The elastic moduli

1 Introduction

2 Definition of stress

3 Definition of strain

4 Hooke’s law

5 Measurement of Young’s modulus

6 Data for Young’s modulus

Examples

1. Introduction

The modulus is a property that measures the resistance of a material to elastic deformation.

If rods of identical cross section are laid on two widely spaced supports and then identical weights are hung at their centers, they bend elastically by very different amounts depending on the material of which they are made.

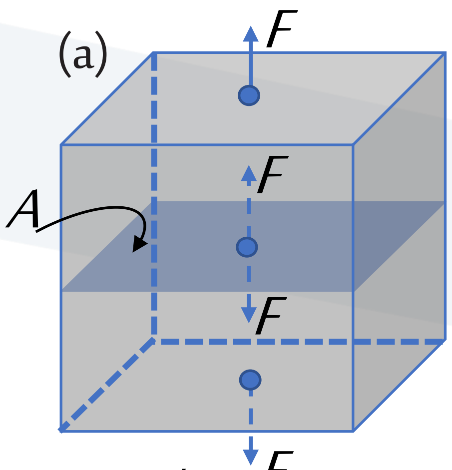
Low modulus materials are floppy and deflect a lot when they are loaded. Sometimes this is desirable, of course: springs, cushions, vaulting poles 2.

But in the great majority of mechanical applications, deflection is undesirable, and the engineer seeks a material with a high modulus.

The modulus is reflected, too, in the natural frequency of vibration of a structure. A beam of low modulus has a lower natural frequency than one of higher modulus (although the density matters also) and this, as well as the deflection, is important in design calculations.

Before we look in detail at the modulus, we must first define stress and strain

2. Definition of stress

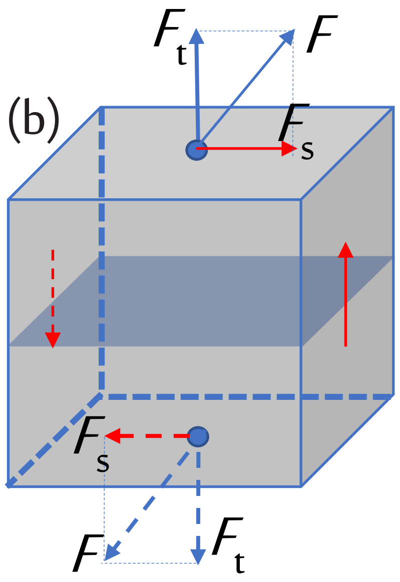


$\sigma = F/A$

A block of material with a force F applied normally at its upper face, as in (Fig.a). The force is transmitted through the block and is balanced by the equal, opposite force acting at the lower face.

The whole of the block is said to be in a state of stress. The intensity of the stress, is measured by the force F divided by the area, A, of the block face, giving: $\sigma = F/A$

This the normal stress (tension or compression) caused by a force normal to the face.



$\sigma = F_t/A$ & $\tau = F_s/A$

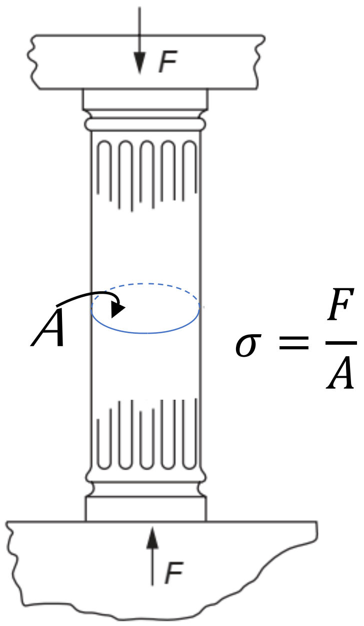
Suppose now that the force acted not normal to the face but at an angle to it, as shown in (Fig.b). The force can be resolved into two components: F_t, normal to the face and F_s parallel to it. The normal component creates a normal stress in the block. Its magnitude, as before, is F_t/A. The other component, F_s, creates a tangential stress in the block parallel to the direction of F_s, knowns as shear stress and given by: $\tau = F_s/A$

The magnitude of a stress is always equal to the magnitude of a force divided by the area of the face on which it acts. stresses are measured in units of force per area.
For engineering applications (Pa ≡ N m^{-2}) is very small, instead (MPa ≡MN m^{-2}≡N mm^{-2}).

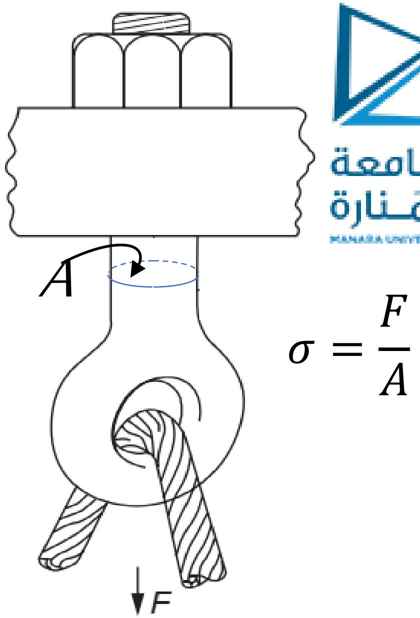
The general case of stress state will be studied in later courses.

Here in the next figures are shown four simple commonly occurring states of stress.

The simplest is that of simple tension or compression (as in a tension member loaded by pin joints at its ends or in a pillar supporting a structure in compression).

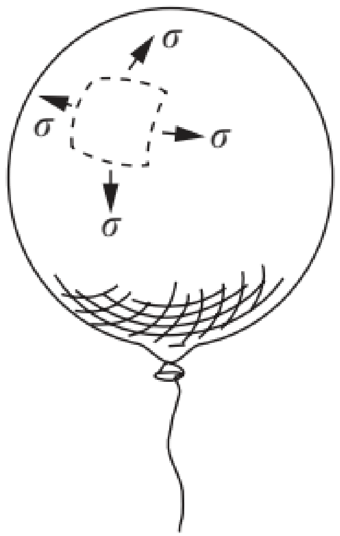


Simple compression

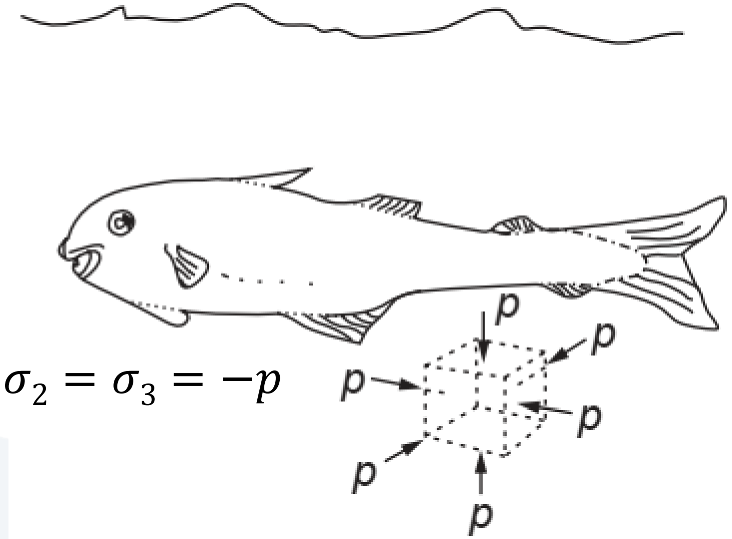


Simple tension

The second common state of stress is that of biaxial tension. If a spherical shell (like a balloon) contains an internal pressure, then the skin of the shell is loaded in two directions, not one, as shown in next figure.

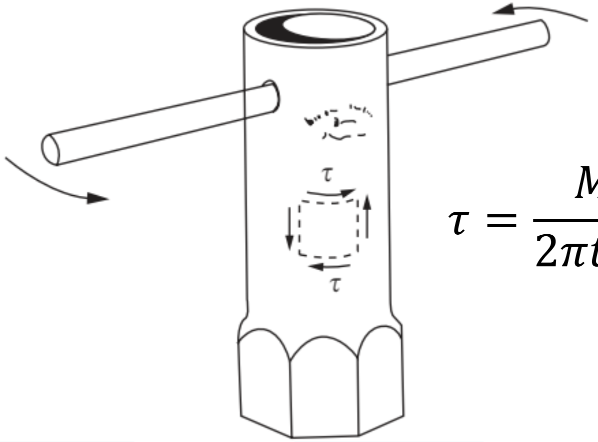


The third common state of stress is that of hydrostatic pressure. This occurs deep in the earth's crust, or deep in the ocean, when a solid is subjected to equal compression on all sides.



$$\sigma_1 = \sigma_2 = \sigma_3 = -p$$

The final common state of stress is that of pure shear. If you try to twist a tube, then elements of it are subjected to pure shear, as shown. This shear stress is simply the shearing force divided by the area of the face on which it acts.

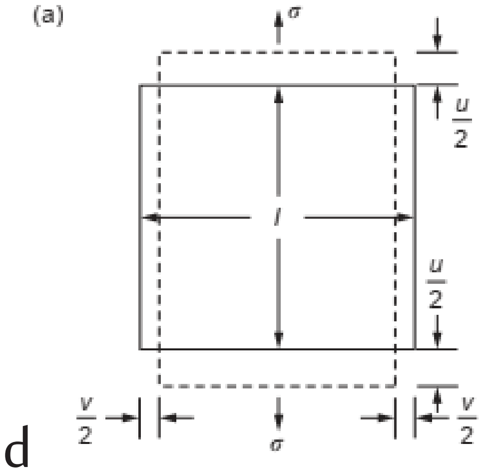


$$\tau = \frac{M}{2\pi t r^2}$$

Remember one final thing; if you know the stress in a body, then the force acting across any face of it is the stress resultant over this area. If the stress in uniform the force is the stree times the area.

The elastic moduli

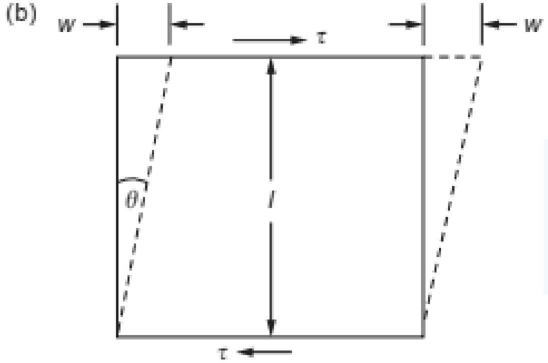
3. Definition of strain [Dimensionless]



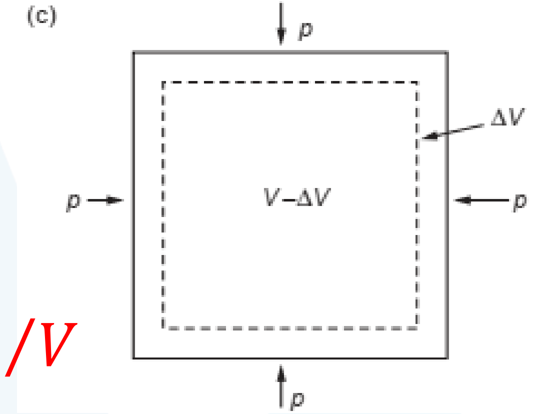
Materials respond to stress (or loading) by *straining*. Under a given stress, a stiff material (like steel) strains only slightly; a floppy material strains much more.

The modulus of the material describes this property, but before we can measure it, or even define it, we must define strain properly.

The kind of stress that we called a tensile stress induces an extension. If the stressed cube of side l, shown in (Fig.a) extends by an amount u parallel to the tensile stress, the nominal tensile strain is: $\varepsilon_n = u/l$.



When it strains in this way, the cube usually gets thinner. The amount by which it shrink inwards is described by Poisson's ratio, , which is the negative of the ratio of the inward strain to the original tensile strain: $\nu = -(laterla\ strain/lomgitudinal\ strain)$



A shear stress induces a shear strain (Fig.b). If a cube shears sideways by an amount w then the shear strain is defined by: $\gamma = w/l = \tan\theta \approx \theta$ (for small strains)

Finally, hydrostatic pressure induces a volume change called dilatation (Fig.c). If the volume change is ΔV and the cube volume is V, the dilatation is defined: $\varepsilon_V = \Delta V/V$

4. Hooke's law

The elastic moduli can now be defined through Hooke's law, which is merely a description of the experimental observation that, when strains are small, the strain is very nearly proportional to the stress; that is, they are linear elastic.

The nominal tensile strain, for example, is proportional to the tensile stress; for simple tension: $\sigma = E\varepsilon_n$.

where E is called Young's modulus. For metals, the same relationship also holds for stresses and strains in simple compression.

In the same way, the shear strain is proportional to the shear stress, with: $\tau = G\gamma$.

where G is the shear modulus.

Finally, the negative of the dilatation is proportional to the pressure (because positive pressure causes a shrinkage of volume) so that: $p = -K\varepsilon_V$.

Wher K is called the bulk (dilation) modulus.

Because strains are dimensionless, the moduli have the same dimensions as those of stress: force per unit area (N m^{-2}). In those units, the moduli are enormous, so they are usually reported instead in units of GPa.

This linear relationship between stress and strain is a very handy one when calculating the response of a solid to stress, but it must be remembered that most solids are elastic only to very small strains: up to about 0.001. Beyond that some break and some become plastic - and this will be discussed later.

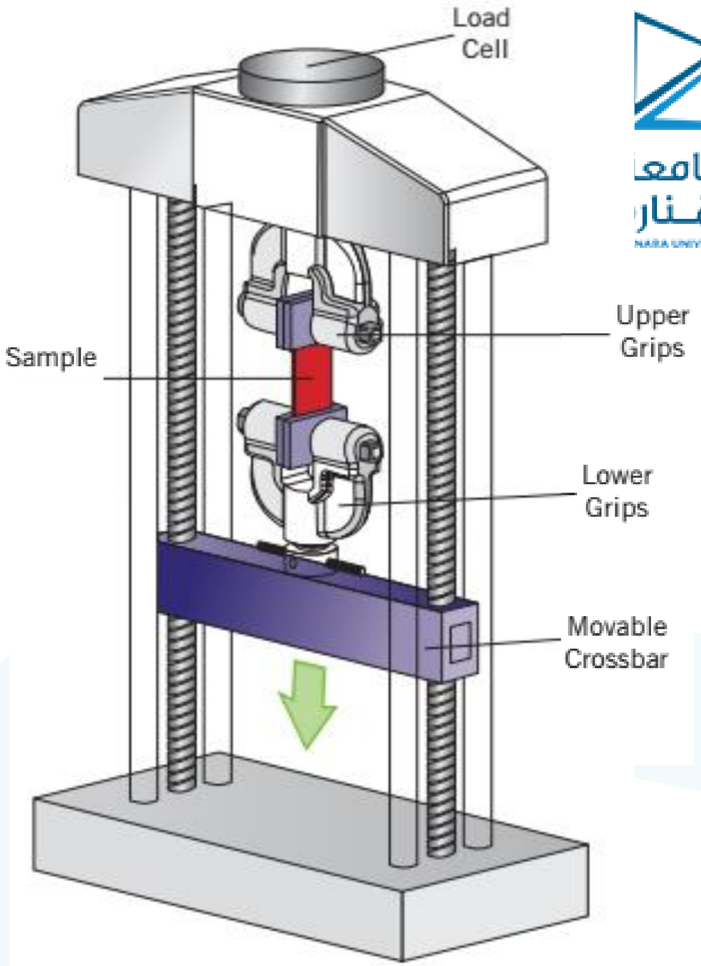
A few solids like rubber are elastic up to very much larger strains of order 4 or 5, but they cease to be linearly elastic (that is the stress is no longer proportional to the strain) after a strain of about 0.01.

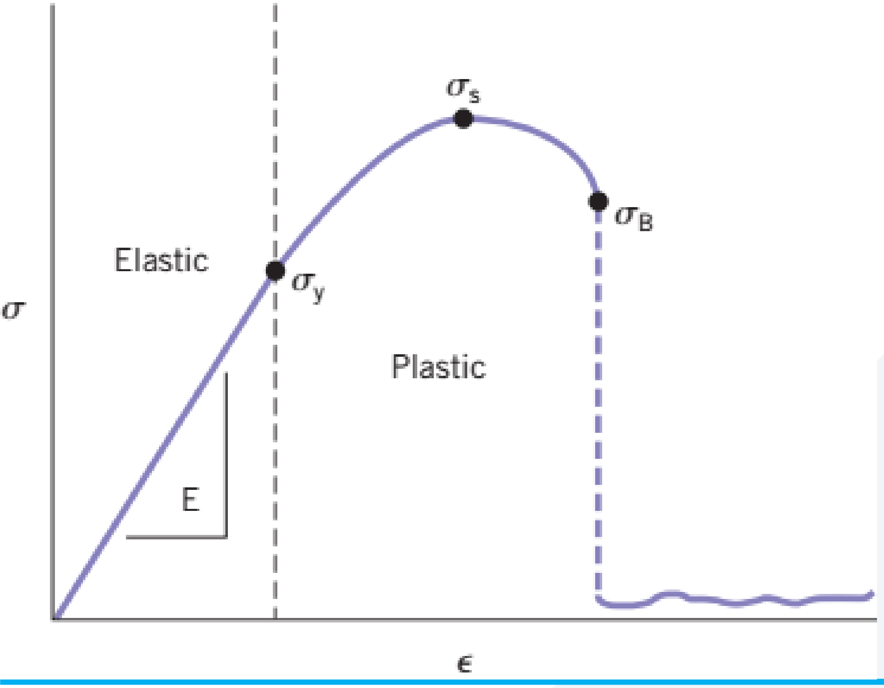
One final point. Poisson's ratio was earlier defined as the negative of the lateral shrinkage strain to the tensile strain. This quantity, Poisson's ratio, is also an elastic constant, so there are four elastic constants: $E, G, K, and\ \nu$. In a moment when the elastic constants are given we only E will be listed. For many materials it is useful to know that

$$K \approx E, \quad G \approx \frac{3}{8}E \quad \text{and} \quad \nu \approx 0.33$$

Although for some these relationships can be more complicated as it will be seen in later courses.

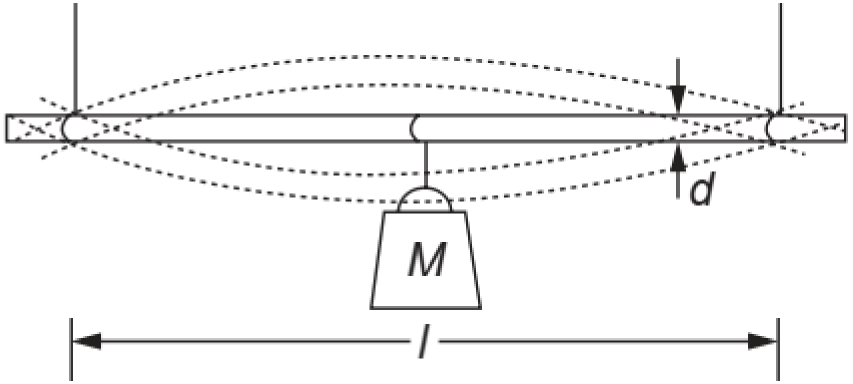
How is Young's modulus measured? One way is to apply a tension or compression to the material with a known force, and measure the strain. Young's modulus is then given by $E = \sigma/\varepsilon_n$; defined as described earlier.





But this is not so good way to measure the modulus. For one thing, if the modulus is large, the extension u may be too small to measure with precision. And, for another, if anything else contributes to the strain, like creep (which we will discussed later), or deflection of the testing machine itself, then it will lead to an incorrect value for E.

A better way of measuring E is to measure the natural frequency of vibration of a round rod of the material, simply supported at its ends and heavily loaded by a mass M at the middle (so that we may neglect the mass of the rod itself). The frequency of oscillation of the rod, f cycles per second (or hertz), is given by



$$f = \frac{1}{2\pi}\left\{\frac{3\pi E d^4}{4l^3 M}\right\}^{1/2}$$

where l is the distance between the supports and d is the diameter of the rod. From this, $E = \frac{16\pi M l^3 f^2}{3d^4}$

Use of stroboscopic techniques and carefully designed apparatus can make this sort of method very accurate

The best of all methods of measuring E is to measure the velocity of sound in the material.
The velocity of longitudinal waves, l, depends on Young's modulus and the density, :

$$v_1 = \left(\frac{E}{\rho}\right)^{1/2}$$

v_1 is measured by "striking" one end of a bar of the material (by glueing a piezoelectric crystal there and applying a charge-difference to the crystal surfaces) and measuring the time sound takes to reach the other end(by attaching a second piezoelectric crystal there). Most moduli are measured by one of these last two methods.

6. Data for Young's modulus

A good perspective of the spread of moduli is given by the bar chart shown in next figure.

Ceramics & metals, even the floppiest of them like lead, lie near the top of this range.

Polymers & elastomers are much more compliant, the common ones (polyethylene, PVC, & polypropylene) lying several decades lower.

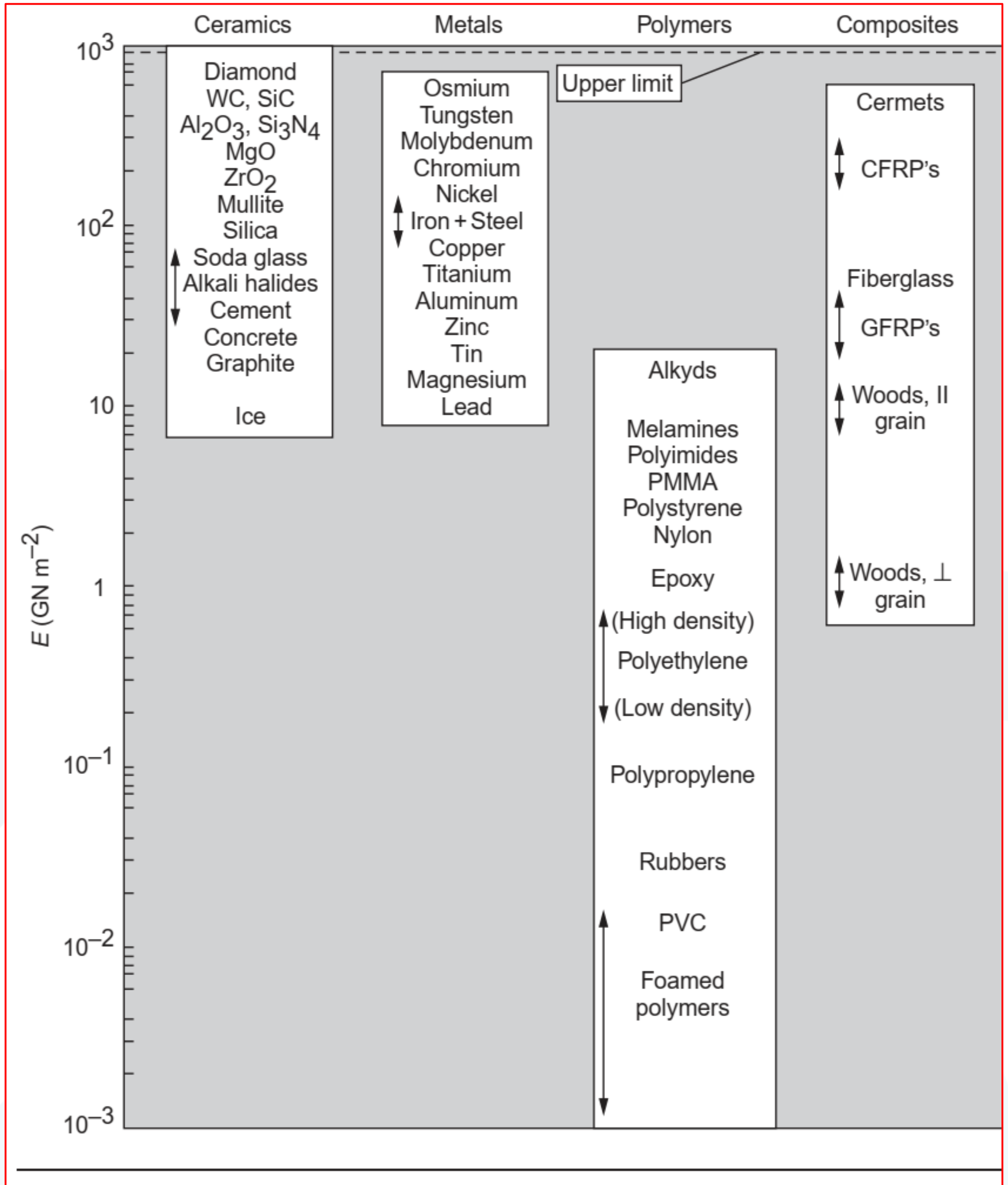
Composites span the range between polymers and ceramics.

Table 1 is a ranked list of Young's modulus of materials, to be used in problems & in particular applications.

Diamond is at the top, with a modulus of 10^{+3} GPa; soft rubbers and foamed polymers are at the bottom with moduli as low as 10^{-3} GPa.

Lower modulus can be made!!: jelly, for instance, has a modulus of about 10^{-6} GPa.

Practical engineering materials lie in the range 10^{+3} to 10^{-3} GPa: a range of 10^{6}. This is the range you have to choose from when selecting a material for a given application.



Bar chart of data for Young's modulus, E.

Table 3.1 Data for Young's modulus, E

Material	E ($GN\,m^{-2}$)
Diamond	1000
Tungsten carbide, WC	450–650
Osmium	551
Cobalt/tungsten carbide cermets	400–530
Borides of Ti, Zr, Hf	450–500
Silicon carbide, SiC	430–445
Boron	441
Tungsten and alloys	380–411
Alumina, Al_2O_3	385–392
Beryllia, BeO	375–385
Titanium carbide, TiC	370–380
Tantalum carbide, TaC	360–375
Molybdenum and alloys	320–365
Niobium carbide, NbC	320–340
Silicon nitride, Si_3N_4	280–310
Beryllium and alloys	290–318
Chromium	285–290
Magnesia, MgO	240–275
Cobalt and alloys	200–248
Zirconia, ZrO_2	160–241
Nickel	214
Nickel alloys	130–234
CFRP	70–200

Table 3.1 Data for Young's modulus, E. (*Continued*)

Material	E ($GN\,m^{-2}$)
Iron	196
Iron-based super-alloys	193–214
Ferritic steels, low-alloy steels	196–207
Stainless austenitic steels	190–200
Mild steel	200
Cast irons	170–190
Tantalum and alloys	150–186
Platinum	172
Uranium	172
Boron/epoxy composites	80–160
Copper	124
Copper alloys	120–150
Mullite	145
Vanadium	130
Titanium	116
Titanium alloys	80–130
Palladium	124
Brasses and bronzes	103–124
Niobium and alloys	80–110
Silicon	107
Zirconium and alloys	96

The elastic moduli

Table 3.1 Data for Young's modulus, E. (*Continued*)

Material	E ($GN\,m^{-2}$)
Silica glass, SiO_2 (quartz)	94
Zinc and alloys	43–96
Gold	82
Calcite (marble, limestone)	70–82
Aluminum	69
Aluminum and alloys	69–79
Silver	76
Soda glass	69
Alkali halides (NaCl, LiF, etc.)	15–68
Granite (Westerly granite)	62
Tin and alloys	41–53
Concrete, cement	30–50
Fiberglass (glass-fiber/epoxy)	35–45
Magnesium and alloys	41–45
GFRP	7–45
Calcite (marble, limestone)	31
Graphite	27
Shale (oil shale)	18
Common woods, ‖ to grain	9–16
Lead and alloys	16–18
Alkyds	14–17
Ice, H_2O	9.1
Melamines	6–7

Table 3.1 Data for Young's modulus, E. (*Continued*)

Material	E ($GN\,m^{-2}$)
Polyimides	3–5
Polyesters	1.8–3.5
Acrylics	1.6–3.4
Nylon	2–4
PMMA	3.4
Polystyrene	3–3.4
Epoxies	2.6–3
Polycarbonate	2.6
Common woods, ⊥ to grain	0.6–1.0
Polypropylene	0.9
PVC	0.2–0.8
Polyethylene, high density	0.7
Polyethylene, low density	0.2
Rubbers	0.01–0.1
Cork	0.01–0.03
Foamed polymers	0.001–0.01

To understand the origin of the modulus, why it has the values it does, why polymers are much less stiff than metals, and what we can do about it, we have to examine the structure of materials, and the nature of the forces holding the atoms together.

Examples

3.1 (a) Define Poisson's ratio, ν ; and the dilatation, ε_V, in the straining of an elastic solid.

(b) Calculate the dilatation ε_V in the uniaxial elastic extension of a bar of material, assuming strains are small, in terms of ν and the tensile strain ε. Hence find the value of ν for which the volume change during elastic deformation is zero.

(c) Poisson's ratio for most metals is about 0.3. For cork it is close to zero; for rubber it is close to 0.5. What are the approximate volume changes in each of these materials during an elastic tensile strain of ε?

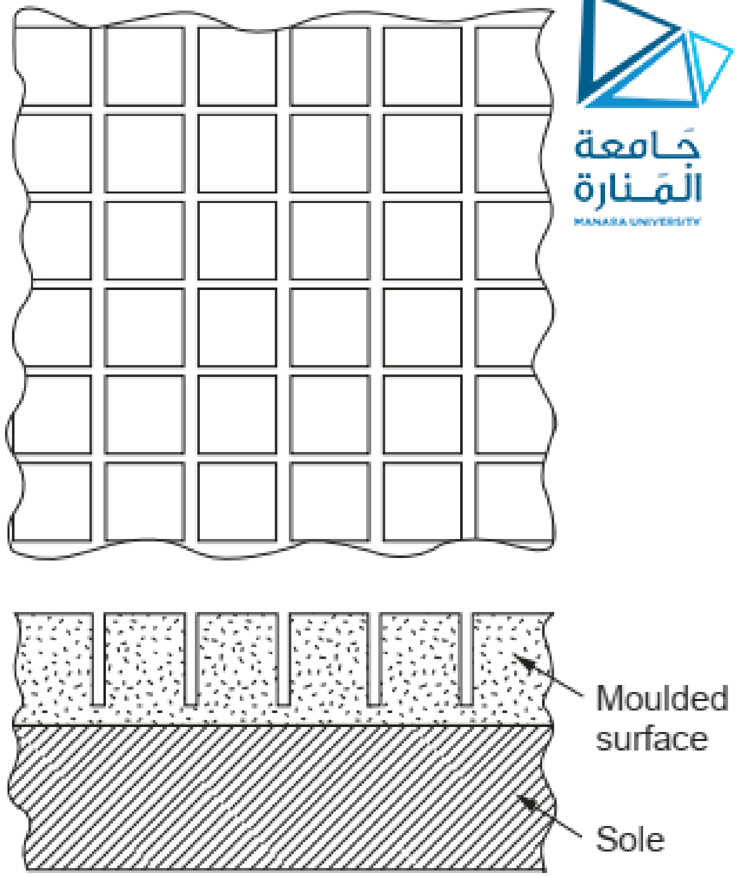
Answers

(b) 0.5;

(c) "most metals": 0.4, cork: ε , rubber: 0.

Examples

3.2 The sole of a shoe is to be surfaced with soft synthetic rubber having a Poisson's ratio of 0.5. The cheapest solution is to use a solid rubber slab of uniform thickness. However, a colleague suggests that the sole would give better cushioning if it were moulded as shown in the diagram. Is your colleague correct? If so, why?



3.3 Explain why it is much easier to push a cork into a wine bottle than a rubber bung. Poisson's ratio is zero for cork and 0.5 for rubber.

problems

3.1 A steel bolt 12 mm in diameter carries a tensile load of 2 tonne (2 metric tons, or 2 × 1,000 kgf). What is the stress in the bolt in MPa? 173 MPa

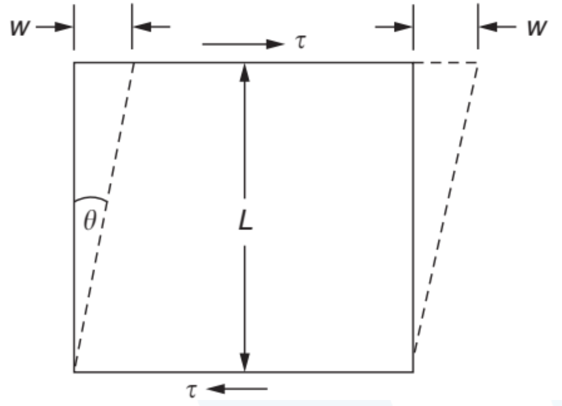
3.2 Calculate the hydrostatic pressure (in tonne m^{-2}) at the bottom of a swimming pool 2 m deep. The density of water is approximately 1 gram per cubic centimeter (1 g cm^{-3}). Is it a suitable unit for pressure?

200 g cm^{-2}= 2000 kg m^{-2}= 2 tonne m^{-2}= 0.0196 Mpa=0.196 bar

3.3 A metal rod 100 cm long is stretched in tension by 1 mm. What is the nominal tensile strain? $\varepsilon_n = \frac{\Delta l}{l}$=0.001

If Poisson's ratio for the metal is 0.3. The initial diameter of the rod is 10 mm. Calculate the reduction in diameter when the rod is stretched. $\Delta D = D\varepsilon_D = D(-\upsilon\varepsilon_n) = -3 \times 10^{-3} mm = -0.003\ mm$

3.4 A cube of polymer foam of side 10 cm is sheared as shown in next figure. The shear displacement w is 1 mm. Calculate the engineering shear strain. $\gamma = \frac{w}{l} = \frac{1\,[mm]}{10\times10[mm]}$=0.01



3.5 A cube of closed-cell (waterproof) polymer foam of side 100 mm is immersed in water to a given depth. The sides of the cube decrease by 1 mm as a result. Calculate the dilatation. $\varepsilon_V = 3\varepsilon_n = -0.03$

3.6 Young's modulus for steel is 200 GN m^{-2}. Calculate the tensile stress required to produce a tensile strain of 0.1%. $\sigma = E\varepsilon = 200 \times 0.001 = 0.2\ GPa = 200\ MPa$

3.7 A cylindrical test piece of metal 5 mm in diameter is loaded in tension to 600 kg. Strain gauges glued to the surface of the test piece register a strain of 0.00435 at this load. Calculate Young's modulus for the metal. Compare your value with the values for E in Table 3.1, and say what the metal is likely to be.

$$E = \frac{\sigma}{\varepsilon} = \frac{F}{\varepsilon A} = 68.9\ GPa$$

aluminum (or one of its alloys).

3.8 A strain gauge is glued on to the surface of an aluminum bridge girder. A heavy vehicle is then driven across the bridge, causing the strain gauge reading to increase by 0.0005. Calculate the change in stress caused by the vehicle. $\sigma = E\varepsilon = 69 \times 10^3 \times 0.0005 = 34.5\ MPa$

3.9 In order to minimize mistakes when reading or recording small strains, strain gauge outputs are often given as "microstrain," or strain 10^{-6}. Rewrite the strain of 0.0005 as microstrain. $\varepsilon = 0.0005 = 5 \times 10^{-4} = 500\ microstrain$

3-10 Explain why the units of strain are dimensionless. Use this to explain why the units of elastic moduli are stress units. Why are moduli usually given in GPa, whereas stress units are usually given in MPa?