Predicates and Quantifiers

Epp, Sections 2.1 and 2.2

Terminology review

- Proposition: a statement that is either true or false
  - Must always be one or the other!
  - Example: “The sky is red”
  - Not a proposition: $x + 3 > 4$

- Boolean variable: A variable (usually p, q, r, etc.) that represents a proposition

Propositional functions

- Consider P(x) = x < 5
  - P(x) has no truth values (x is not given a value)
  - P(1) is true
    - The proposition 1<5 is true
  - P(10) is false
    - The proposition 10<5 is false
- Thus, P(x) will create a proposition when given a value

Propositional functions 2

- Let P(x) = "x is a multiple of 5"
  - For what values of x is P(x) true?
- Let P(x) = x+1 > x
  - For what values of x is P(x) true?
- Let P(x) = x + 3
  - For what values of x is P(x) true?

Anatomy of a propositional function

$$\underset{\text{variable}}{P(x)} = \underbrace{x + 5 > x}_{\text{predicate}}$$

Propositional functions 3

- Functions with multiple variables:
  - $P(x,y) = x + y == 0$
    - $P(1,2)$ is false, $P(1,-1)$ is true
  - $P(x,y,z) = x + y == z$
    - $P(3,4,5)$ is false, $P(1,2,3)$ is true
  - $P(x_1,x_2,x_3 \ldots x_n) = \ldots$

So, why do we care about quantifiers?

- Many things (in this course and beyond) are specified using quantifiers
  - In some cases, it's a more accurate way to describe things than Boolean propositions

Quantifiers

- A quantifier is “an operator that limits the variables of a proposition”

- Two types:
  - Universal
  - Existential

Universal quantifiers 1

- Represented by an upside-down A: $\forall$
  - It means "for all"
  - Let $P(x) = x+1 > x$

- We can state the following:

  $\forall x\ P(x)$

  - English translation: "for all values of x, P(x) is true"
  - English translation: "for all values of x, x+1>x is true"

Universal quantifiers 2

- But is that always true?

  $\forall x\ P(x)$
- Let x = the character ‘a’
  - Is ‘a’+1 > ‘a’?
- Let x = the state of Virginia
  - Is Virginia+1 > Virginia?
- You need to specify your universe!
  - What values x can represent
  - Called the “domain” or “universe of discourse” by the textbook

Universal quantifiers 3

- Let the universe be the real numbers.
  - Then, $\forall x\, P(x)$ is true

- Let $P(x) = x/2 < x$
  - Not true for the negative numbers!
  - Thus, $\forall x\, P(x)$ is false
    - When the domain is all the real numbers

- In order to prove that a universal quantification is true, it must be shown for ALL cases
- In order to prove that a universal quantification is false, it must be shown to be false for only ONE case

Universal quantification 4

- Given some propositional function P(x)
- And values in the universe $x_1 .. x_n$
- The universal quantification $\forall x\, P(x)$ implies:

$$P(x_1) \land P(x_2) \land \ldots \land P(x_n)$$

Universal quantification 5

- Think of ∀ as a for loop:

∀ $\forall x\, P(x)$, where $1 \le x \le 10$

- … can be translated as …

```
for ( x = 1; x <= 10; x++ )
    is P(x) true?
```

- If $P(x)$ is true for all parts of the for loop, then $\forall x\, P(x)$
  – Consequently, if $P(x)$ is false for any one value of the for loop, then $\forall x\, P(x)$ is false

Existential quantification 1

- Represented by an bacwards E: ∃
  - It means "there exists"
  - Let $P(x) = x+1 > x$

- We can state the following:

  $\exists x\ P(x)$

  - English translation: "there exists (a value of) x such that P(x) is true"
  - English translation: "for at least one value of x, x+1>x is true"

Existential quantification 2

- Note that you still have to specify your universe
  - If the universe we are talking about is all the states in the US, then $\exists x\, P(x)$ is not true
- Let $P(x) = x+1 < x$
  - There is no numerical value x for which $x+1<x$
  - Thus, $\exists x\, P(x)$ is false

Existential quantification 3

- Let P(x) = x+1 > x
  - There is a numerical value for which x+1>x
    - In fact, it's true for all of the values of x!
  - Thus, $\exists x\, P(x)$ is true

- In order to show an existential quantification is true, you only have to find ONE value
- In order to show an existential quantification is false, you have to show it's false for ALL values

Existential quantification 4

- Given some propositional function P(x)
- And values in the universe $x_1 .. x_n$
- The existential quantification $\exists x\, P(x)$ implies:

$$P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$$

A note on quantifiers

- Recall that P(x) is a propositional function
  - Let P(x) be “x == 0”
- Recall that a proposition is a statement that is either true or false
  - P(x) is not a proposition
- There are two ways to make a propositional function into a proposition:
  - Supply it with a value
    - For example, P(5) is false, P(0) is true
  - Provide a quantifiaction
    - For example, $\forall x$ P(x) is false and $\exists x$ P(x) is true
      - Let the universe of discourse be the real numbers

Binding variables

- Let $P(x,y)$ be $x > y$
- Consider: $\forall x\ P(x,y)$
  - This is not a proposition!
  - What is y?
    - If it's 5, then $\forall x\ P(x,y)$ is false
    - If it's $x-1$, then $\forall x\ P(x,y)$ is true
- Note that y is not "bound" by a quantifier

Binding variables 2

- $(\exists x\ P(x)) \vee Q(x)$
  - The x in Q(x) is not bound; thus not a proposition
- $(\exists x\ P(x)) \vee (\forall x\ Q(x))$
  - Both x values are bound; thus it is a proposition
- $(\exists x\ P(x) \wedge Q(x)) \vee (\forall y\ R(y))$
  - All variables are bound; thus it is a proposition
- $(\exists x\ P(x) \wedge Q(y)) \vee (\forall y\ R(y))$
  - The y in Q(y) is not bound; this not a proposition

Negating quantifications

- Consider the statement:
  - All students in this class have red hair
- What is required to show the statement is false?
  - There exists a student in this class that does NOT have red hair
- To negate a universal quantification:
  - You negate the propositional function
  - AND you change to an existential quantification
  - $\neg \forall x\, P(x) = \exists x\, \neg P(x)$

Negating quantifications 2

- Consider the statement:
  - There is a student in this class with red hair
- What is required to show the statement is false?
  - All students in this class do not have red hair
- Thus, to negate an existential quantification:
  - Tou negate the propositional function
  - AND you change to a universal quantification
  - $\neg\exists x\ P(x) = \forall x\ \neg P(x)$

Translating from English

- Consider “For every student in this class, that student has studied calculus”
- Rephrased: “For every student x in this class, x has studied calculus”
  - Let C(x) be “x has studied calculus”
  - Let S(x) be “x is a student”

∀ ∀x C(x)

  - True if the universe of discourse is all students in this class

Translating from English 2

- What about if the unvierse of discourse is all students (or all people?)

  $\forall x\ (S(x) \wedge C(x))$

  - This is wrong! Why?

  $\forall x\ (S(x) \rightarrow C(x))$

- Another option:
  - Let Q(x,y) be “x has stuided y”

  $\forall x\ (S(x) \rightarrow Q(x, \text{calculus}))$

Translating from English 3

- Consider:
  - “Some students have visited Mexico”
  - “Every student in this class has visited Canada or Mexico”
- Let:
  - S(x) be “x is a student in this class”
  - M(x) be “x has visited Mexico”
  - C(x) be “x has visited Canada”

Translating from English 4

- Consider: “Some students have visited Mexico”
  - Rephrasing: “There exists a student who has visited Mexico”

$\forall$ $\exists x\ M(x)$

  - True if the universe of discourse is all students

- What about if the universe of discourse is all people?

  $\exists x\ (S(x) \rightarrow M(x))$

  - This is wrong! Why?

  $\exists x\ (S(x) \wedge M(x))$

Translating from English 5

- Consider: “Every student in this class has visited Canada or Mexico”

$\forall\ \forall x\ (M(x) \vee C(x))$

  – When the universe of discourse is all students

$\forall\ \forall x\ (S(x) \rightarrow (M(x) \vee C(x)))$

  – When the universe of discourse is all people

- Why isn’t $\forall x\ (S(x) \wedge (M(x) \vee C(x)))$ correct?

Translating from English 6

- Note that it would be easier to define V(x, y) as “x has visited y”

  $\forall x\, (S(x) \wedge V(x,\text{Mexico}))$

  $\forall x\, (S(x) \rightarrow (V(x,\text{Mexico}) \vee V(x,\text{Canada}))$

Translating from English 7

- Translate the statements:
  - “All hummingbirds are richly colored”
  - “No large birds live on honey”
  - “Birds that do not live on honey are dull in color”
  - “Hummingbirds are small”
- Assign our propositional functions
  - Let P(x) be “x is a hummingbird”
  - Let Q(x) be “x is large”
  - Let R(x) be “x lives on honey”
  - Let S(x) be “x is richly colored”
- Let our universe of discourse be all birds

Translating from English 8

- Our propositional functions
  - Let P(x) be “x is a hummingbird”
  - Let Q(x) be “x is large”
  - Let R(x) be “x lives on honey”
  - Let S(x) be “x is richly colored”
- Translate the statements:
  - “All hummingbirds are richly colored”

    $\forall\forall x\ (P(x) \rightarrow S(x))$
  - “No large birds live on honey”
    - $\neg\exists x\ (Q(x) \wedge R(x))$
    - Alternatively: $\forall x\ (\neg Q(x) \vee \neg R(x))$
  - “Birds that do not live on honey are dull in color”

    $\forall\forall x\ (\neg R(x) \rightarrow \neg S(x))$
  - “Hummingbirds are small”

    $\forall\forall x\ (P(x) \rightarrow \neg Q(x))$

Prolog

- A programming language using logic!
- Entering facts:

```
instructor (bloomfield, cs202)
enrolled (alice, cs202)
enrolled (bob, cs202)
enrolled (claire, cs202)
```

- Entering predicates:

```
teaches (P,S) :- instructor (P,C), enrolled (S,C)
```

- Extracting data

```
?enrolled (alice, cs202)
```

  - Result:

```
yes
```

Prolog 2

- Extracting data

```
?enrolled (X, cs202)
```

  - Result:

```
alice
bob
claire
```

- Extracting data

```
?teaches (X, alice)
```

  - Result:

```
bloomfield
```

Multiple quantifiers

- You can have multiple quantifiers on a statement

- $\forall x \exists y\ P(x, y)$
  - "For all x, there exists a y such that P(x,y)"
  - Example: $\forall x \exists y\ (x+y == 0)$

- $\exists x \forall y\ P(x,y)$
  - There exists an x such that for all y P(x,y) is true"
  - Example: $\exists x \forall y\ (x*y == 0)$

Order of quantifiers

- $\exists x \forall y$ and $\forall x \exists y$ are not equivalent!

- $\exists x \forall y\ P(x,y)$
  - $P(x,y) = (x+y == 0)$ is false

- $\forall x \exists y\ P(x,y)$
  - $P(x,y) = (x+y == 0)$ is true

Negating multiple quantifiers

- Recall negation rules for single quantifiers:
  - $\neg\forall x\ P(x) = \exists x\ \neg P(x)$
  - $\neg\exists x\ P(x) = \forall x\ \neg P(x)$
  - Essentially, you change the quantifier(s), and negate what it's quantifying

- Examples:
  - $\neg(\forall x\exists y\ P(x,y))$

    $= \exists x\ \neg\exists y\ P(x,y)$

    $= \exists x\forall y\ \neg P(x,y)$
  - $\neg(\forall x\exists y\forall z\ P(x,y,z))$

    $= \exists x\neg\exists y\forall z\ P(x,y,z)$

    $= \exists x\forall y\neg\forall z\ P(x,y,z)$

    $= \exists x\forall y\exists z\ \neg P(x,y,z)$

Negating multiple quantifiers 2

- Consider $\neg(\forall x \exists y\, P(x,y)) = \exists x \forall y\, \neg P(x,y)$
  - The left side is saying “for all x, there exists a y such that P is true”
  - To disprove it (negate it), you need to show that “there exists an x such that for all y, P is false”
- Consider $\neg(\exists x \forall y\, P(x,y)) = \forall x \exists y\, \neg P(x,y)$
  - The left side is saying “there exists an x such that for all y, P is true”
  - To disprove it (negate it), you need to show that “for all x, there exists a y such that P is false”

Translating between English and quantifiers

- The product of two negative integers is positive

  $\forall x \forall y\ ((x<0) \wedge (y<0) \rightarrow (xy > 0))$

  – Why conditional instead of and?
- The average of two positive integers is positive

  $\forall x \forall y\ ((x>0) \wedge (y>0) \rightarrow ((x+y)/2 > 0))$
- The difference of two negative integers is not necessarily negative

  $\exists x \exists y\ ((x<0) \wedge (y<0) \wedge (x-y \geq 0))$

  – Why and instead of conditional?
- The absolute value of the sum of two integers does not exceed the sum of the absolute values of these integers

  $\forall x \forall y\ (|x+y| \leq |x| + |y|)$

Translating between English and quantifiers

- $\exists x \forall y\ (x+y = y)$
  - There exists an additive identity for all real numbers
- $\forall x \forall y\ (((x \geq 0) \wedge (y<0)) \rightarrow (x-y > 0))$
  - A non-negative number minus a negative number is greater than zero
- $\exists x \exists y\ (((x \leq 0) \wedge (y \leq 0)) \wedge (x-y > 0))$
  - The difference between two non-positive numbers is not necessarily non-positive (i.e. can be positive)
- $\forall x \forall y\ (((x \neq 0) \wedge (y \neq 0)) \leftrightarrow (xy \neq 0))$
  - The product of two non-zero numbers is non-zero if and only if both factors are non-zero

Negation examples

- Rewrite these statements so that the negations only appear within the predicates

α) $\neg\exists y\exists x\, P(x,y)$

- $\forall y\neg\exists x\, P(x,y)$
- $\forall y\forall x\, \neg P(x,y)$

β) $\neg\forall x\exists y\, P(x,y)$

- $\exists x\neg\exists y\, P(x,y)$
- $\exists x\forall y\, \neg P(x,y)$

χ) $\neg\exists y\, (Q(y) \wedge \forall x\, \neg R(x,y))$

- $\forall y\, \neg(Q(y) \wedge \forall x\, \neg R(x,y))$
- $\forall y\, (\neg Q(y) \vee \neg(\forall x\, \neg R(x,y)))$
- $\forall y\, (\neg Q(y) \vee \exists x\, R(x,y))$

Negation examples

- Express the negations of each of these statements so that all negation symbols immediately precede predicates.

α) $\forall x \exists y \forall z\ T(x,y,z)$
- $\neg(\forall x \exists y \forall z\ T(x,y,z))$
- $\neg \forall x \exists y \forall z\ T(x,y,z)$
- $\exists x \neg \exists y \forall z\ T(x,y,z)$
- $\exists x \forall y \neg \forall z\ T(x,y,z)$
- $\exists x \forall y \exists z\ \neg T(x,y,z)$

β) $\forall x \exists y\ P(x,y) \vee \forall x \exists y\ Q(x,y)$
- $\neg(\forall x \exists y\ P(x,y) \vee \forall x \exists y\ Q(x,y))$
- $\neg \forall x \exists y\ P(x,y) \wedge \neg \forall x \exists y\ Q(x,y)$
- $\exists x \neg \exists y\ P(x,y) \wedge \exists x \neg \exists y\ Q(x,y)$
- $\exists x \forall y\ \neg P(x,y) \wedge \exists x \forall y\ \neg Q(x,y)$

Rules of inference for the universal quantifier

- Assume that we know that $\forall x\ P(x)$ is true
  - Then we can conclude that $P(c)$ is true
    - Here c stands for some specific constant
  - This is called “universal instantiation”

- Assume that we know that $P(c)$ is true for any value of c
  - Then we can conclude that $\forall x\ P(x)$ is true
  - This is called “universal generalization”

Rules of inference for the existential quantifier

- Assume that we know that $\exists x\, P(x)$ is true
  - Then we can conclude that $P(c)$ is true for some value of c
  - This is called "existential instantiation"

- Assume that we know that $P(c)$ is true for some value of c
  - Then we can conclude that $\exists x\, P(x)$ is true
  - This is called "existential generalization"

Example of proof

- Given the hypotheses:
  - "Linda, a student in this class, owns a red convertible."
  - "Everybody who owns a red convertible has gotten at least one speeding ticket"
- Can you conclude: "Somebody in this class has gotten a speeding ticket"?

$C(\text{Linda})$

$R(\text{Linda})$

$\forall x\ (R(x) \rightarrow T(x))$

$\exists x\ (C(x) \wedge T(x))$

Example of proof

1.	$\forall x\ (R(x) \rightarrow T(x))$	3rd hypothesis
2.	$R(Linda) \rightarrow T(Linda)$	Universal instantiation using step 1
3.	$R(Linda)$	2nd hypothesis
4.	$T(Linda)$	Modes ponens using steps 2 & 3
5.	$C(Linda)$	1st hypothesis
6.	$C(Linda) \wedge T(Linda)$	Conjunction using steps 4 & 5
7.	$\exists x\ (C(x) \wedge T(x))$	Existential generalization using step 6

Thus, we have shown that “Somebody in this class has gotten a speeding ticket”

Example of proof

- Given the hypotheses:
  - "There is someone in this class who has been to France" $\exists x\,(C(x) \wedge F(x))$
  - "Everyone who goes to France visits the Louvre" $\forall x\,(F(x) \rightarrow L(x))$
- Can you conclude: "Someone in this class has visited the Louvre"? $\exists x\,(C(x) \wedge L(x))$

Example of proof

1.	$\exists x\,(C(x) \wedge F(x))$	1st hypothesis
2.	$C(y) \wedge F(y)$	Existential instantiation using step 1
3.	$F(y)$	Simplification using step 2
4.	$C(y)$	Simplification using step 2
5.	$\forall x\,(F(x) \rightarrow L(x))$	2nd hypothesis
6.	$F(y) \rightarrow L(y)$	Universal instantiation using step 5
7.	$L(y)$	Modus ponens using steps 3 & 6
8.	$C(y) \wedge L(y)$	Conjunction using steps 4 & 7
9.	$\exists x\,(C(x) \wedge L(x))$	Existential generalization using step 8

Thus, we have shown that "Someone in this class has visited the Louvre"