Valid and Invalid Arguments

Epp Section 1.3

Argument

An argument is a sequence of statements.

All statements but the final one are called assumptions or hypothesis.

The final statement is called the conclusion.

An argument is valid if:

> whenever all the assumptions are true, then the conclusion is true.

If today is Wednesday, then yesterday is Tuesday.

Today is Wednesday.

$\therefore$ Yesterday is Tuesday.

Modus Ponens

طريقة التأكيد

	If p then q.	
	p	
∴	q	

	If typhoon, then class cancelled.
	Typhoon.
∴	Class cancelled.

assumptions ($p \to q$, p) — conclusion (q)

p	q	p→q	p	q
T	T	T	T	T
T	F	F	T	F
F	T	T	F	T
F	F	T	F	F

Modus ponens is Latin meaning “method of affirming”.

Modus Tollens

طريقة الرفض

	If p then q.	
	~q	
∴	~p	

	If typhoon, then class cancelled.
	Class not cancelled.
∴	No typhoon.

assumptions (p→q, ~q) — conclusion (~p)

p	q	p→q	~q	~p
T	T	T	F	F
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Modus tollens is Latin meaning "method of denying".

Equivalence

A student is trying to prove that propositions *P*, *Q*, and *R* are all true.

She proceeds as follows.

First, she proves three facts:

- *P* implies *Q*
- *Q* implies *R*
- *R* implies *P*.

Then she concludes,

``Thus *P*, *Q*, and *R* are all true.''

Proposed argument:

assumption

$$\frac{(P \to Q),\ (Q \to R),\ (R \to P)}{P \wedge Q \wedge R}$$

conclusion

Is it valid?

Valid Argument?

$$\frac{(P \to Q),\ (Q \to R),\ (R \to P)}{P \wedge Q \wedge R}$$

Is it valid?

assumptions

conclusion

P	Q	R
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

$P \to Q$	$Q \to R$	$R \to P$
T	T	T
T	F	T
F	T	T
F	T	T
T	T	F
T	F	T
T	T	F
T	T	T

$P \wedge Q \wedge R$	OK?
T	yes
F	yes
F	yes
F	yes
F	yes
F	yes
F	yes
F	no

To prove an argument is not valid, we just need to find a counterexample.

Valid Arguments?

assumptions (p→q, q) — conclusion (p)

p	q	p→q	q	p
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	F

If p then q.
q
∴ p

Assumptions are true, but not the conclusion.

If you are a fish, then you drink water.
You drink water.
You are a fish.

Valid Arguments?

If p then q.
~p
∴ ~q

		assumptions		conclusion
p	q	p→q	~p	~q
T	T	T	F	F
T	F	F	F	T
F	T	T	T	F
F	F	T	T	T

If you are a fish, then you drink water.
You are not a fish.
You do not drink water.

Modus Ponens example

- Assume you are given the following two statements:
  - "you are in this class"
  - "if you are in this class, you will get a grade"

$$\begin{array}{r} p \\ p \rightarrow q \\ \hline \therefore q \end{array}$$

- Let p = "you are in this class"
- Let q = "you will get a grade"

- By Modus Ponens, you can conclude that you will get a grade

Modus Ponens

- Consider $(p \wedge (p \rightarrow q)) \rightarrow q$

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q))$	$(p \wedge (p \rightarrow q)) \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

$$\begin{array}{l} p \\ \underline{p \rightarrow q} \\ \therefore q \end{array}$$

Generalization & Specialization

- Generalization: If you know that p is true, then p ∨ q will ALWAYS be true

$$\frac{p}{\therefore p \vee q}$$

- Specialization: If p ∧ q is true, then p will ALWAYS be true

$$\frac{p \wedge q}{\therefore p}$$

Example of proof

p
q
r
s
t

- We have the hypotheses:
  - "It is not sunny this afternoon and it is colder than yesterday"
  - "We will go swimming only if it is sunny"
  - "If we do not go swimming, then we will take a canoe trip"
  - "If we take a canoe trip, then we will be home by sunset"
- Does this imply that "we will be home by sunset"?

$\neg p \wedge q$

$r \rightarrow p$

$\neg r \rightarrow s$

$s \rightarrow t$

---

t

Example of proof

1.	$\neg p \wedge q$	1st hypothesis
2.	$\neg p$	Simplification using step 1
3.	$r \rightarrow p$	2nd hypothesis
4.	$\neg r$	Modus tollens using steps 2 & 3
5.	$\neg r \rightarrow s$	3rd hypothesis
6.	s	Modus ponens using steps 4 & 5
7.	$s \rightarrow t$	4th hypothesis
8.	t	Modus ponens using steps 6 & 7

$$\frac{p \wedge q}{\therefore p} \qquad \frac{\begin{array}{c} p \\ p \rightarrow q \end{array}}{\therefore q} \qquad \frac{\begin{array}{c} \neg q \\ p \rightarrow q \end{array}}{\therefore \neg p}$$

So what did we show?

- We showed that:
  - $[(\neg p \wedge q) \wedge (r \rightarrow p) \wedge (\neg r \rightarrow s) \wedge (s \rightarrow t)] \rightarrow t$
  - That when the 4 hypotheses are true, then the implication is true
  - In other words, we showed the above is a tautology!

- To show this, enter the following into the truth table generator at

  http://sciris.shu.edu/~borowski/Truth/:

  ((~P ^ Q) ^ (R => P) ^ (~R => S) ^ (S => T)) => T

More rules of inference

- Conjunction: if p and q are true separately, then p∧q is true

$$\begin{array}{l} p \\ \underline{q} \\ \therefore p \wedge q \end{array}$$

- Elimination: If p∨q is true, and p is false, then q must be true

$$\begin{array}{l} p \vee q \\ \underline{\neg p} \\ \therefore q \end{array}$$

- Transitivity: If p→q is true, and q→r is true, then p→r must be true

$$\begin{array}{l} p \to q \\ \underline{q \to r} \\ \therefore p \to r \end{array}$$

Even more rules of inference

- Proof by division into cases: if at least one of p or q is true, then r must be true

$$\begin{array}{l} p \vee q \\ p \rightarrow r \\ \underline{q \rightarrow r} \\ \therefore r \end{array}$$

- Contradiction rule: If ¬p→c is true, we can conclude p (via the contra-positive)

$$\begin{array}{l} \underline{\neg p \rightarrow c} \\ \therefore p \end{array}$$

- Resolution: If p∨q is true, and ¬p∨r is true, then q∨r must be true
  - Not in the textbook

$$\begin{array}{l} p \vee q \\ \underline{\neg p \vee r} \\ \therefore q \vee r \end{array}$$

Example of proof

- Given the hypotheses:
  - "If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on"
  - "If the sailing race is held, then the trophy will be awarded"
  - "The trophy was not awarded"
- Can you conclude: "It rained"?

$(\neg r \vee \neg f) \rightarrow (s \wedge l)$

$s \rightarrow t$

$\neg t$

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r

Example of proof

1.	¬t	3rd hypothesis
2.	s → t	2nd hypothesis
3.	¬s	Modus tollens using steps 2 & 3
4.	(¬r∨¬f)→(s∧l)	1st hypothesis
5.	¬(s∧l)→¬(¬r∨¬f)	Contrapositive of step 4
6.	(¬s∨¬l)→(r∧f)	DeMorgan's law and double negation law
7.	¬s∨¬l	Addition from step 3
8.	r∧f	Modus ponens using steps 6 & 7
9.	r	Simplification using step 8

$$\frac{\begin{array}{c} p \\ p \rightarrow q \end{array}}{\therefore q} \qquad \frac{p}{\therefore p \vee q} \qquad \frac{p \wedge q}{\therefore p} \qquad \frac{\begin{array}{c} \neg q \\ p \rightarrow q \end{array}}{\therefore \neg p}$$

Fallacy of affirming the conclusion

~~Modus Badus~~

- Consider the following:

$$q \qquad\qquad q$$
$$\underline{p \to q} \qquad \underline{\neg q \to \neg p}$$
$$\therefore p \qquad\qquad \therefore p$$

- Is this true?

p	q	p→q	q∧(p→q))	(q∧(p→q)) → p
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

Not a valid rule!

Modus Badus example

- Assume you are given the following two statements:
  - "you will get a grade"
  - "if you are in this class, you will get a grade"

$$\begin{array}{l} q \\ p \rightarrow q \\ \hline \therefore p \end{array}$$

- Let p = "you are in this class"
- Let q = "you will get a grade"

- You CANNOT conclude that you are in this class
  - You could be getting a grade for another class

~~Modus Badus~~

Fallacy of denying the hypothesis

- Consider the following:

$$\begin{array}{l} \neg p \\ \underline{p \rightarrow q} \\ \therefore \neg q \end{array}$$

- Is this true?

p	q	p→q	¬p∧(p→q))	(¬p∧(p→q)) → ¬q
T	T	T	F	T
T	F	F	F	T
F	T	T	T	F
F	F	T	T	T

Not a valid rule!

Modus Badus example

- Assume you are given the following two statements:
  - "you are not in this class"
  - "if you are in this class, you will get a grade"

$$\begin{array}{r} \neg p \\ p \rightarrow q \\ \hline \therefore \neg q \end{array}$$

- Let p = "you are in this class"
- Let q = "you will get a grade"

- You CANNOT conclude that you will not get a grade
  - You could be getting a grade for another class