المحاضرة الخامسة

ENGINEERING ECONOMY

MONEY-TIME[2] RELATIONSHIPS AND EQUIVALENCE

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INTEREST FORMULAS FOR ALL OCCASIONS

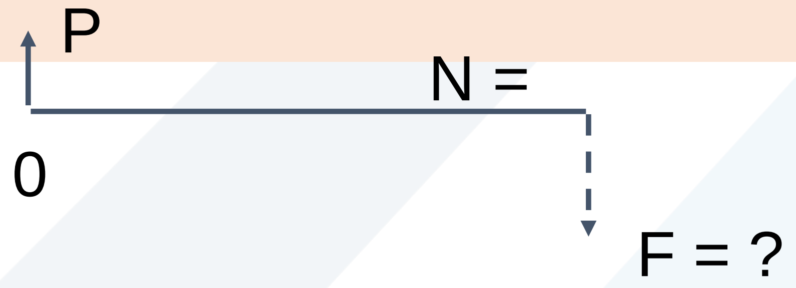
- relating present and future values of single cash flows;
- relating a uniform series (annuity) to present and future equivalent values;
  - for discrete compounding and discrete cash flows;
  - for deferred annuities (uniform series);
- equivalence calculations involving multiple interest;
- relating a uniform gradient of cash flows to annual and present equivalents;
- relating a geometric sequence of cash flows to present and annual equivalents;

INTEREST FORMULAS FOR ALL OCCASIONS

- relating nominal and effective interest rates;
- relating to compounding more frequently than once a year;
- relating to cash flows occurring less often than compounding periods;
- for continuous compounding and discrete cash flows;
- for continuous compounding and continuous cash flows;

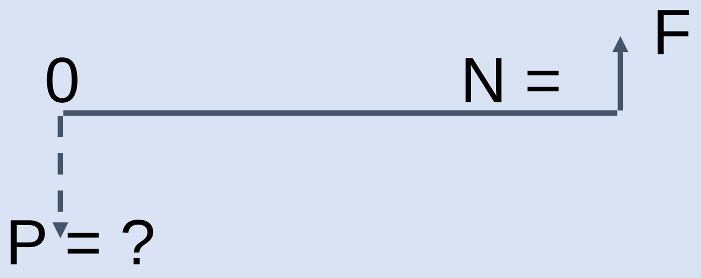
RELATING PRESENT AND FUTURE EQUIVALENT VALUES OF SINGLE CASH FLOWS

- Finding F when given P:
- Finding future value when given present value
- $F = P\ (1+i)^{N}$
  - $(1+i)^N$ single payment compound amount factor
  - functionally expressed as F = (F / P, i%,N)
  - predetermined values of this are presented in column 2 of Appendix C of text.



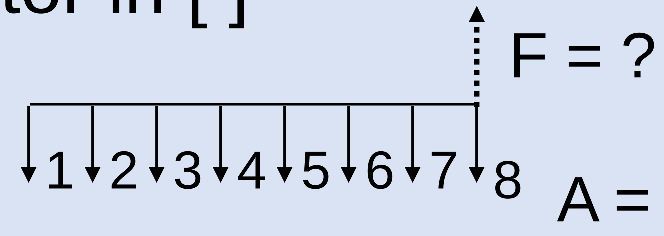
RELATING PRESENT AND FUTURE EQUIVALENT VALUES OF SINGLE CASH FLOWS

- Finding P when given F:
- Finding present value when given future value
- $P = F\ [1 / (1 + i)]^{N}$
  - $(1+i)^{-N}$ single payment present worth factor
  - functionally expressed as P = F (P / F, i%, N)
  - predetermined values of this are presented in column 3 of Appendix C of text;



RELATING A UNIFORM SERIES (ORDINARY ANNUITY) TO PRESENT AND FUTURE EQUIVALENT VALUES

- Finding F given A:
- Finding future equivalent income (inflow) value given a series of uniform equal Payments
- $F = A \left[\dfrac{(1+i)^N - 1}{1} \right]$
- functionally expressed as F = A (F / A,i%,N)
- uniform series compound amount factor in [ ]
  - predetermined values are in column 4 of Appendix C of text

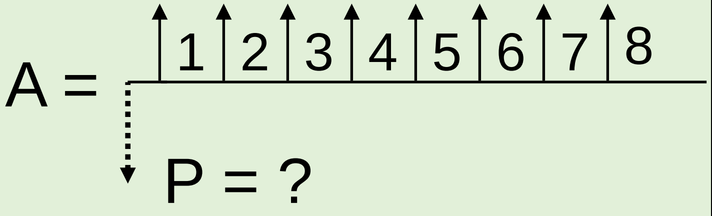


$$(F / A,i\%,N) = (P / A,i,N) (F / P,i,N)$$

$$(F / A,i\%,N) = (F / P,i,N-k)$$

RELATING A UNIFORM SERIES (ORDINARY ANNUITY) TO PRESENT AND FUTURE EQUIVALENT VALUES

- Finding P given A:
- Finding present equivalent value given a series of uniform equal receipts
- $$P = A \left[\frac{(1+i)^N - 1}{i(1+i)^N} \right]$$
  - uniform series present worth factor in [ ]
  - functionally expressed as P = A (P / A,i%,N)
  - predetermined values are in column 5 of Appendix C of text



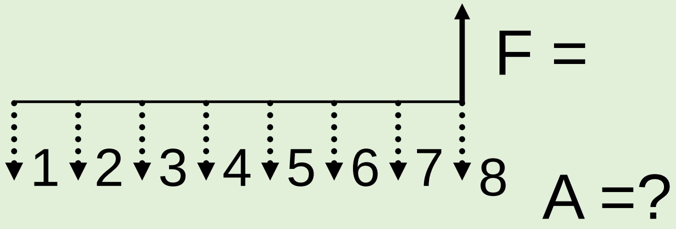
$$(P / A, i\%, N) = (P / F, i, k)$$

RELATING A UNIFORM SERIES (ORDINARY ANNUITY) TO PRESENT AND FUTURE EQUIVALENT VALUES

- Finding A given F:
- Finding amount A of a uniform series when given the equivalent future value

$$A = F\left[\frac{i}{(1+i)^N - 1}\right]$$

  - sinking fund factor in [ ]
  - functionally expressed as A = F (A / F,i%,N)
  - predetermined values are in column 6 of Appendix C of text



$$(A / F,i\%,N) = 1 / (F / A,i\%,N)$$

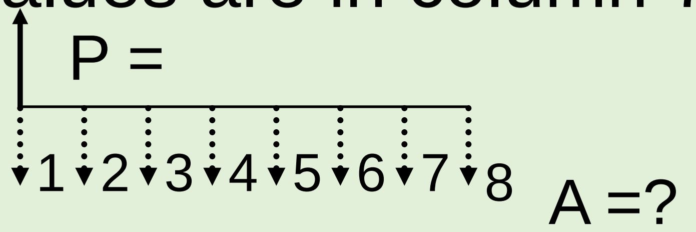
$$(A / F,i\%,N) = (A / P,i\%,N) - i$$

RELATING A UNIFORM SERIES (ORDINARY ANNUITY) TO PRESENT AND FUTURE EQUIVALENT VALUES

- Finding A given P:
- Finding amount A of a uniform series when given the equivalent present value

$$A = P\left[\frac{i(1+i)^N}{(1+i)^N - 1}\right]$$

  - – capital recovery factor in [ ]
  - – functionally expressed as A = P (A / P,i%,N)
  - – predetermined values are in column 7 of Appendix C of text



$$(A / P, i\%, N) = 1 / (P / A, i\%, N)$$

RELATING A UNIFORM GRADIENT OF CASH FLOWS TO ANNUAL AND PRESENT EQUIVALENTS

- Find P when given G:
- Find the present equivalent value when given the uniform gradient amount
- $$P = G\left\{\frac{1}{i}\left[\frac{(1+i)^N - 1}{i(1+i)^N} - \frac{N}{(1+i)^N}\right]\right\}$$
- Functionally represented as P = G (P / G, i%,N)
- The value shown in{ } is the gradient to present equivalent conversion factor and is presented in column 8 of Appendix C (represented in the above parenthetical expression).

RELATING GE0METRIC SEQUENCE OF CASH FLOWS TO PRESENT AND ANNUAL EQUIVALENTS

Projected cash flow patterns changing at an average rate of *f* each period;

Resultant end-of-period cash-flow pattern is referred to as a geometric gradient series;

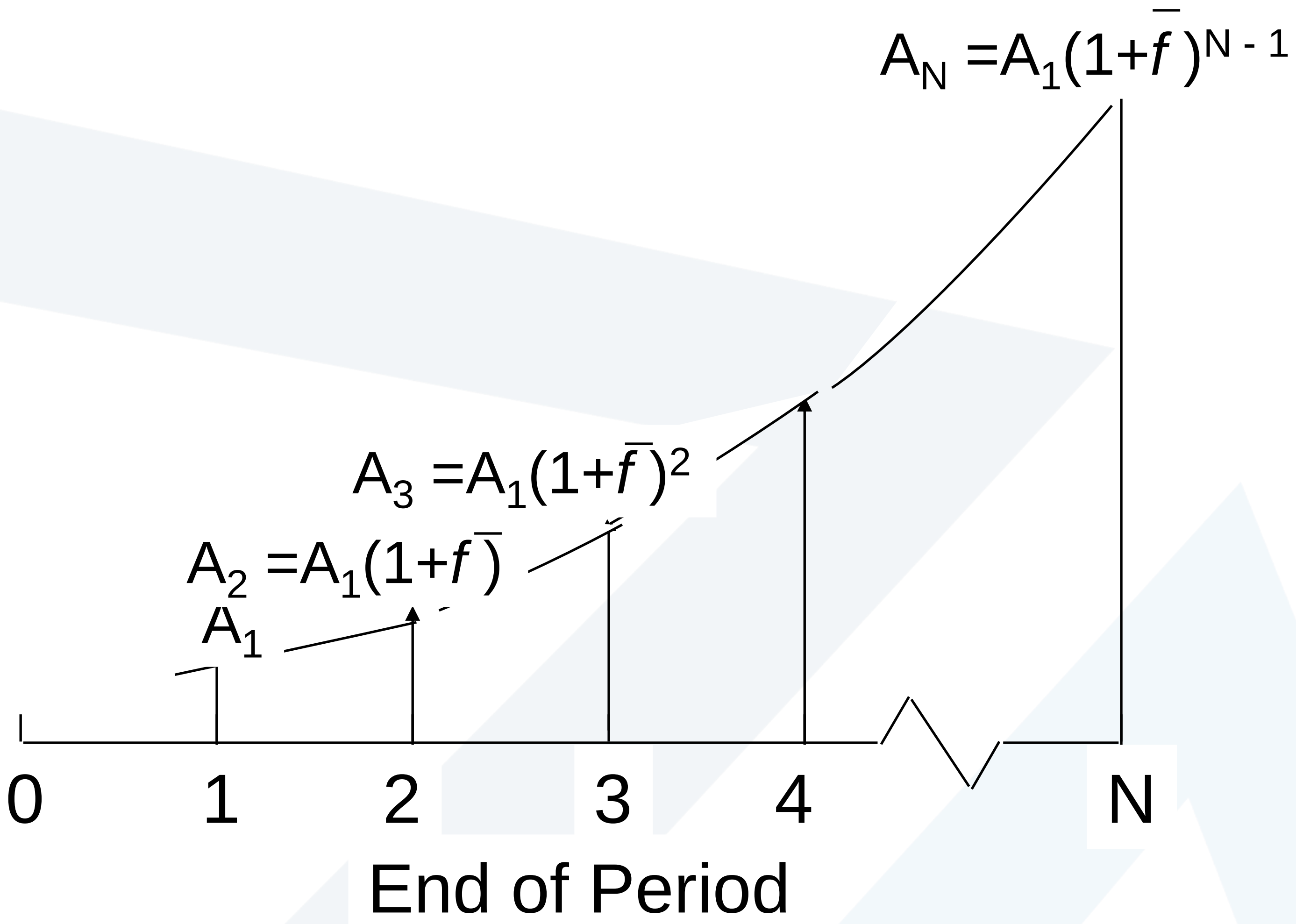
A_1 is cash flow at end of period 1

$A_k = (A_{k-1})(1 + f), 2 < k < N$

$A_N = A_1 (1 + f)^{N-1}$

$f = (A_k - A_{k-1}) / A_{k-1}$

f may be either positive *or* negative



Cash-flow diagram for a Geometric Sequence of Cash Flows

RELATING A GEOMETRIC SEQUENCE OF CASH FLOWS TO ANNUAL AND PRESENT EQUIVALENTS

- Find P when given A:
- Find the present equivalent value when given the annual equivalent value (i = *f*)

$$P = \frac{A_1[1 - (1+i)^{-N}(1+f)^{N}]}{1 - f}$$

which may also be written as

$$P = \frac{A_1[1 - (P/F,i\%,N)(F/P,f\%,N)]}{i - f}$$

RELATING A GEOMETRIC SEQUENCE OF CASH FLOWS TO ANNUAL AND PRESENT EQUIVALENTS

- Note that the foregoing is mathematically equivalent to the following ($i \neq \bar{f}$):

$$P = \frac{A_1}{1+\bar{f}} \left(P/A\ \frac{1+i}{1+\bar{f}} - 1,\ N \right)$$

RELATING A GEOMETRIC SEQUENCE OF CASH FLOWS TO ANNUAL AND PRESENT EQUIVALENTS

- The foregoing may be functionally represented as A = P (A / P, i%,N)
- The year zero “base” of annuity, increasing at constant rate *f* % is A_0 = P (A / P, f %, N)
- The future equivalent of this geometric gradient is F = P (F / P, i%, N)

RELATING A GEOMETRIC SEQUENCE OF CASH FLOWS TO ANNUAL AND PRESENT EQUIVALENTS

- Find P when given A:
- Find the present equivalent value when given the annual equivalent value (i = *f*)

  $P = A_1N$ (i+i)-1 *which may be written as*

  $P = A_1N$ (P/F,i%,1)

  Functionally represented as A = P (A / P, i%,N)
- The year zero “base” of annuity, increasing at constant rate *f* % is A_0 = P (A / P, f %, N)
- The future equivalent of this geometric gradient is F = P (F / P, i%, N)

INTEREST RATES THAT VARY WITH TIME

Find P given F and interest rates that vary over N

Find the present equivalent value given a future value and a varying interest rate over the period of the loan

$$P = \frac{F_N}{\prod^{N} (1 + i_k)}$$

NOMINAL AND EFFECTIVE INTEREST RATES

Nominal Interest Rate - r - For rates compounded more frequently than one year, the stated annual interest rate.

Effective Interest Rate - i - For rates compounded more frequently than one year, the actual amount of interest paid.

$$i = (1 + r / M)^{M} - 1 = (F / P, r / M, M) - 1$$

M - the number of compounding periods per year

Annual Percentage Rate - APR - percentage rate per period times number of periods.

$$APR = r \times M$$

COMPOUNDING MORE OFTEN THAN ONCE A YEAR

- <u>Single Amounts</u>
- Given nominal interest rate and total number of compounding periods, P, F or A can be determined by
- $F = P\ (\ F\ /\ P,\ i\%,\ N\ )$
- $i\% = (\ 1 + r\ /\ M\ )^{M} - 1$
- <u>Uniform and / or Gradient Series</u>

Given nominal interest rate, total number of compounding periods, and existence of a cash flow at the end of each period, P, F or A may be determined by the formulas and tables for uniform annual series and uniform gradient series.

CASH FLOWS LESS OFTEN THAN COMPOUNDING PERIODS

- Find A, given i, k and X, where:
  - i is the effective interest rate per interest period
  - k is the period at the end of which cash flow occurs
  - X is the uniform cash flow amount
- Use: A = X (A / F,i%, k)
- Find A, given i, k and X, where:
  - i is the effective interest rate per interest period
  - k is the period at the beginning of which cash flow occurs
  - X is the uniform cash flow amount
- Use: A = X (A / P, i%, k)

CONTINUOUS COMPOUNDING AND DISCRETE CASH FLOWS

- Continuous compounding assumes cash flows occur at discrete intervals, but compounding is continuous throughout the interval.
- Given nominal per year interest rate -- r,
- compounding per year -- M
- one unit of principal = $[1 + (r / M)]^{M}$
- Given $M / r = p$, $[1 + (r / M)]^{M} = [1 + (1/p)]^{rp}$
- Given $\lim [1 + (1 / p)]^{p} = e^{1} = 2.71828$ $(F / P, r\%, N) = e^{rN}$
- $i = e^{r} - 1$

CONTINUOUS COMPOUNDING AND DISCRETE CASH FLOWS
Single Cash Flow

- Finding F given P
- Finding future equivalent value given present value
- $F = P\ (e^{rN})$
- Functionally expressed as (F / P, r%, N)
- e^{rN} is continuous compounding compound amount
- Predetermined values are in column 2 of appendix D of text

CONTINUOUS COMPOUNDING AND DISCRETE CASH FLOWS
Single Cash Flow

- Finding P given F
- Finding present equivalent value given future value
- $P = F\,(e^{-rN})$
- Functionally expressed as (P / F, r%, N)
- e^{-rN} is continuous compounding present equivalent
- Predetermined values are in column 3 of appendix D of text

CONTINUOUS COMPOUNDING AND DISCRETE CASH FLOWS

Uniform Series

- Finding F given A
- Finding future equivalent value given a series of uniform equal receipts
- $F = A\,(e^{rN} - 1)/(e^{r} - 1)$
- Functionally expressed as (F / A, r%, N)
- $(e^{rN} - 1)/(e^{r} - 1)$ is continuous compounding compound amount
- Predetermined values are in column 4 of appendix D of text

CONTINUOUS COMPOUNDING AND DISCRETE CASH FLOWS
Uniform Series

- Finding P given A
- Finding present equivalent value given a series of uniform equal receipts
- $P = A\,(e^{rN} - 1) / (e^{rN})\,(e^{r} - 1)$
- Functionally expressed as (P / A, r%, N)
- $(e^{rN} - 1) / (e^{rN})\,(e^{r} - 1)$ is continuous compounding present equivalent
- Predetermined values are in column 5 of appendix D of text

CONTINUOUS COMPOUNDING AND DISCRETE CASH FLOWS
Uniform Series

- Finding A given F
- Finding a uniform series given a future value
- $A = F\,(e^{r} - 1) / (e^{rN} - 1)$
- Functionally expressed as (A / F, r%, N)
- $(e^{r} - 1) / (e^{rN} - 1)$ is continuous compounding sinking fund
- Predetermined values are in column 6 of appendix D of text

CONTINUOUS COMPOUNDING AND DISCRETE CASH FLOWS
Uniform Series

- Finding A given P
- Finding a series of uniform equal receipts given present equivalent value
- $A = P\,[e^{rN}(e^{r} - 1) / (e^{rN} - 1)]$
- Functionally expressed as (A / P, r%, N)
- $[e^{rN}(e^{r} - 1) / (e^{rN} - 1)]$ is continuous compounding capital recovery
- Predetermined values are in column 7 of appendix D of text

CONTINUOUS COMPOUNDING AND CONTINUOUS CASH FLOWS

Continuous flow of funds suggests a series of cash flows occurring at infinitesimally short intervals of time

Given:

a nominal interest rate or $\underline{r}$

p is payments per year

$$P = \frac{[1 + (r/p)]^{p} - 1}{r[1 + (r/p)]^{p}}$$

Given $\lim [1 + (r/p)]^{p} = e^{r}$

For one year $(P / A, \underline{r}\%, 1) = (e^{r} - 1) / re^{r}$

CONTINUOUS COMPOUNDING AND CONTINUOUS CASH FLOWS

Finding F given A

Finding the future equivalent given the continuous funds flow

$F = A\ [\ (\ e^{rN} - 1\ )\ /\ r\ ]$

Functionally expressed as (F / A, *r*%, N)

$(\ e^{rN} - 1\ )\ /\ r$ is continuous compounding compound amount

Predetermined values are found in column 6 of appendix D of text.

CONTINUOUS COMPOUNDING AND CONTINUOUS CASH FLOWS

- Finding $\bar{P}$ given $\bar{A}$
- Finding the present equivalent given the continuous funds flow
- $\bar{P} = \bar{A} [(e^{rN} - 1) / re^{rN}]$
- Functionally expressed as ($\bar{P}$ / $\bar{A}$, r%, N)
- $(e^{rN} - 1) / re^{rN}$ is continuous compounding present equivalent
- Predetermined values are found in column 7 of appendix D of text.

CONTINUOUS COMPOUNDING AND CONTINUOUS CASH FLOWS

- Finding $\bar{A}$ given F
- Finding the continuous funds flow given the future equivalent
- $\bar{A} = F\,[\,r / (\,e^{rN} - 1\,)]$
- Functionally expressed as ($\bar{A}$ / F, r%, N)
- $r / (\,e^{rN} - 1\,)$ is continuous compounding sinking fund

CONTINUOUS COMPOUNDING AND CONTINUOUS CASH FLOWS

- Finding $\bar{A}$ given P
- Finding the continuous funds flow given the present equivalent
- $\bar{A} = F\, [\, re^{rN} / (\, e^{rN} - 1\,)]$
- Functionally expressed as ($\bar{A}$ / P, r%, N)
- $re^{rN} / (\, e^{rN} - 1\,)$ is continuous compounding capital recovery