

Linear Algebraic Equations

Part-01

Linear Algebraic Equations

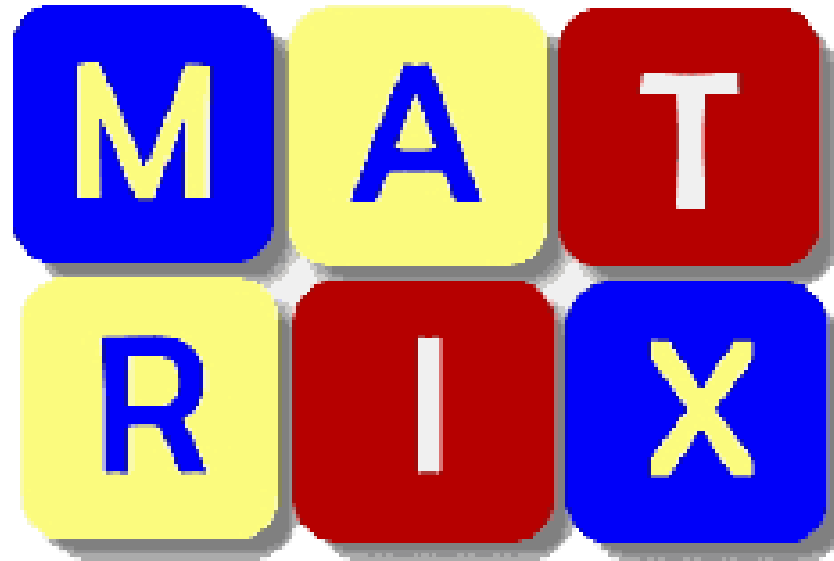
- An equation of the form $ax+by+c=0$ or equivalently $ax+by=-c$ is called a linear equation in x and y variables.
- $ax+by+cz=d$ is a linear equation in three variables, x, y , and z .
- Thus, a linear equation in n variables is $a_1x_1+a_2x_2+ \dots +a_nx_n = b$
- A solution of such an equation consists of real numbers $c_1, c_2, c_3, \dots , c_n$. If you need to work more than one linear equations, a system of linear equations must be solved simultaneously.

Mathematical Background

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Numerical Analysis



VECTORS

Vector : a one dimensional array of numbers

Examples :

row vector $[1 \ 4 \ 2]$ column vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Identity vectors $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $e_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

MATRECIS

Matrix : a two dimensional array of numbers

Examples :

zero matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

diagonal $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$,

Tridiagonal $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$

MATRECIS

Examples :

symmetric $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 5 \\ -1 & 5 & 4 \end{bmatrix}$, upper triangular

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Adding and Multiplying Matrices

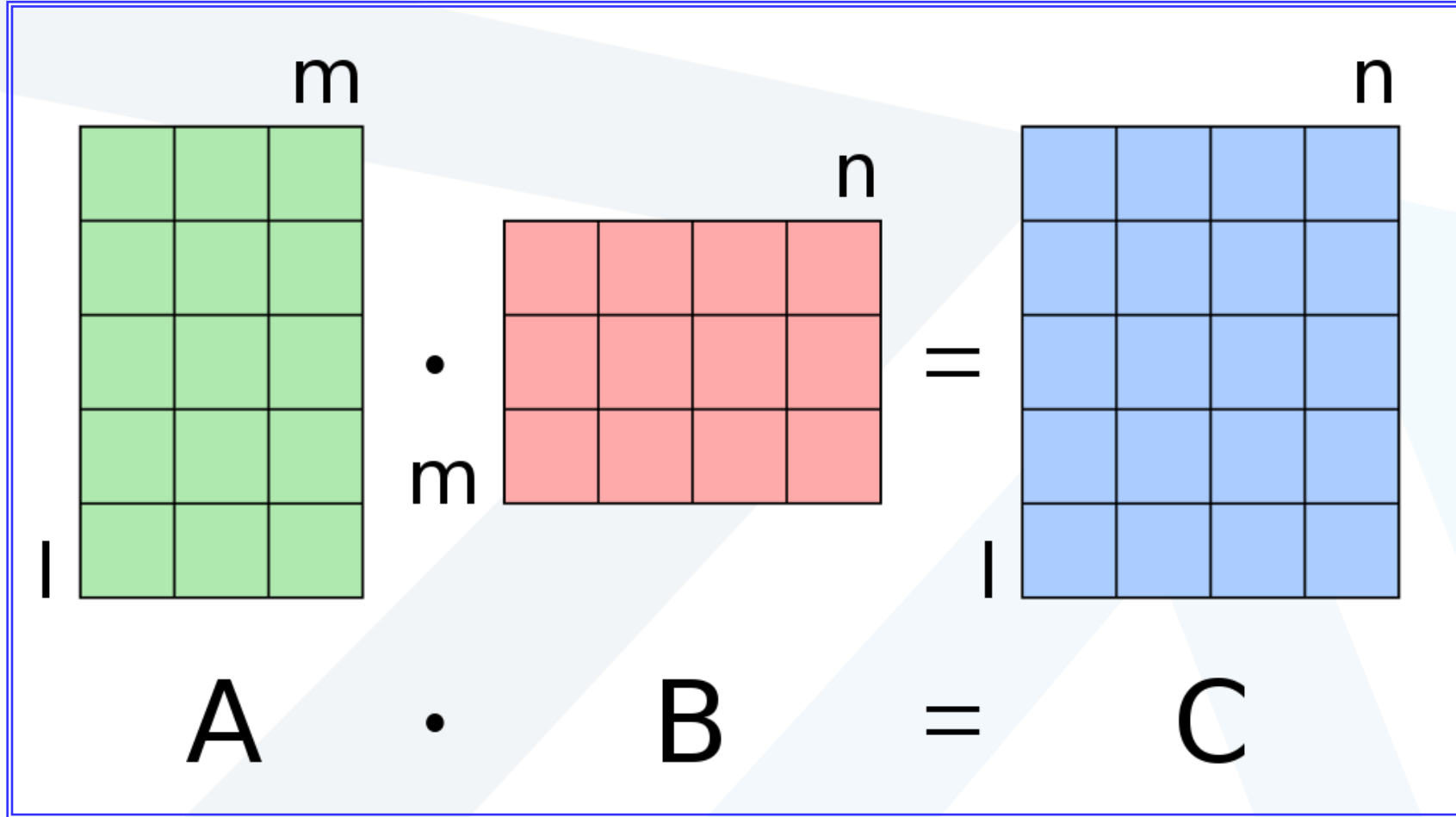
The addition of two matrices A and B

- * Defined only if they have the same size
- * $C = A + B \Leftrightarrow c_{ij} = a_{ij} + b_{ij} \quad \forall i, j$

Multiplication of two matrices $A(n \times m)$ and $B(p \times q)$

- * The product $C = AB$ is defined only if $m = p$
- * $C = AB \Leftrightarrow c_{ij} = \sum_{k=1}^m a_{ik} b_{kj} \quad \forall i, j$

Multiplying Matrices



Multiplying Matrices

$$\begin{array}{c}
 \text{A} \qquad \qquad \text{B} \\
 \left[\begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array} \right] \times \left[\begin{array}{cc} 5 & 6 \\ 7 & 8 \end{array} \right] = \left[\begin{array}{cc} 19 & 22 \\ 43 & 50 \end{array} \right]
 \end{array}$$

$$\begin{array}{l}
 1 \times 6 + 2 \times 8 = 22 \\
 1 \times 5 + 2 \times 7 = 19 \\
 3 \times 5 + 4 \times 7 = 43 \\
 3 \times 6 + 4 \times 8 = 50
 \end{array}$$

8 multiplications

DETERMINANT of MATRECSIS

Determinant Of A Matrix

⁺ a	⁺ b	⁺ c	⁻ a	⁻ b	⁻ c
d	e	f	d	e	f
g	h	i	g	h	i

$aei + bfg + cdh - afh - bdi - ceg$

DETERMINANT of MATRECSIS

Defined for square matrices only

Examples :

$$\det \begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 5 \\ -1 & 5 & 4 \end{bmatrix} = 2 \begin{vmatrix} 0 & 5 \\ 5 & 4 \end{vmatrix} - 1 \begin{vmatrix} 3 & -1 \\ 5 & 4 \end{vmatrix} - 1 \begin{vmatrix} 3 & -1 \\ 0 & 5 \end{vmatrix}$$

$$= 2(-25) - 1(12 + 5) - 1(15 - 0) = -82$$

Inverse of Matrices


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Inverse of 2x2 Matrix

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then

 $A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Inverse
of A

Determinant
of A

Adjoint
of A

Note: A^{-1} exists only when $ad - bc \neq 0$

Systems of Linear Equations

A system of linear equations can be presented
in different forms

$$\left. \begin{array}{l} 2x_1 + 4x_2 - 3x_3 = 3 \\ 2.5x_1 - x_2 + 3x_3 = 5 \\ x_1 - 6x_3 = 7 \end{array} \right\} \Leftrightarrow \begin{bmatrix} 2 & 4 & -3 \\ 2.5 & -1 & 3 \\ 1 & 0 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$$

Standard form Matrix form

Noncomputer Methods for Solving System of Equations

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Numerical Analysis

- For small number of equations ($n \leq 3$) linear equations can be solved readily by simple techniques such as “**method of elimination.**”
 - Linear algebra provides the tools to solve such systems of linear equations.
 - There are many ways to solve a system of linear equations:
 - Graphical Methods.
 - Cramer’s Rule.
 - Method of Elimination.
- For $n \leq 3$
- Nowadays, easy access to computers makes the solution of large sets of linear algebraic equations possible and practical.

Graphical Methods

- Consider a set of two equations:

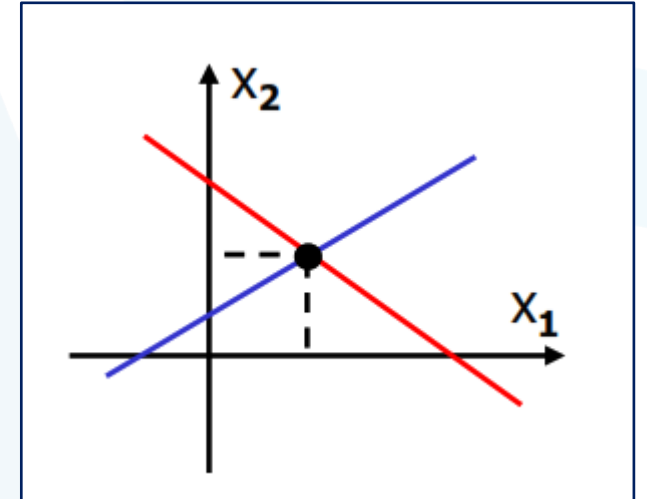
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

- Plot these on the Cartesian coordinate system with axes x_1 and x_2 .

Solve for (x_2)

$$x_2 = -\left(\frac{a_{11}}{a_{12}}\right)x_1 + \frac{b_1}{a_{12}} \Rightarrow x_2 = (\text{slope})x_1 + \text{intercept}$$

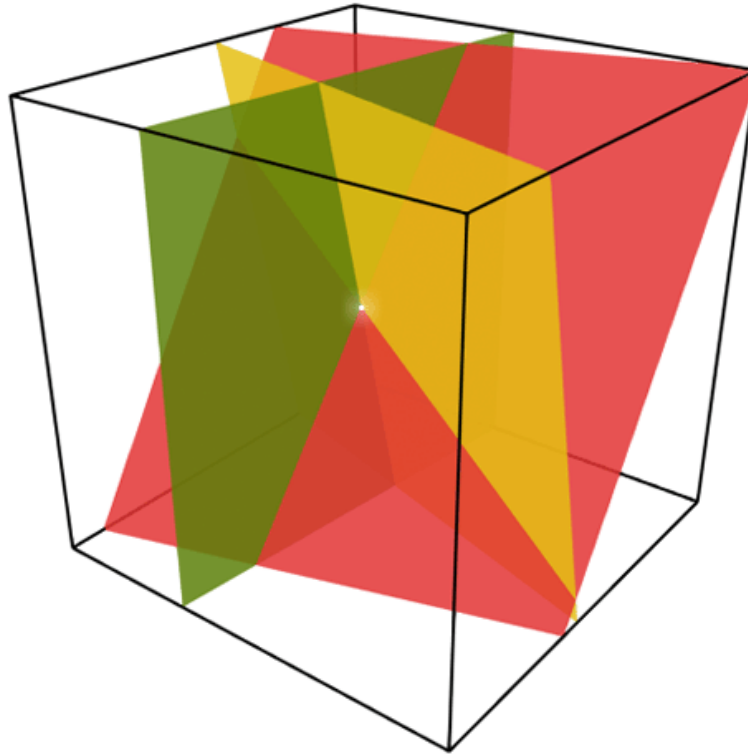
$$x_2 = -\left(\frac{a_{21}}{a_{22}}\right)x_1 + \frac{b_2}{a_{22}}$$



Graphical Methods

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- For $n=3$, each equation will be a plane on a 3D coordinate system. Solution is the point where these planes intersect.



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- For $n>3$, graphical solution is not practical.

Graphical Methods

- For example : solve

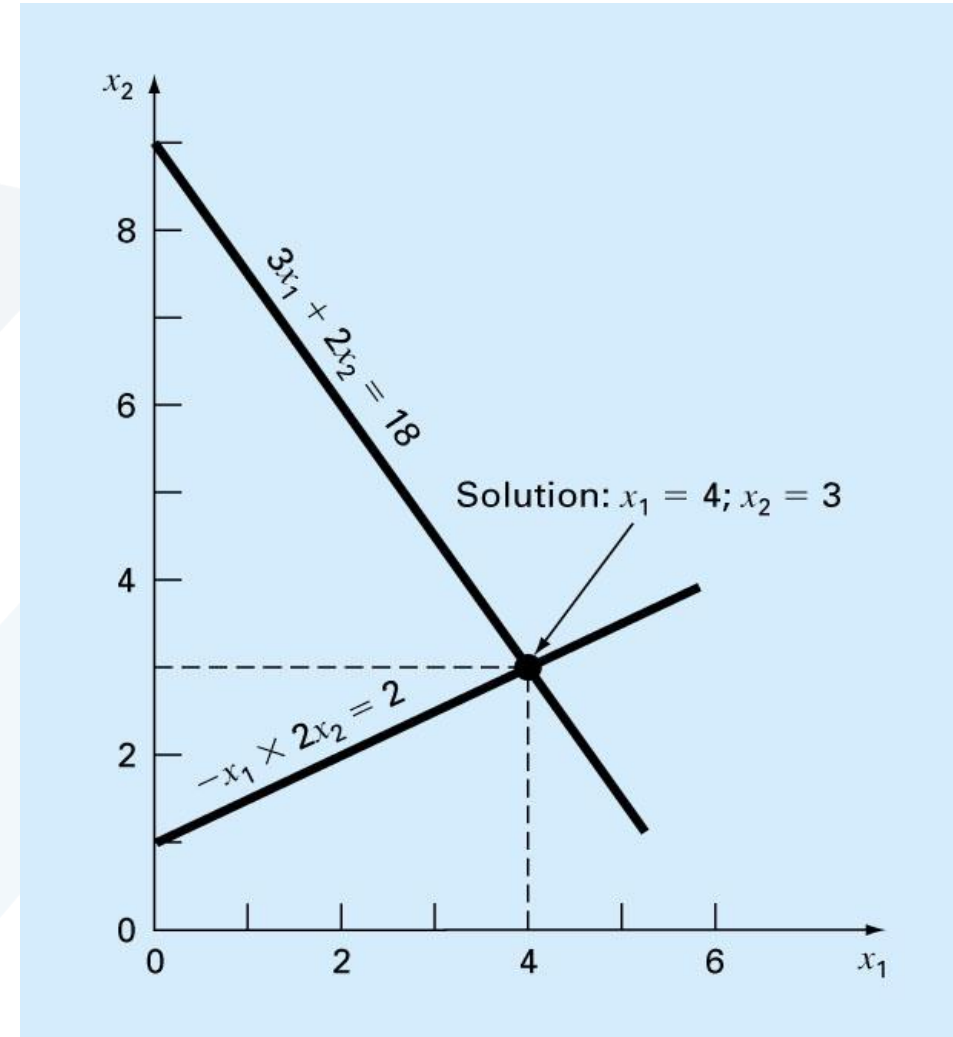
$$3x_1 + 2x_2 = 18$$

$$-x_1 + 2x_2 = 2$$

- The solution is the intersection of the two lines at $x_1=4$ and $x_2=3$.
- This result can be checked by substituting these values into the original equations to yield

$$3(4)+2(3)=18$$

$$-(4)+2(3) = 2$$



Cramer's Rule

- Cramer's rule is another solution technique that is best suited to small numbers of equations.
- This rule states that each unknown in a system of linear algebraic equations may be expressed as a fraction of two determinants with denominator D and with the numerator obtained from D by replacing the column of coefficients of the unknown in question by the constants b_1, b_2, \dots, b_n .

For a 3x3 system

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{D} \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{D} \quad x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{D}$$

Cramer's Rule- Example

- Example: Cramer's Rule can be used to solve the system

$$x_1 = \frac{\begin{vmatrix} 3 & 1 \\ 5 & 2 \\ 1 & 1 \\ 1 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}} = 1, \quad x_2 = \frac{\begin{vmatrix} 1 & 3 \\ 1 & 5 \\ 1 & 1 \\ 1 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}} = 2$$

$$x_1 + x_2 = 3$$

$$x_1 + 2x_2 = 5$$

Cramer's Rule is not practical for large systems .

To solve N by N system requires $(N + 1)(N - 1)N!$ multiplications.

To solve a 30 by 30 system, 2.38×10^{35} multiplications are needed.

It can be used if the determinants are computed in efficient way

Cramer's Rule- Example

- Example:

$$\begin{aligned}x_1 + x_2 - x_3 &= 1 \\x_1 + 2x_2 - 2x_3 &= 0 \\-2x_1 + x_2 + x_3 &= 1\end{aligned}$$

$$x_1 = \frac{\begin{vmatrix} 1 & 1 & -1 \\ 0 & 2 & -2 \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{vmatrix}} = \frac{4}{2} = 2$$

$$x_2 = \frac{\begin{vmatrix} 1 & 1 & -1 \\ 1 & 0 & -2 \\ -2 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{vmatrix}} = \frac{4}{2} = 2$$

$$x_3 = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ -2 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{vmatrix}} = \frac{6}{2} = 3$$

$$\{x\} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$$

The Elimination of Unknowns

- The elimination of unknowns by combining equations is an algebraic approach that can be illustrated for a set of two equations:

$$a_{11}x_1 + a_{12}x_2 = b_1 \quad (1)$$

$$a_{21}x_1 + a_{22}x_2 = b_2 \quad (2)$$

- Eq. (1) might be multiplied by a_{21} and Eq. (2) by a_{11} to give

$$a_{11}a_{21}x_1 + a_{12}a_{21}x_2 = b_1a_{21} \quad (3)$$

$$a_{21}a_{11}x_1 + a_{22}a_{11}x_2 = b_2a_{11} \quad (4)$$

Subtracting Eq. (3) from Eq. (4) will, therefore, eliminate the x_1 term from the equations to yield

$$a_{22}a_{11}x_2 - a_{12}a_{21}x_2 = b_2a_{11} - b_1a_{21}$$

The Elimination of Unknowns

- Which can be solved for:

$$x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}$$

$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} =$$

- x_2 can then be substituted into Eq. (1), which can be solved for:

$$x_1 = \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}$$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

The Elimination of Unknowns - Example

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Numerical Analysis

- Use the elimination to solve:

$$3x_1 + 2x_2 = 18$$

$$-x_1 + 2x_2 = 2$$

- The elimination of unknowns can be extended to systems with more than two or three equations. However, the numerous calculations that are required for larger systems make the method extremely tedious to implement by hand. However, as described in the next section, the technique can be formalized and readily programmed for the computer.

Naïve Gauss Elimination

- In the previous section, the elimination of unknowns was used to solve a pair of simultaneous equations. The procedure consisted of two steps:
 1. The equations were manipulated to eliminate one of the unknowns from the equations. The result of this **elimination** step was that we had one equation with one unknown.
 2. Consequently, this equation could be solved directly and the result **back-substituted** into one of the original equations to solve for the remaining unknown.
- This basic approach can be extended to large sets of equations by developing a systematic scheme or algorithm to eliminate unknowns and to back-substitute. Gauss elimination is the most basic of these schemes.

Naïve Gauss Elimination

- The method consists of two steps:
 - **Forward Elimination:** the system is reduced to **upper triangular form**. A sequence of **elementary operations** is used.
 - **Backward Substitution:** Solve the system starting from the last variable.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & 0 & a_{33}' \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2' \\ b_3' \end{bmatrix}$$

Naïve Gauss Elimination

The two phases of Gauss elimination:

Forward elimination and back substitution.

The primes indicate the number of times that the coefficients and constants have been modified.

$$\begin{array}{c} \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & c_1 \\ a_{21} & a_{22} & a_{23} & c_2 \\ a_{31} & a_{32} & a_{33} & c_3 \end{array} \right] \\ \Downarrow \\ \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & c_1 \\ & a'_{22} & a'_{23} & c'_2 \\ & & a''_{33} & c''_3 \end{array} \right] \\ \Downarrow \\ \begin{array}{l} x_3 = c''_3 / a''_{33} \\ x_2 = (c'_2 - a'_{23}x_3) / a'_{22} \\ x_1 = (c_1 - a_{12}x_2 - a_{13}x_3) / a_{11} \end{array} \end{array}$$

Forward elimination

Back substitution

Naïve Gauss Elimination - Example

Example

Use Gauss elimination to solve

$$x_1 - 2x_2 + 2x_3 = 1$$

$$2x_1 + x_2 - 3x_3 = -3$$

$$-3x_1 + x_2 - x_3 = 4$$

Carry six significant figures during the computation.

1	-2	2		1
2	1	-3		-3
-3	1	-1		4
1	-2	2		1
0	5	-7		-5
0	-5	5		7
1	-2	2		1
0	5	-7		-5
0	0	-2		2
			x3	-1
			x2	-2,4
			x1	-1,8

Naïve Gauss Elimination - Homework

Example: Solve the following system using Naïve Gauss Elimination.

$$\begin{array}{r} 6x_1 - 2x_2 + 2x_3 + 4x_4 = 16 \\ 12x_1 - 8x_2 + 6x_3 + 10x_4 = 26 \\ 3x_1 - 13x_2 + 9x_3 + 3x_4 = -19 \\ -6x_1 + 4x_2 + x_3 - 18x_4 = -34 \end{array}$$