General Case Let us now consider a bar with only a *slight* taper (See Example 1). In this case the normal stress may be calculated from $\sigma = N/A$, with a sufficient accuracy.



But the area A and the stress σ depend on the location along the axis. If volume forces act in the direction of the axis in addition to the concentrated forces, then the normal force N also depends on the location. Introducing the coordinate x in the direction of the axis we can write:

$$\sigma(x) = \frac{N(x)}{A(x)}$$

Here it is also assumed that the stress is uniformly distributed over the cross section at x.

In applications structures have to be designed in such a way that a given maximum stressing is not exceeded. In the case of a bar this requirement means that the absolute value of the stress σ must not exceed a given *allowable stress* σ_{allow}: $|\sigma| \leq \sigma_{allow}$. The required cross section A_{req} of a bar for a given load and thus a known normal force N can then be determined from :

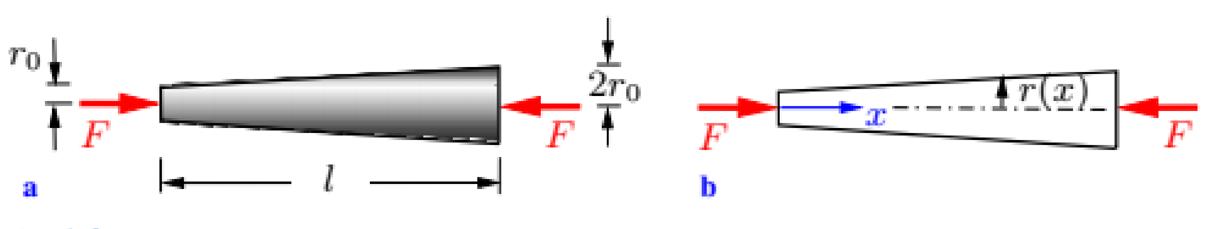
$$A_{req} = |N|/\sigma_{allow}$$

This is referred to as *dimensioning* of the bar. Alternatively, the allowable load can be calculated from $|N| \leq A\sigma_{allow}$ in the case of a given cross-sectional area A.

Note that a slender bar which is subjected to compression may fail due to buckling before the stress attains an inadmissibly large value. We will investigate buckling problems in Mechanics of Materials 2.

Example 1

A bar (length l) with a circular cross section and a slight taper (linearly varying from radius r_0 to $2r_0$) is subjected to the compressive forces F as shown in Fig.a. Determine the normal stress σ in an arbitrary cross section perpendicular to the axis of the bar.



Solution

We introduce the coordinate x, see Fig. b. Then the radius of an arbitrary cross section is given by

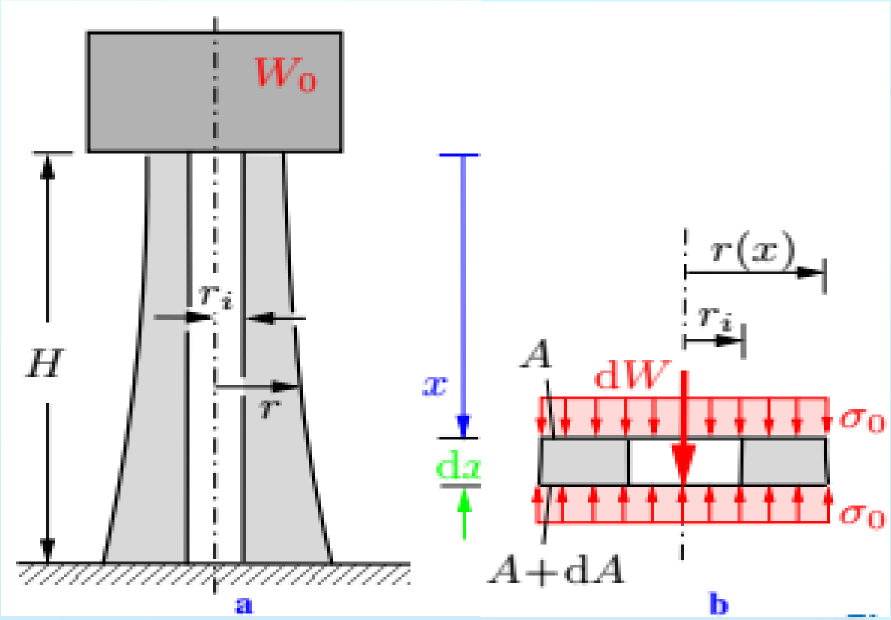
$$r(x) = r_0 + \frac{r_0}{l} x = r_0 (1 + \frac{x}{l})$$

Using $\sigma = N/A$ with the cross section $A(x) = \pi\, r^2(x)$ and the constant normal force $N = -F$, yields

$$\sigma(x) = \frac{N}{A(x)} = \frac{-F}{\pi r_0^2 (1 + \frac{x}{l})^2}$$

The minus sign indicates that σ is a compressive stress. Its value at the left end ($x = 0$) is four times the value at the right end ($x = l$).

Example 2 A water tower (height H, density ρ) with a cross section in the form of a circular ring carries a tank (weight W_0) as shown in Fig. a. The inner radius r_i of the ring is constant. Determine the outer radius r in such a way that the normal stress σ_0 in the tower is constant along its height. The weight of the tower cannot be neglected.

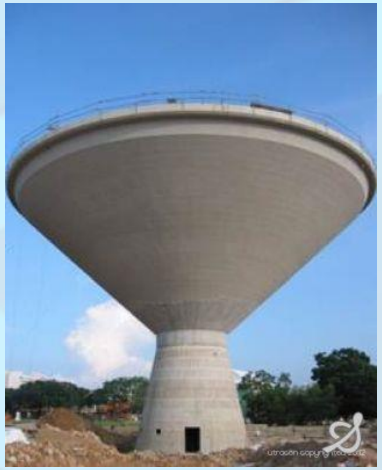


Solution: Consider the tower to be a slender bar. The relation between stress, normal force and cross-sectional area is given by $\sigma = N/A$.

In this example the constant compressive stress $\sigma = \sigma_0$ is given; the normal force (here counted positive as compressive force) and the area A are unknown.

The equilibrium condition furnishes a second equation.

We introduce the coordinate x as shown in Fig.b and consider a slice element of length dx. The cross-sectional area of the circular ring as a function of x is: $A = \pi(r^2 - r_i^2)$ where $r = r(x)$ is the unknown outer radius. The normal force at the location x is given by $N = \sigma_0 A$. At the location $x + dx$, the area and the normal force are $A + dA$ and $N + dN = \sigma_0(A + dA)$.

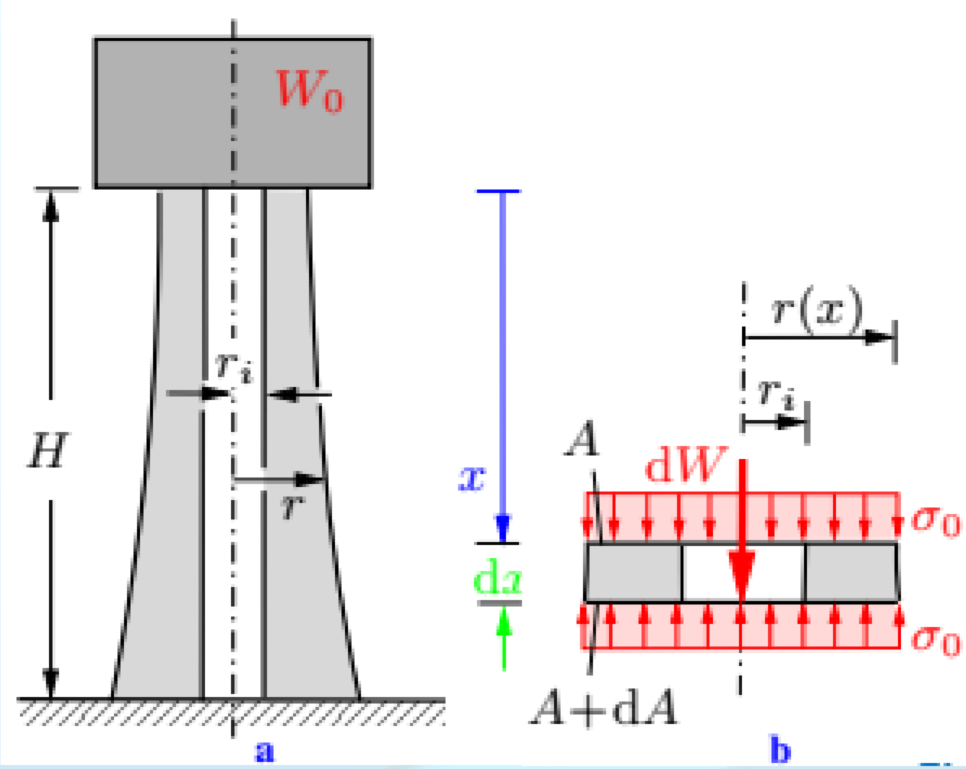


The weight of the element is $dW = \rho g\, dV$ where $dV = A\, dx$ is the volume of the element. Note that terms of higher order are neglected. Equilibrium in the vertical direction yields

$$\uparrow:\ \sigma_0(A + dA) - \rho g dV - \sigma_0 A = 0 \ \Rightarrow\ \sigma_0 dA - \rho g A dx = 0\,.$$

Separation of variables and integration lead to

$$\int \frac{dA}{A} = \int \frac{\rho g}{\sigma_0} dx \ \Rightarrow \ln\frac{A}{A_0} = \frac{\rho g x}{\sigma_0} \ \Rightarrow A = A_0 e^{\frac{\rho g x}{\sigma_0}}\,.$$



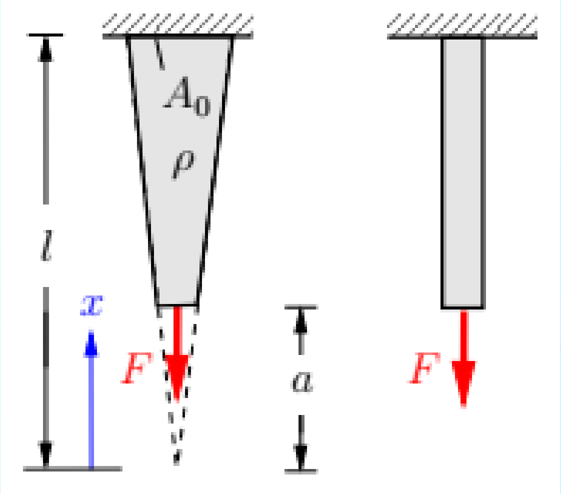
The constant of integration A_0 follows from the condition that the stress at the upper end of the tower (for $x = 0$ we have $N = W_0$) also has to be equal to σ_0:

$$\frac{W_0}{A_0} = \sigma_0 \ \Rightarrow\ A_0 = \frac{W_0}{\sigma_0}.$$

Substituting this into the above two Equations yield the outer radius:

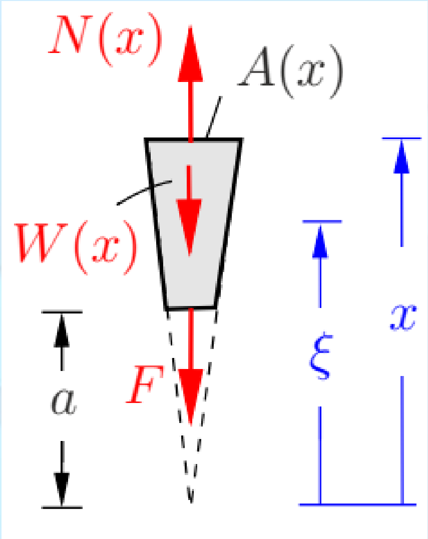
$$r^2(x) = r_i^2 + \frac{W_0}{\pi\sigma_0} e^{\frac{\rho g x}{\sigma_0}}$$

Example 3 A slender bar (density ρ) is suspended from its upper end as shown in Fig. It has a rectangular cross section with a constant depth and a linearly varying width. The cross section at the upper end is A_0.
Determine the stress $\sigma(x)$ due to the force F and the weight of the bar. Calculate the minimum stress $\sigma_{\min}$ and its location.



Solution It is reasonable to introduce the x-coordinate at the intersection of the extended edges of the trapezoid. The x dependent cross section area follows then as: $A(x) = A_0 x/h$

With the weight: $W(x) = \rho g V(x) = \rho g \int_a^x A(\xi) d\xi = \rho g A_0 \dfrac{x^2 - a^2}{2h}$



of the lower part equilibrium provides:

$$N(x) = F + W(x) = F + \rho g A_0 \frac{x^2 - a^2}{2h}$$

This leads to the stress $$\sigma(x) = \frac{N(x)}{A(x)} = \frac{Fh + \frac{1}{2}\rho g A_0 (x^2 - a^2)}{A_0 x}$$

The location x^* of the minimum is determined by condition: $\frac{d\sigma}{dx} = 0$

$$\frac{d\sigma}{dx} = \frac{\rho g A_0 x(A_0 x) - A_0 F h - \frac{1}{2}\rho g A_0^2(x^2 - a^2)}{A_0^2 x^2} = \frac{\rho g A_0(x^2 + a^2) - 2Fh}{2A_0 x^2}$$

$$\frac{d\sigma}{dx} = 0 \Rightarrow \frac{\rho g A_0(x^2 + a^2) - 2Fh}{2A_0 x^2} = 0 \Rightarrow x^* = \sqrt{\frac{2Fh}{\rho g A_0} - a^2}$$

Where the value of the minimum stress is

$$\sigma_{min} = \sigma(x^*) = \rho g \sqrt{\frac{2Fh}{\rho g A_0} - a^2} = \rho g x^*$$

Note:

- For $\rho g = 0$ ("weightless bar") no minimum exists. The largest stress occurs at $x = a$.

- The minimum will be located within the bar, only if $a < x^* < h$ or $\rho g A_0 a^2/(2h) < F < \rho g A_0(h^2 + a^2)/(2h)$ holds.