

Programming and Numerical Analysis



Roots Of Equations

Part-02

Root Finding Problems

21/11/2022

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Numerical Analysis

Many problems in Science and Engineering are expressed as:

Given a continuous function $f(x)$,
find the value r such that $f(r) = 0$

These problems are called root finding problems.

Nonlinear Equation Solvers

Graphical
Solutions

Numerical Solutions

Analytical
Solutions

Bracketing Methods

Open Methods

Bisection
Method

False
Position
Method

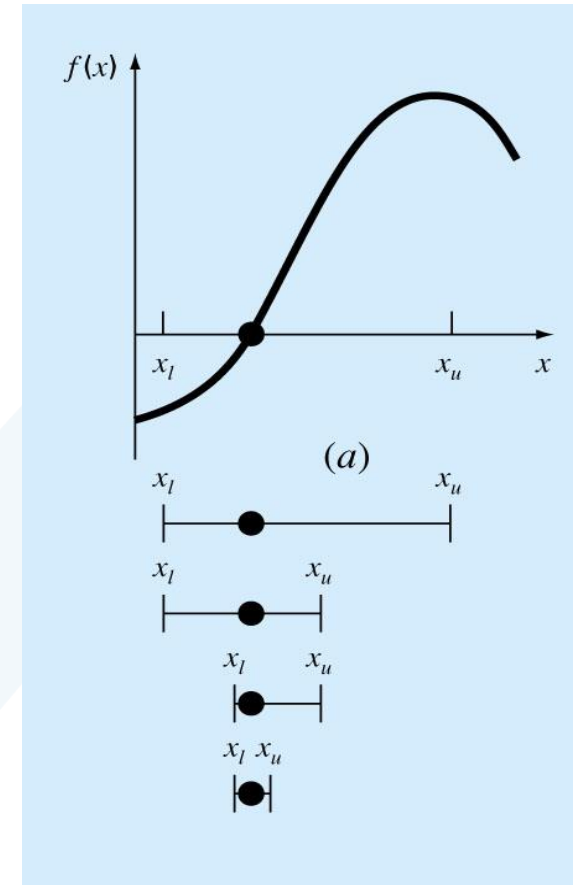
Fixed Point
Iteration

Newton
Raphson
Method

Secant
Method

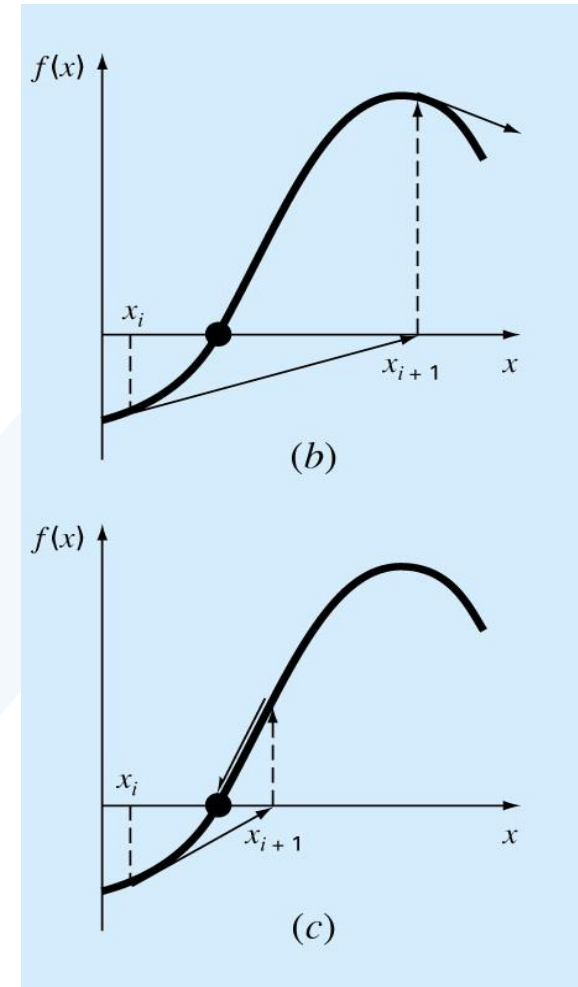
Open Methods

- For the **bracketing methods** which is discussed in previous lecture, the root is located within an **interval** prescribed by a **lower** and an **upper bound**. Repeated application of these methods always results in closer estimates of the true value of the root. Such methods are said to be **convergent** because they move closer to the truth as the computation progresses.



Open Methods

- In contrast, the **open methods** described in this lecture are based on formulas necessarily bracket the root. As such, they sometimes diverge or move away from the true root as the computation progresses (Fig. b). However, when the open methods converge (Fig. c), they usually do so much more quickly than the bracketing methods.
- We will begin our discussion of open techniques with a simple version that is useful for illustrating their general form and also for demonstrating the concept of convergence.



Simple Fixed Point Iteration

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Numerical Analysis

- Also known as **one-point iteration** or **successive substitution**
- To find the root for $f(x) = 0$, we **reformulate** $f(x) = 0$ so that **there is an x on one side** of the equation.

$$f(x) = 0 \Leftrightarrow g(x) = x$$

- If we can solve $g(x) = x$, we solve $f(x) = 0$.
 - x is known as the fixed point of $g(x)$.
- We solve $g(x) = x$ by computing

$$x_{i+1} = g(x_i) \quad \text{with } x_0 \text{ given}$$

until x_{i+1} converges to x .

Simple Fixed Point Iteration

$$f(x) = x^2 + 2x - 3 = 0$$

$$x^2 + 2x - 3 = 0 \Rightarrow 2x = 3 - x^2 \Rightarrow x = \frac{3 - x^2}{2}$$

$$\Rightarrow x_{i+1} = g(x_i) = \frac{3 - x_i^2}{2}$$

Reason: **If** x converges, i.e. $x_{i+1} \rightarrow x_i$

$$x_{i+1} = \frac{3 - x_i^2}{2} \rightarrow x_i = \frac{3 - x_i^2}{2}$$

$$\Rightarrow x_i^2 + 2x_i - 3 = 0$$

$\sin x = 0$???????????

Simple Fixed Point Iteration

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Numerical Analysis

- **Example :** Use simple fixed-point iteration to locate the root of $f(x) = e^{-x} - x$.
- **Solution:** The function can be separated directly and expressed in Equation as: $x_{i+1} = e^{-x_i}$. Starting with an initial guess of $x_i = 0$, the iterative equation can be applied to compute:

Thus, each iteration brings the estimate closer to the true value of the root: 0.56714329.

i	x_i	ε_a (%)	ε_t (%)
0	0		100.0
1	1.000000	100.0	76.3
2	0.367879	171.8	35.1
3	0.692201	46.9	22.1
4	0.500473	38.3	11.8
5	0.606244	17.4	6.89
6	0.545396	11.2	3.83
7	0.579612	5.90	2.20
8	0.560115	3.48	1.24
9	0.571143	1.93	0.705
10	0.564879	1.11	0.399

Newton Raphson Method

- Given an initial guess of the root x_0 , Newton-Raphson method uses information about the function and its derivative at that point to find a better guess of the root.
- **Based on Taylor series expansion:**

$$f(x_{i+1}) = f(x_i) + f'(x_i)\Delta x + f''(x_i)\frac{\Delta x^2}{2!} + O\Delta x^3$$

The root is the value of x_{i+1} when $f(x_{i+1}) = 0$

Rearranging,

$$0 = f(x_i) + f'(x_i)(x_{i+1} - x_i)$$

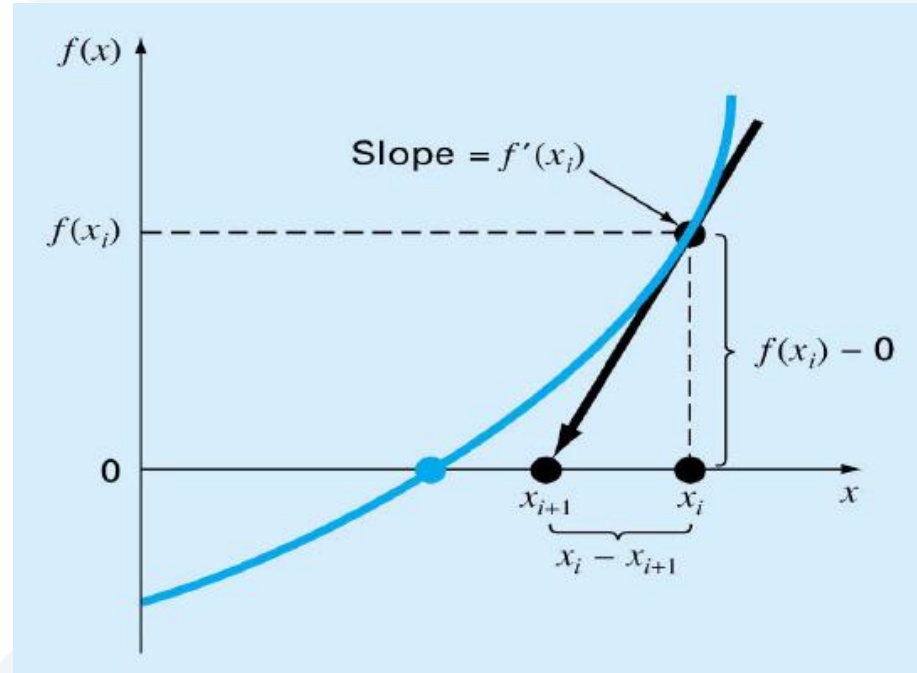
Solve for

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Newton-Raphson formula

Newton Raphson Method

- **Graphical Depiction:** If the initial guess at the root is x_i , then a tangent to the function of x_i that is $f'(x_i)$ is extrapolated down to the x -axis to provide an estimate of the root at x_{i+1} .



A convenient method for functions whose derivatives can be evaluated analytically.

Newton Raphson Method

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Numerical Analysis

- **Example:** Use the Newton-Raphson method to estimate the root of $f(x) = e^{-x} - x$, employing an initial guess of $x_0 = 0$
- **Solution:** The first derivative of the function can be evaluated as: $f'(x) = -e^{-x} - 1$ which can be substituted along with the original function: $x_{i+1} = x_i - \frac{e^{-x_i} - x_i}{-e^{-x_i} - 1}$
Starting with an initial guess of $x_0 = 0$, the iterative equation can be applied to compute:

Pitfalls of Newton Raphson Method

*Assumptions: $f(x)$, $f'(x)$, x_0 are available,
 $f'(x_0) \neq 0$*

Newton's Method new estimate:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Problem :

$f'(x_i)$ is not available,
or difficult to obtain analytically.

It may not be convenient for functions whose derivatives cannot be evaluated analytically.

The Secant Method

- A slight variation of Newton's method for functions whose **derivatives are difficult to evaluate**. For these cases the derivative can be approximated by a backward finite divided difference.

The Secant Method - Derivation

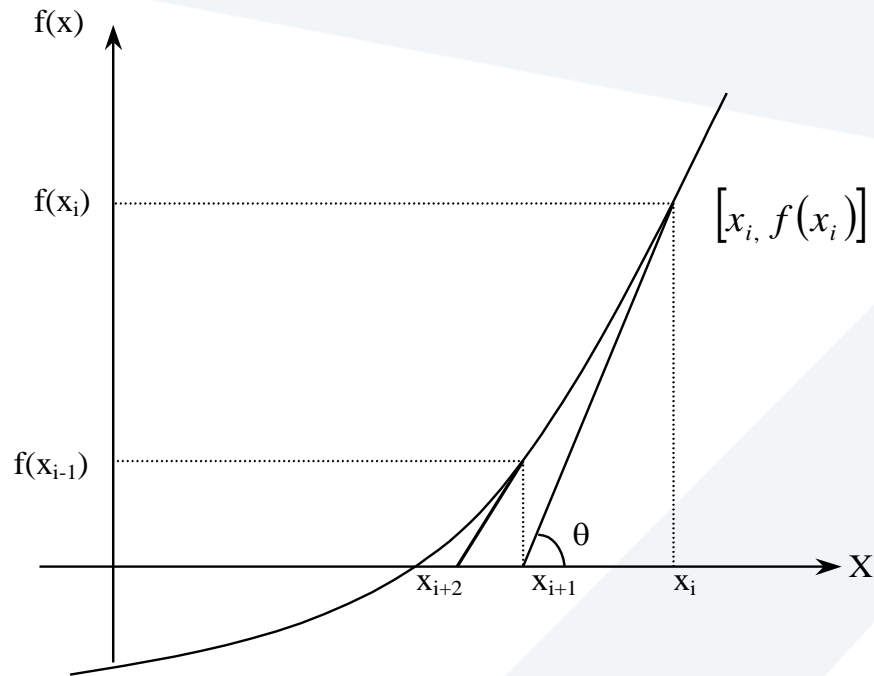


Figure 1 Geometrical illustration of the Newton-Raphson method.

Newton's Method

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad (1)$$

Approximate the derivative

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \quad (2)$$

Substituting Equation (2) into Equation (1) gives the Secant method

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

The Secant Method - Derivation

The secant method can also be derived from geometry:

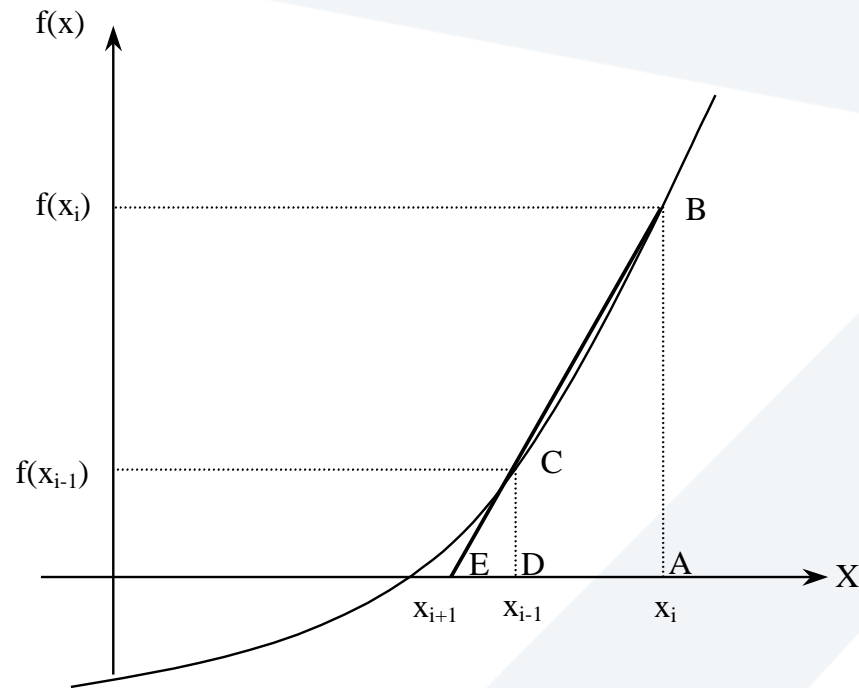


Figure 2 Geometrical representation of the Secant method.

The Geometric Similar Triangles

$$\frac{AB}{AE} = \frac{DC}{DE}$$

can be written as

$$\frac{f(x_i)}{x_i - x_{i+1}} = \frac{f(x_{i-1})}{x_{i-1} - x_{i+1}}$$

On rearranging, the secant method is given as

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

The Secant Method

- **Example:** Use the secant method to estimate the root of $f(x) = e^{-x} - x$, start with initial estimates of $x_{-1} = 0$ and $x_0 = 1.0$
- **Solution:**

Homework

Problem Statement: Determine the highest real root of:

$$f(x) = -6 + 17.5x - 11.6x^2 + 2.1x^3$$

- Graphically.
- Fixed Point iteration method (three iterations, $x_0=3$).
- Newton Raphson method (three iterations, $x_0=3$).
- Secant method (three iterations, $x_{-1}=3$, $x_0=4$).

Compute the approximate percent relative errors for your solutions.