Exercises 8: Inner Product Spaces

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(a) Determine whether the set of vectors in $\boldsymbol{R}^n$ is orthogonal, (b) if the set is orthogonal, then determine whether it is also orthonormal, and (c) determine whether the set is a basis for $\boldsymbol{R}^n$

① $\boldsymbol{u} = (2, -4),\ \boldsymbol{v} = (2, 1)$

(a) $(2,-4)\cdot(2,\ 1) = 2(2) + (-4)(1) = 0$, the set is orthogonal

(b) $\|(2,-4)\| = \sqrt{2^2 + (-4)^2} = \sqrt{20} \neq 1$, the set is not orthonormal

(c) 2 independents vectors, the set is a basis for R^2

② $\{\boldsymbol{u} = (3/5,\ 4/5),\ \boldsymbol{v} = (-4/5,\ 3/5)\}$

(a) $(\frac{3}{5}, \frac{4}{5})\cdot(-\frac{4}{5},\ \frac{3}{5}) = \frac{3}{5}(-\frac{4}{5}) + \frac{4}{5}(\frac{3}{5}) = 0$, the set is orthogonal

(b) $\|(\frac{3}{5}, \frac{4}{5})\| = \sqrt{(\frac{3}{5})^2 + (\frac{4}{5})^2} = 1,\quad \|(-\frac{4}{5}, \frac{3}{5})\| = \sqrt{(-\frac{4}{5})^2 + (\frac{3}{5})^2} = 1$, the set is orthonormal

(c) 2 independents vectors, the set is a basis for R^2

$$\left\|\left(\frac{3}{5},\frac{4}{5}\right)\right\| = \sqrt{\left(\frac{3}{5}\right)^2+\left(\frac{4}{5}\right)^2} = 1, \quad \left\|\left(-\frac{4}{5},\frac{3}{5}\right)\right\| = \sqrt{\left(-\frac{4}{5}\right)^2+\left(\frac{3}{5}\right)^2} = 1$$

3 $$\left\{ \boldsymbol{u} = \left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right), \boldsymbol{v} = \left(-\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6}\right), \boldsymbol{w} = \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right) \right\}$$

(a) The set is orthogonal because

$$\left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right) \cdot \left(-\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6}\right) = -\frac{\sqrt{12}}{12} + 0 + \frac{\sqrt{12}}{12} = 0$$

$$\left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right) \cdot \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right) = \frac{\sqrt{6}}{6} + 0 - \frac{\sqrt{6}}{6} = 0$$

$$\left(-\frac{\sqrt{6}}{6},\frac{\sqrt{6}}{3},\frac{\sqrt{6}}{6}\right)\cdot\left(\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},-\frac{\sqrt{3}}{3}\right)=-\frac{\sqrt{18}}{18}+\frac{\sqrt{18}}{9}-\frac{\sqrt{18}}{18}=0$$

(b) The set is orthonormal because

$$\left\|\left(\frac{\sqrt{2}}{2},0,\frac{\sqrt{2}}{2}\right)\right\|=\sqrt{\frac{2}{4}+0+\frac{2}{4}}=1,\quad \left\|\left(\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},-\frac{\sqrt{3}}{3}\right)\right\|=\sqrt{\frac{3}{9}+\frac{3}{9}+\frac{3}{9}}=1$$

$$\left\|\left(-\frac{\sqrt{6}}{6},\frac{\sqrt{6}}{3},\frac{\sqrt{6}}{6}\right)\right\|=\sqrt{\frac{6}{36}+\frac{6}{9}+\frac{6}{36}}=1$$

(c) 3 independents vectors, the set is a basis for R^3

④ $\left\{ \boldsymbol{u} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right), \boldsymbol{v} = (0,1,0) \right\}$

(a) The set is orthogonal because $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \cdot (0,1,0) = 0$

(b) The set is orthonormal because

$$\left\| \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \right\| = \sqrt{\frac{1}{2} + 0 + \frac{1}{2}} = 1, \quad \|(0,1,0)\| = \sqrt{0+1+0} = 1$$

(c) Only 2 independents vectors, the set is not a basis for R^3

(a) Show that the set of vectors in R^n is orthogonal, and (b) normalize the set to produce an orthonormal set

① $\boldsymbol{u} = (-1, 4),\ \boldsymbol{v} = (8, 2)$

(a) $(-1,4)\cdot(8,\ 2) = -1(8)+4(2) = 0$, the set is orthogonal

(b) $\|(-1,4)\| = \sqrt{(-1)^2+4^2} = \sqrt{17} \neq 1$, the set is not orthonormal. Normalizing

$$\boldsymbol{u}_1 = \frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|} = \frac{1}{\sqrt{17}}(-1,4) = \left(-\frac{\sqrt{17}}{17}, \frac{4\sqrt{17}}{17}\right)$$

$$\boldsymbol{u}_2 = \frac{\boldsymbol{v}_2}{\|\boldsymbol{v}_2\|} = \frac{1}{2\sqrt{17}}(8,2) = \left(\frac{4\sqrt{17}}{17}, \frac{\sqrt{17}}{17}\right)$$

② $\left\{v_1=\left(\sqrt{3},\sqrt{3},\sqrt{3}\right),\ v_2=\left(-\sqrt{2},0,\sqrt{2}\right)\right\}$

(a) $(\sqrt{3},\sqrt{3},\sqrt{3})\cdot(-\sqrt{2},0,\sqrt{2})=-\sqrt{6}+0+\sqrt{6}=0$, the set is orthogonal

(b) $\left\|(\sqrt{3},\sqrt{3},\sqrt{3})\right\|=\sqrt{\sqrt{3}^2+\sqrt{3}^2+\sqrt{3}^2}=\sqrt{9}=3\neq 1$

the set is not orthonormal. Normalizing

$$u_1=\frac{1}{3}(\sqrt{3},\sqrt{3},\sqrt{3})=\left(\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}\right)$$

$$u_2=\frac{1}{2}(-\sqrt{2},0,\sqrt{2})=\left(-\frac{\sqrt{2}}{2},0,\frac{\sqrt{2}}{2}\right)$$

Find the coordinate matrix of w relative to the orthonormal basis $\boldsymbol{B}$ in $\boldsymbol{R}^n$

① $w = (-3, 4),\ B = \left\{ \left(\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5} \right), \left(-\frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5} \right) \right\}$

$$(-3,4) \cdot \left(\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5} \right) = \frac{-3\sqrt{5}}{5} + \frac{8\sqrt{5}}{5} = \frac{5\sqrt{5}}{5} = \sqrt{5}$$

$$(-3,4) \cdot \left(-\frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5} \right) = \frac{6\sqrt{5}}{5} + \frac{4\sqrt{5}}{5} = \frac{10\sqrt{5}}{5} = 2\sqrt{5}$$

$$\text{So, } [w]_B = \begin{bmatrix} \sqrt{5} \\ 2\sqrt{5} \end{bmatrix}$$

② $$w=(2,-2,1),\ B=\left\{\left(\frac{\sqrt{10}}{10},0,\frac{3\sqrt{10}}{10}\right),(0,1,0),\left(-\frac{3\sqrt{10}}{10},0,\frac{\sqrt{10}}{10}\right)\right\}$$

$$(2,-2,1)\cdot\left(\frac{\sqrt{10}}{10},0,\frac{3\sqrt{10}}{10}\right)=\frac{2\sqrt{10}}{10}+0+\frac{3\sqrt{10}}{10}=\frac{5\sqrt{10}}{10}=\frac{\sqrt{10}}{2}$$

$$(2,-2,1)\cdot(0,1,0)=0-2+0=-2$$

$$(2,-2,1)\cdot\left(-\frac{3\sqrt{10}}{10},0,\frac{\sqrt{10}}{10}\right)=-\frac{6\sqrt{5}}{5}+0+\frac{\sqrt{10}}{10}=-\frac{5\sqrt{5}}{10}=-\frac{\sqrt{10}}{2}$$

So, $$[w]_B=\begin{bmatrix}\frac{\sqrt{10}}{2}\\ -2\\ -\frac{\sqrt{10}}{2}\end{bmatrix}$$

Apply the Gram-Schmidt orthonormalization process to transform the given basis for $\boldsymbol{R}^n$ into an orthonormal basis

① $\boldsymbol{v}_1 = (3, 4),\ \boldsymbol{v}_2 = (1, 0)$

$$\boldsymbol{w}_1 = \boldsymbol{v}_1 = (3, 4)$$

$$\boldsymbol{w}_2 = \boldsymbol{v}_2 - \frac{\boldsymbol{v}_2 \cdot \boldsymbol{w}_1}{\boldsymbol{w}_1 \cdot \boldsymbol{w}_1}\boldsymbol{w}_1 = (1,0) - \frac{1(3)+0(4)}{3^2+4^2}(3,4) = \left(\frac{16}{25}, -\frac{12}{25}\right)$$

Normalize the vectors

$$\boldsymbol{u}_1 = \frac{\boldsymbol{w}_1}{\|\boldsymbol{w}_1\|} = \frac{1}{\sqrt{3^2+4^2}}(3,4) = \left(\frac{3}{5}, \frac{4}{5}\right)$$

$$\boldsymbol{u}_2 = \frac{\boldsymbol{w}_2}{\|\boldsymbol{w}_2\|} = \frac{1}{\sqrt{\left(\frac{16}{25}\right)^2 + \left(-\frac{12}{25}\right)^2}}\left(\frac{16}{25}, -\frac{12}{25}\right) = \left(\frac{4}{5}, -\frac{3}{5}\right)$$

② $B = \{v_1 = (1, -2, 2),\ v_2 = (2, 2, 1),\ v_3 = (2, -1, -2)\}$

Because $v_i \cdot v_j = 0$ for $i \neq j$, the given vectors are orthogonal. Normalize the vectors

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{1^2 + (-2)^2 + 2^2}}(1, -2, 2) = \left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right)$$

$$u_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{2^2 + 2^2 + 1^2}}(2, 2, 1) = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$$

$$u_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{1^2 + (-2)^2 + 2^2}}(2, -1, -2) = \left(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right)$$

③ $B = \{v_1 = (4, -3, 0),\ v_2 = (1, 2, 0),\ v_3 = (0, 0, 4)\}$

Orthogonalize each vector in B vectors

$$w_1 = v_1 = (4, -3, 0)$$

$$w_2 = v_2 - \frac{v_2 \cdot w_1}{w_1 \cdot w_1} w_1 = (1,2,0) - \tfrac{2}{25}(4,-3,0) = \left(\tfrac{33}{25}, \tfrac{44}{25}, 0\right)$$

$$w_3 = v_3 - \frac{v_3 \cdot w_1}{w_1 \cdot w_1} w_1 - \frac{v_3 \cdot w_2}{w_2 \cdot w_2} w_2 = (0,0,4) - 0(4,-3,0) - 0\left(\tfrac{33}{25}, \tfrac{44}{25}, 0\right) = (0,0,4)$$

Normalize the vectors

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{4^2 + (-3)^2 + 0^2}}(4,-3,0) = \left(\frac{4}{5}, -\frac{3}{5}, 0\right)$$

$$u_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{(\tfrac{33}{25})^2 + (\tfrac{44}{25})^2 + 0^2}}(\tfrac{33}{25}, \tfrac{44}{25}, 0) = \left(\frac{3}{5}, \frac{4}{5}, 0\right)$$

$$u_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{0^2 + 0^2 + 4^2}}(0,0,4) = (0,0,1)$$

(a) Determine whether the set of vectors in $\boldsymbol{R}^n$ is orthogonal, (b) if the set is orthogonal, then determine whether it is also orthonormal, and (c) determine whether the set is a basis for $\boldsymbol{R}^n$

1. $\{\boldsymbol{u} = (-3, 5), \boldsymbol{v} = (4, 0)\}$
2. $\{\boldsymbol{u} = (2, 1), \boldsymbol{v} = (1/3, -2/3)\}$
3. $\left\{\boldsymbol{u} = \left(\frac{\sqrt{2}}{3}, 0, -\frac{\sqrt{2}}{6}\right), \boldsymbol{v} = \left(0, \frac{2\sqrt{5}}{5}, -\frac{\sqrt{5}}{5}\right), \boldsymbol{w} = \left(\frac{\sqrt{5}}{5}, 0, \frac{1}{2}\right)\right\}$
4. $\{(2, 5, -3), (4, 2, 6)\}$

(a) Show that the set of vectors in $\boldsymbol{R}^n$ is orthogonal, and (b) normalize the set to produce an orthonormal set

1. $\{\boldsymbol{v}_1 = (-1, 3),\ \boldsymbol{v}_2 = (12, 4)\}$
2. $\{\boldsymbol{v}_1 = (1, 3, 1),\ \boldsymbol{v}_2 = (3, 0, -3)\}$

Find the coordinate matrix of $\boldsymbol{w}$ relative to the orthonormal basis $\boldsymbol{B}$ in $\boldsymbol{R}^n$

1. $\boldsymbol{w} = (1, 2),\ B = \left\{ \left(-\frac{2\sqrt{13}}{13}, \frac{3\sqrt{13}}{13} \right), \left(\frac{3\sqrt{13}}{13}, \frac{2\sqrt{13}}{13} \right) \right\}$
2. $\boldsymbol{w} = (1, -3, 2),\ B = \left\{ \left(\frac{\sqrt{10}}{10}, 0, \frac{3\sqrt{10}}{10} \right), (0, 1, 0), \left(-\frac{3\sqrt{10}}{10}, 0, \frac{\sqrt{10}}{10} \right) \right\}$

Apply the Gram-Schmidt orthonormalization process to transform the given basis for R^n into an orthonormal basis

1. $B = \{v_1 = (4, -3),\ v_2 = (3, 2)\}$
2. $B = \{v_1 = (2, 1, -2),\ v_2 = (1, 2, 2),\ v_3 = (2, -2, 1)\}$
3. $B = \{v_1 = (1, 0, 0),\ v_2 = (1, 1, 1),\ v_3 = (1, 1, -1)\}$