptions:

Given an interval $[a,b]$

$f(x)$ is continuous on $[a,b]$

$f(a)$ and $f(b)$ have opposite signs.

These assumptions ensure the existence of at least one zero in the interval $[a,b]$ and the bisection method can be used to obtain a smaller interval that contains the zero.

Bisection Algorithm

Assumptions:

$f(x)$ is continuous on $[a,b]$

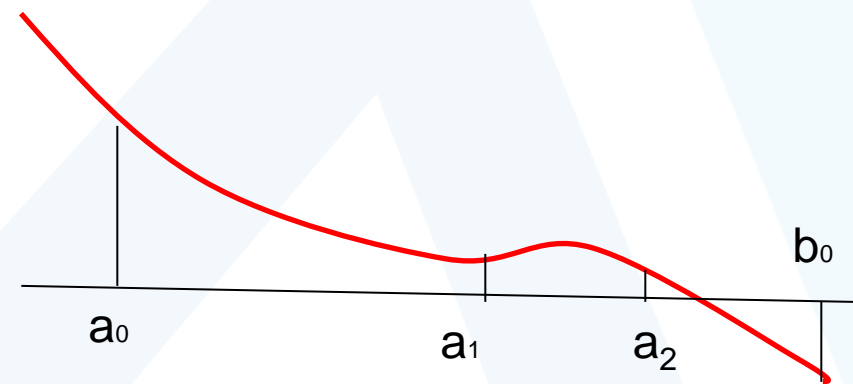
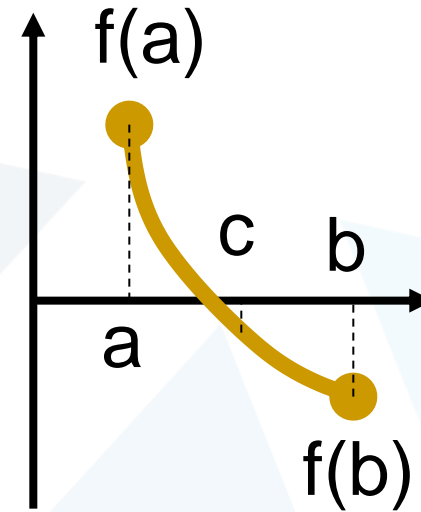
$f(a) \times f(b) < 0$

Algorithm:

Loop

1. Compute the mid point $c=(a+b)/2$
2. Evaluate $f(c)$
3. If $f(a) \times f(c) < 0$ then new interval $[a, c]$
 If $f(a) \times f(c) > 0$ then new interval $[c, b]$

End loop

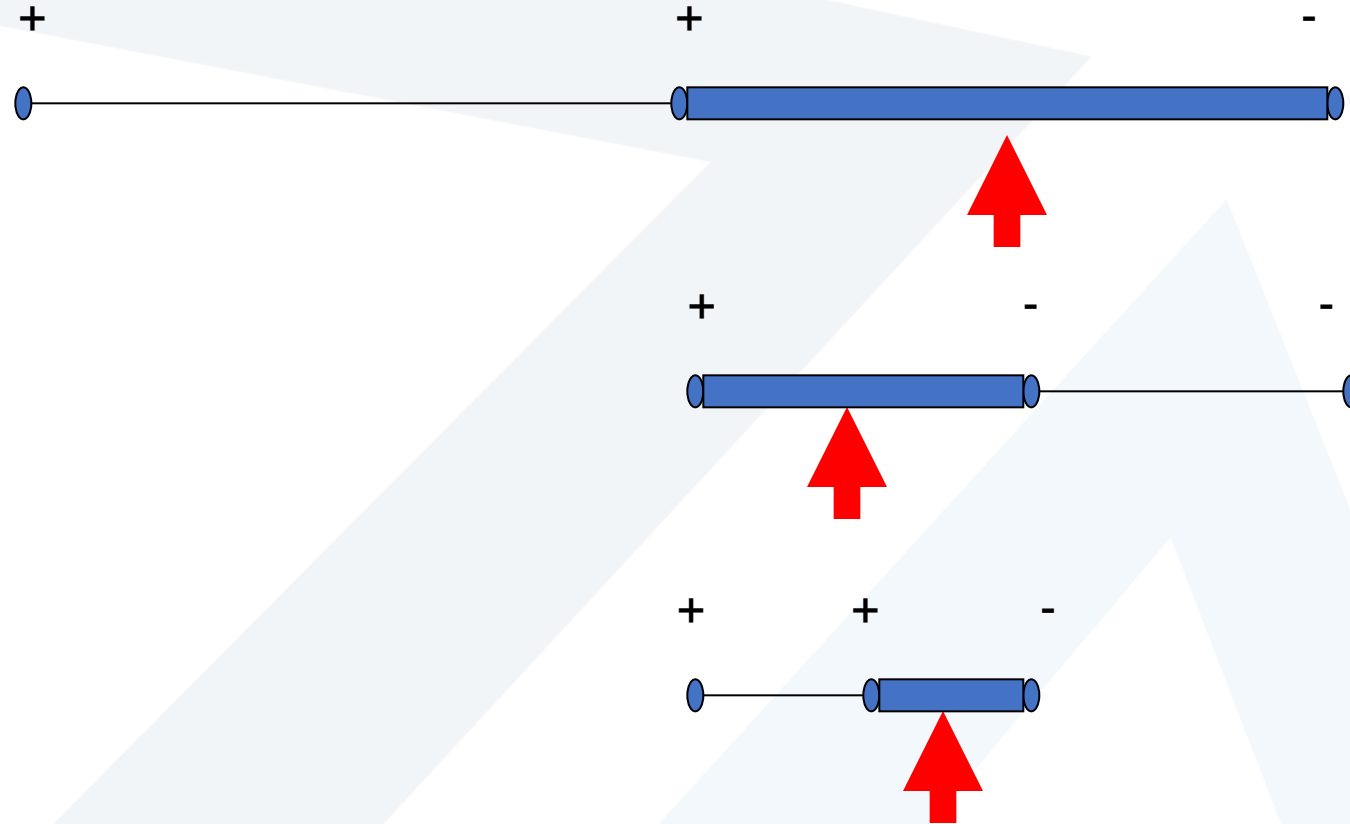


Bisection Method

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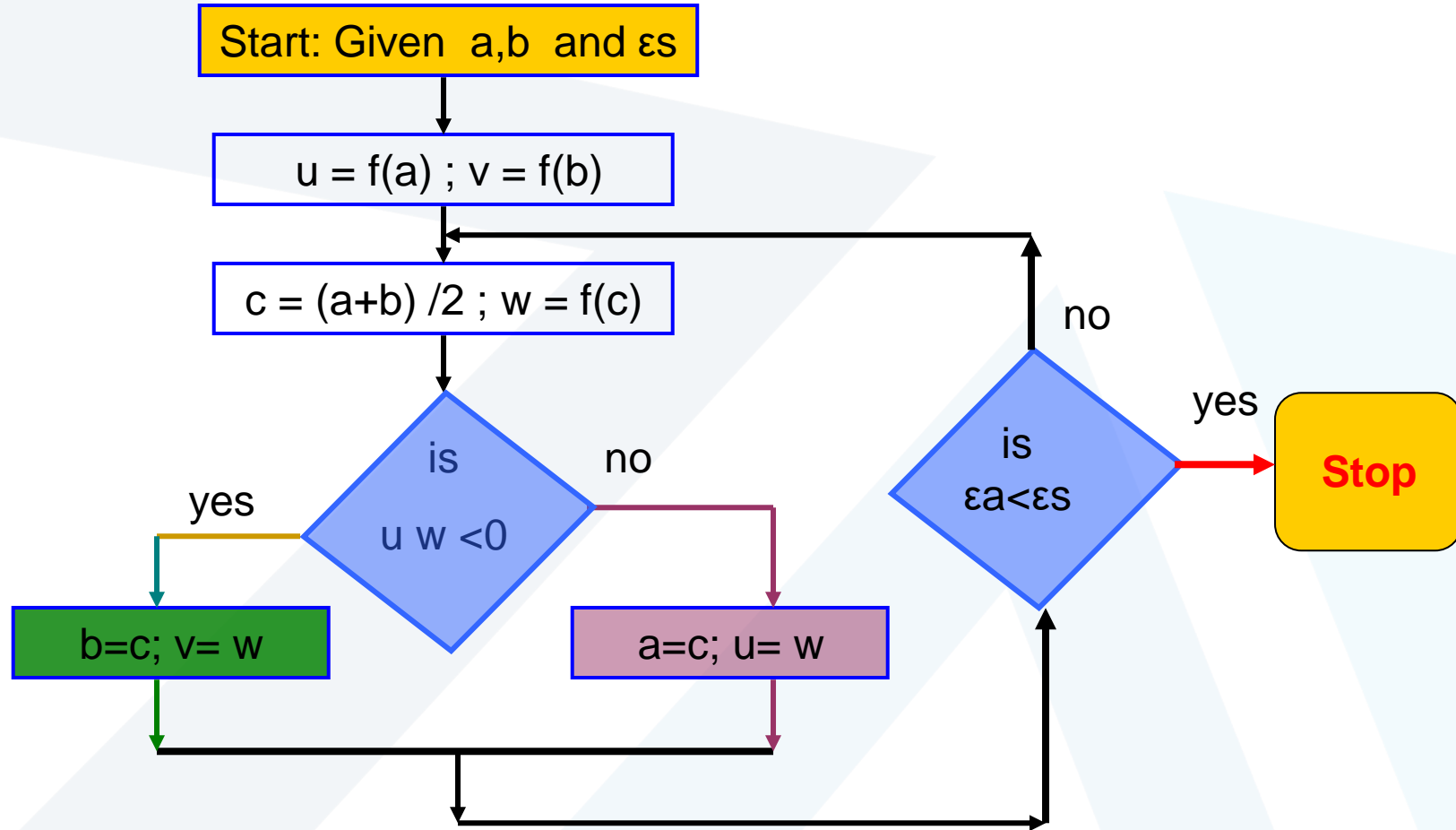


Flow Chart of Bisection Method

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Bisection Method (Examples)

Can you use Bisection method to find a zero of :

$f(x) = x^3 - 3x + 1$ in the interval $[0, 2]$?

Answer:

$f(x)$ is continuous on $[0, 2]$

and $f(0) * f(2) = (1)(3) = 3 > 0$

\Rightarrow Assumptions are not satisfied

\Rightarrow Bisection method can not be used

Bisection Method (Examples)

Can you use Bisection method to find a zero of :

$f(x) = x^3 - 3x + 1$ in the interval $[0,1]$?

Answer:

$f(x)$ is continuous on $[0,1]$

and $f(0) * f(1) = (1)(-1) = -1 < 0$

\Rightarrow Assumptions are satisfied

\Rightarrow Bisection method can be used

Bisection Method (Examples)

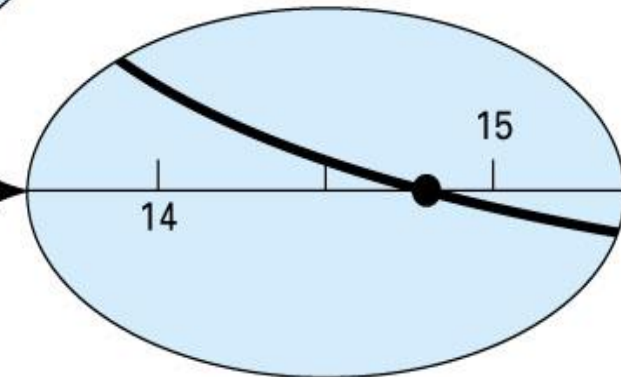
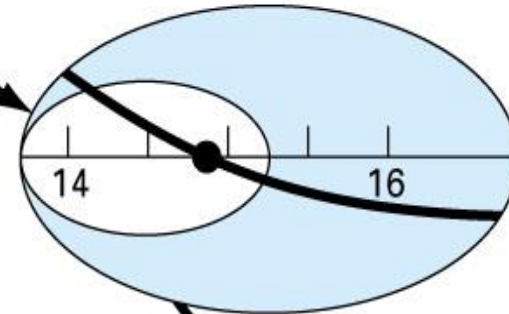
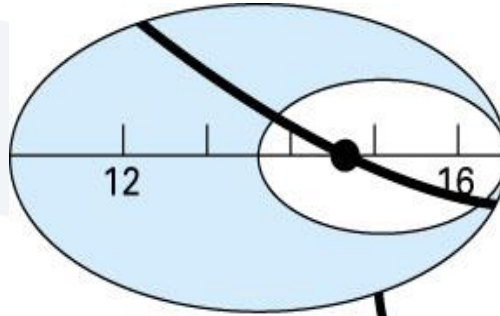
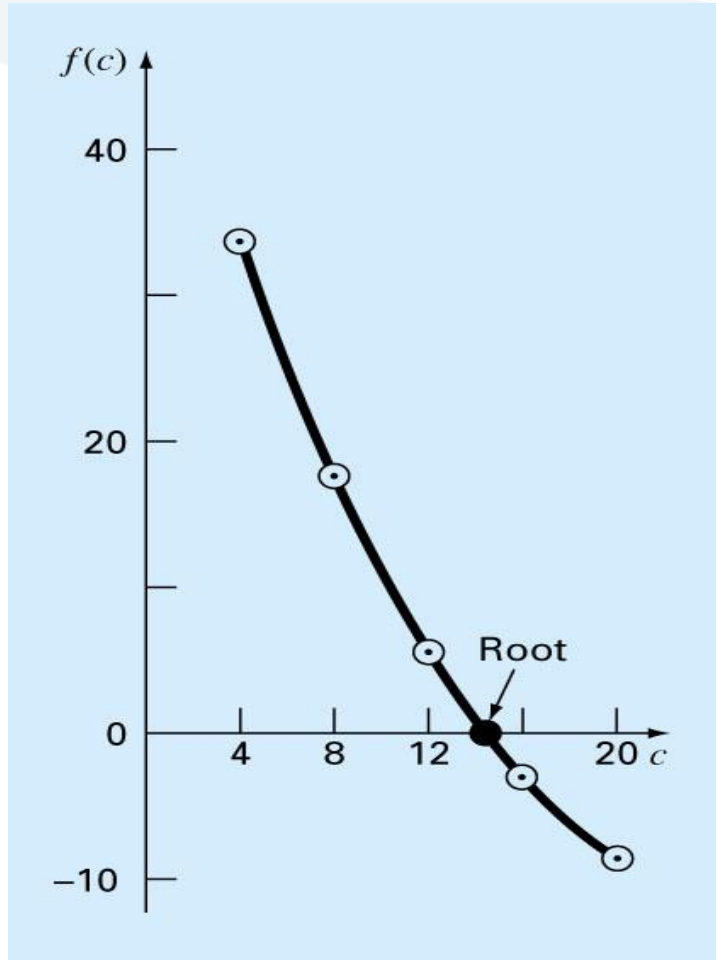
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- **Problem Statement.** Problem Statement. Use bisection to solve the same problem approached graphically in slide 15.

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Bisection Method (Examples)



Bisection Method (Error Estimation)

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- we require an error estimate that is not contingent on foreknowledge of the root. As developed previously, an approximate percent relative error ϵ_a can be calculated, as in

$$\epsilon_a = \left| \frac{x_r^{\text{new}} - x_r^{\text{old}}}{x_r^{\text{new}}} \right| 100\%$$

- where x_{new} is the root for the present iteration and x_{old} is the root from the previous iteration. The absolute value is used because we are usually concerned with the magnitude of ϵ_a rather than with its sign. When ϵ_a becomes less than a prespecified stopping criterion ϵ_s , the computation is terminated.

Bisection Method (Error Estimation)

- **Problem Statement.** Continue previous example until the approximate error falls below a stopping criterion of $\epsilon_s = 2\%$.

Bisection Method (Examples)

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Use Bisection method to find a root of the equation $x = \cos(x)$ with absolute error < 0.02
(assume the initial interval $[0.5, 0.9]$)

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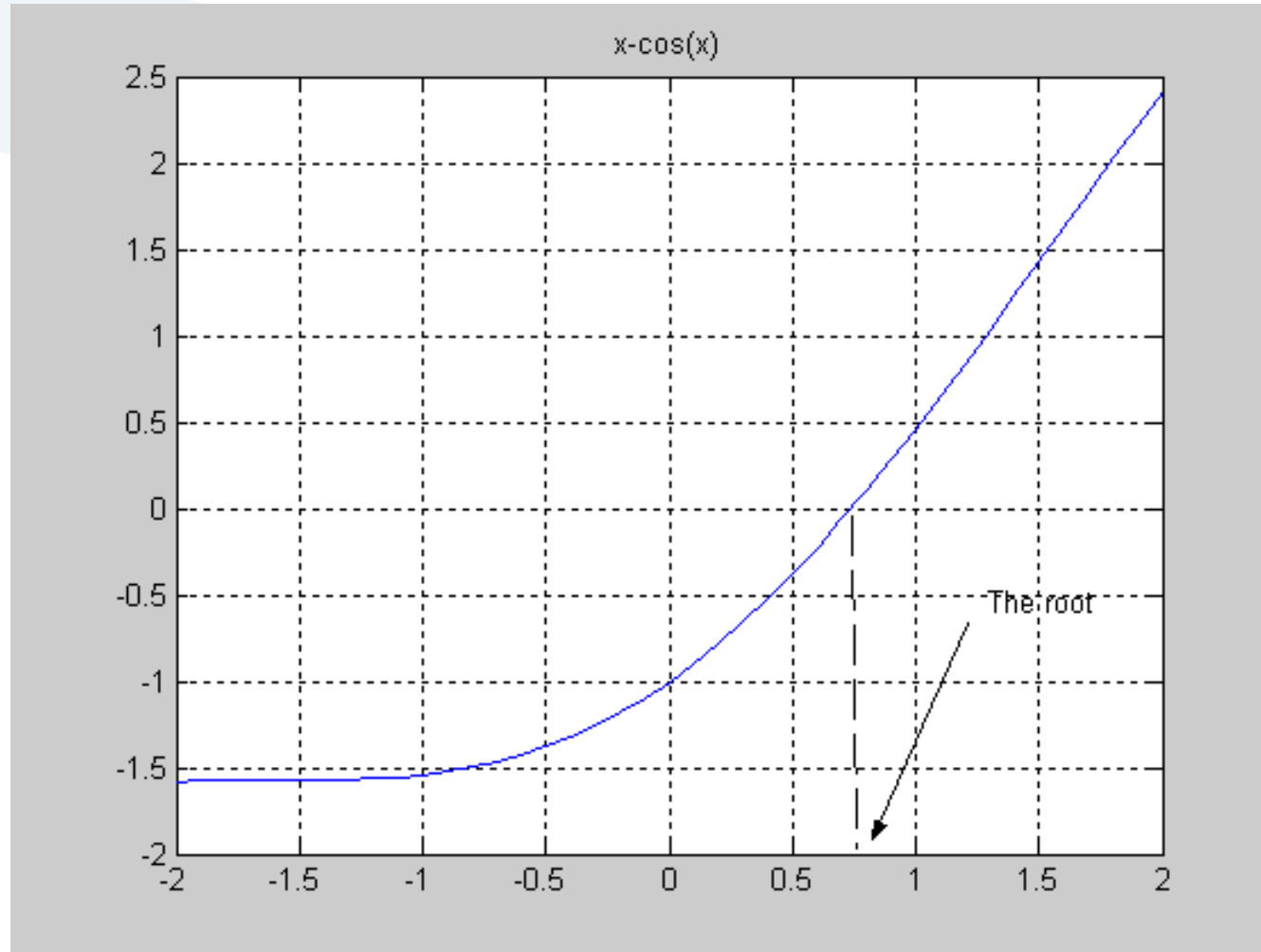
Question 1: What is $f(x)$?

Question 2: Are the assumptions satisfied?

Question 3: How to compute the new estimate?

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Bisection Method (Examples)



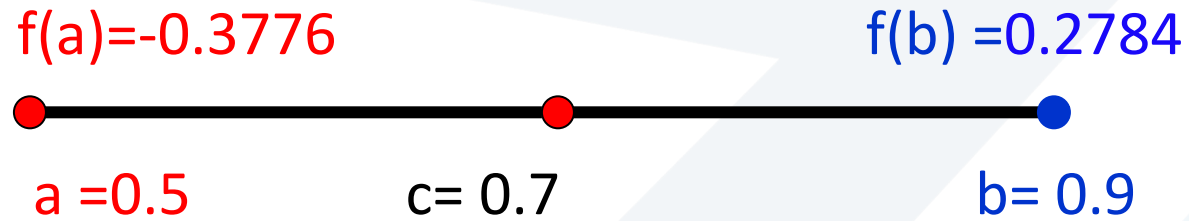
Bisection Method (Examples)

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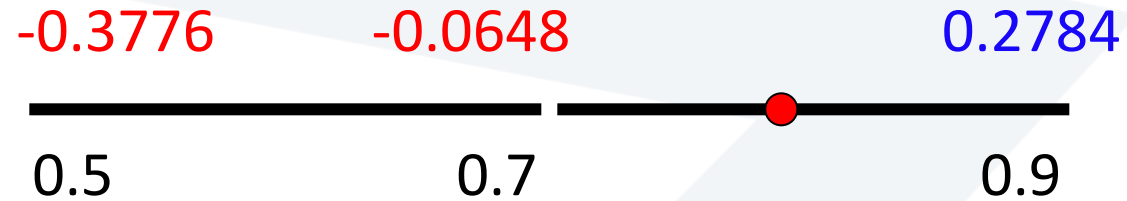
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Numerical Analysis

Initial Interval



Bisection Method (Examples)

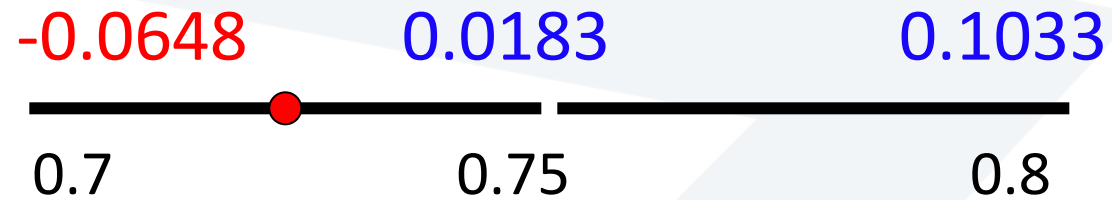


Error = 12.5%



Error = 6.66%

Bisection Method (Examples)



Error = 3.45%



Error = 1.75%

Bisection Method (Examples)

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Initial interval containing the root: $[0.5, 0.9]$

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After 5 iterations:

Interval containing the root: $[0.725, 0.75]$

Best estimate of the root is 0.7375

$|\text{Error}| < 2\%$

Numerical Analysis

Bisection Method

Advantages

- **Simple** and easy to implement
- **One** function evaluation per iteration
- The **size** of the interval containing the zero is reduced by 50% after each iteration
- The **number of iterations** can be determined **a priori**
- **No** knowledge of the **derivative** is needed
- The function does **not** have to be **differentiable**

Disadvantage

- **Slow** to converge
- **Good** intermediate approximations may be **discarded**

Homework

Problem Statement: Determine the real root of :

$$f(x) = -26 + 85x - 91x^2 + 44x^3 - 8x^4 + x^5$$

- Graphically.
- Using bisection method to determine the root to $\epsilon_s = 10\%$. Employ initial guess of $x_l=0.5$ and $x_u=1.0$.
- Resolve the previous questions using Excel.