MATHEMATICAL ANALAYSIS 2

Lecture 2

Prepared by

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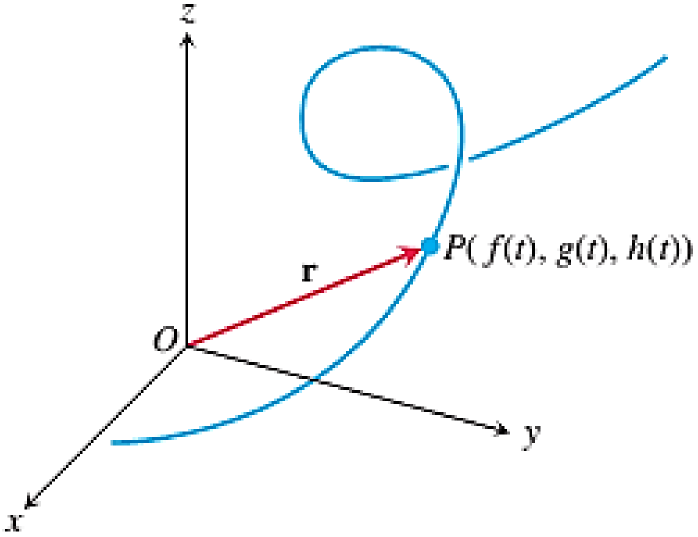
Vector-Valued Functions and Motion in Space

- Curves in Space and Their Tangents
- Integrals of Vector Functions; Projectile Motion
- Arc Length in Space
- Curvature and Normal Vectors of a Curve
- Tangential and Normal Components of Acceleration

Curves in Space and Their Tangents

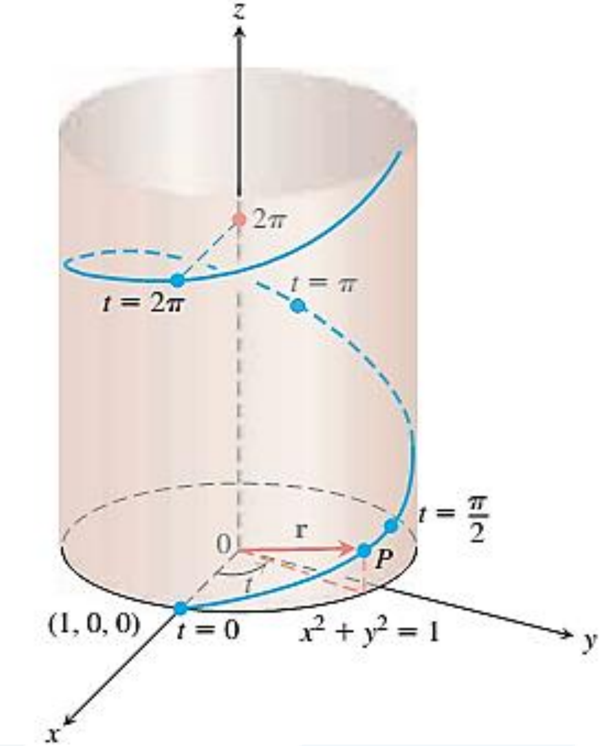
$$\mathbf{r}(t) = \overrightarrow{OP} = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$



EXAMPLE 1 Graph the vector function

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}.$$

Helix



Curves in Space and Their Tangents

Limits and Continuity

DEFINITION Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ be a vector function with domain D, and let $\mathbf{L}$ be a vector. We say that $\mathbf{r}$ has **limit** $\mathbf{L}$ as t approaches t_0 and write

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{L}$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all $t \in D$

$$\left|\mathbf{r}(t) - \mathbf{L}\right| < \varepsilon \quad \text{whenever} \quad 0 < \left|t - t_0\right| < \delta.$$

DEFINITION A vector function $\mathbf{r}(t)$ is **continuous at a point** $t = t_0$ in its domain if $\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$. The function is **continuous** if it is continuous at every point in its domain.

Curves in Space and Their Tangents

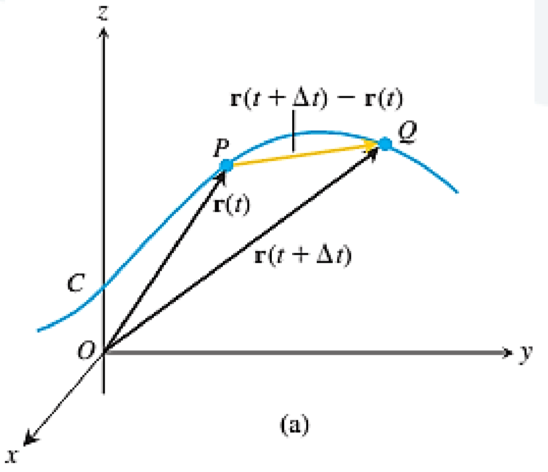
Limits and Continuity

$$\mathbf{g}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + \lfloor t \rfloor \mathbf{k}$$

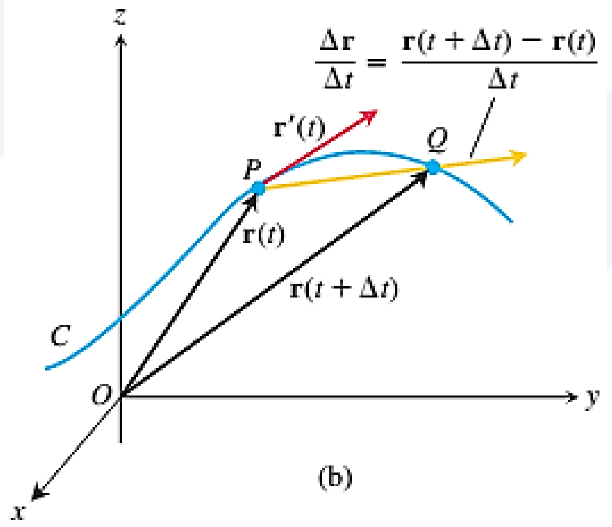
is discontinuous at every integer, because the greatest integer function $\lfloor t \rfloor$ is discontinuous at every integer. ■

Curves in Space and Their Tangents

Derivatives and Motion



(a)



(b)

DEFINITION The vector function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ has a **derivative (is differentiable) at t** if f, g, and h have derivatives at t. The derivative is the vector function

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = \frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k}.$$

Curves in Space and Their Tangents

Derivatives and Motion

DEFINITIONS If $\mathbf{r}$ is the position vector of a particle moving along a smooth curve in space, then

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt}$$

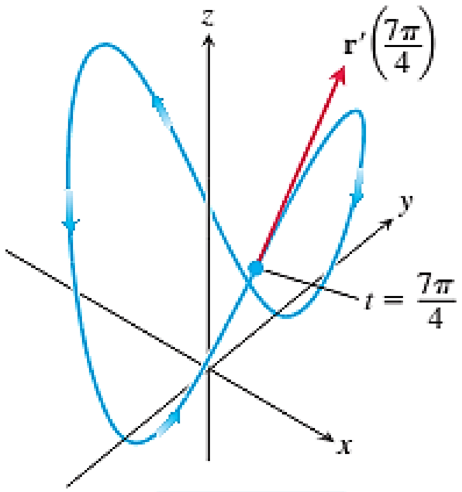
is the particle's **velocity vector**, tangent to the curve. At any time t, the direction of $\mathbf{v}$ is the **direction of motion**, the magnitude of $\mathbf{v}$ is the particle's **speed**, and the derivative $\mathbf{a} = d\mathbf{v}/dt$, when it exists, is the particle's **acceleration vector**. In summary,

1. Velocity is the derivative of position: $\mathbf{v} = \dfrac{d\mathbf{r}}{dt}$.
2. Speed is the magnitude of velocity: $\text{Speed} = |\mathbf{v}|$.
3. Acceleration is the derivative of velocity: $\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = \dfrac{d^2\mathbf{r}}{dt^2}$.
4. The unit vector $\mathbf{v}/|\mathbf{v}|$ is the direction of motion at time t.

Curves in Space and Their Tangents

Derivatives and Motion

EXAMPLE 4 Find the velocity, speed, and acceleration of a particle whose motion in space is given by the position vector $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 5\cos^2 t\,\mathbf{k}$. Sketch the velocity vector $\mathbf{v}(7\pi/4)$.



$$\mathbf{v}(t) = \mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} - 5\sin 2t\,\mathbf{k},$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = -2\cos t\,\mathbf{i} - 2\sin t\,\mathbf{j} - 10\cos 2t\,\mathbf{k},$$

$$|\mathbf{v}(t)| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + (-5\sin 2t)^2} = \sqrt{4 + 25\sin^2 2t}.$$

When $t = 7\pi/4$, we have

$$\mathbf{v}\left(\frac{7\pi}{4}\right) = \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j} + 5\mathbf{k}, \qquad \mathbf{a}\left(\frac{7\pi}{4}\right) = -\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}, \qquad \left|\mathbf{v}\left(\frac{7\pi}{4}\right)\right| = \sqrt{29}.$$

Curves in Space and Their Tangents

Differentiation Rules

Differentiation Rules for Vector Functions

Let $\mathbf{u}$ and $\mathbf{v}$ be differentiable vector functions of t, $\mathbf{C}$ a constant vector, c any scalar, and f any differentiable scalar function.

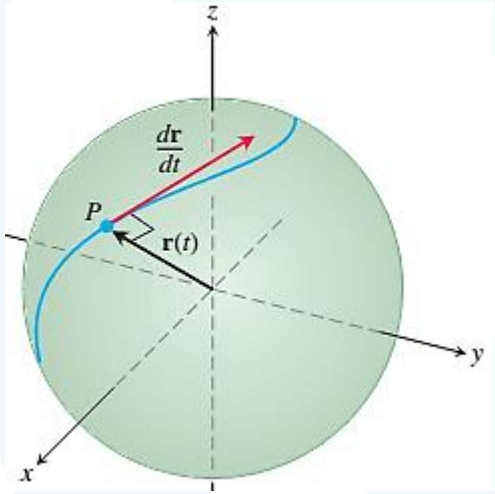
1. *Constant Function Rule:* $\dfrac{d}{dt}\mathbf{C} = \mathbf{0}$
2. *Scalar Multiple Rules:* $\dfrac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

   $\dfrac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
3. *Sum Rule:* $\dfrac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
4. *Difference Rule:* $\dfrac{d}{dt}[\mathbf{u}(t) - \mathbf{v}(t)] = \mathbf{u}'(t) - \mathbf{v}'(t)$
5. *Dot Product Rule:* $\dfrac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$
6. *Cross Product Rule:* $\dfrac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$
7. *Chain Rule:* $\dfrac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

Vector Functions of Constant Length

$$\mathbf{r}(t)\cdot\mathbf{r}(t) = |\mathbf{r}(t)|^2 = c^2$$

$$\mathbf{r}\cdot\frac{d\mathbf{r}}{dt} = 0.$$



Exercises

- $\lim_{t\to 1}\left[\left(\frac{t^2-1}{\ln t}\right)\mathbf{i}-\left(\frac{\sqrt{t}-1}{1-t}\right)\mathbf{j}+(\tan^{-1} t)\mathbf{k}\right]$ $\lim_{t\to 0}\left[\left(\frac{\sin t}{t}\right)\mathbf{i}+\left(\frac{\tan^2 t}{\sin 2t}\right)\mathbf{j}-\left(\frac{t^3-8}{t+2}\right)\mathbf{k}\right]$

- Give the position vectors of particles moving along the curve in the xy-plane, find the particle's velocity and acceleration vectors at the stated times and sketch them as vectors on the curve.

Motion on the circle $x^2 + y^2 = 1$

$\mathbf{r}(t) = (\sin t)\mathbf{i} + (\cos t)\mathbf{j}; \quad t = \pi/4$ and $\pi/2$

$\mathbf{r}(t) = (t+1)\mathbf{i} + (t^2-1)\mathbf{j} + 2t\mathbf{k}, \quad t = 1$

- $\mathbf{r}(t)$ is the position of a particle in space at time t. Find the angle between the velocity and acceleration vectors at time $t = 0$.

$\mathbf{r}(t) = \left(\frac{\sqrt{2}}{2}t\right)\mathbf{i} + \left(\frac{\sqrt{2}}{2}t - 16t^2\right)\mathbf{j}$ $\theta = \frac{3\pi}{4}$

Exercises

- As mentioned in the text, the **tangent line** to a smooth curve $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ at $t = t_0$ is the line that passes through the point $(f(t_0), g(t_0), h(t_0))$ parallel to $\mathbf{v}(t_0)$, the curve's velocity vector at t_0. In Exercises 23–26, find parametric equations for the line that is tangent to the given curve at the given parameter value $t = t_0$.

$$\mathbf{r}(t) = \ln t\,\mathbf{i} + \frac{t-1}{t+2}\mathbf{j} + t\ln t\,\mathbf{k}, \quad t_0 = 1$$

$x = 0 + t = t$, $y = 0 + \frac{1}{3}t = \frac{1}{3}t$, and $z = 0 + t = t$

Integrals of Vector Functions; Projectile Motion

Integrals of Vector Functions

DEFINITION The **indefinite integral** of $\mathbf{r}$ with respect to t is the set of all antiderivatives of $\mathbf{r}$, denoted by $\int \mathbf{r}(t)\,dt$. If $\mathbf{R}$ is any antiderivative of $\mathbf{r}$, then

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{C}.$$

DEFINITION If the components of $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ are integrable over $[a, b]$, then so is $\mathbf{r}$, and the **definite integral** of $\mathbf{r}$ from a to b is

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\mathbf{i} + \left(\int_a^b g(t)\,dt\right)\mathbf{j} + \left(\int_a^b h(t)\,dt\right)\mathbf{k}.$$

$$\int_a^b \mathbf{r}(t)\,dt = \mathbf{R}(t)\Big]_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

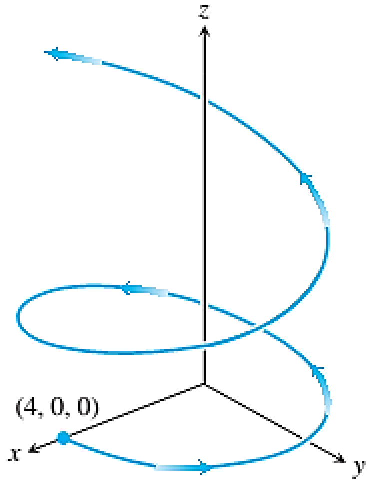
Integrals of Vector Functions; Projectile Motion

Integrals of Vector Functions

EXAMPLE 2

$$\int_0^{\pi} ((\cos t)\mathbf{i} + \mathbf{j} - 2t\mathbf{k})\, dt = \left[\sin t\right]_0^{\pi}\mathbf{i} + \left[t\right]_0^{\pi}\mathbf{j} - \left[t^2\right]_0^{\pi}\mathbf{k} = \pi\mathbf{j} - \pi^2\mathbf{k}$$

EXAMPLE 3 Suppose we do not know the path of a hang glider, but only its acceleration vector $\mathbf{a}(t) = -(3\cos t)\mathbf{i} - (3\sin t)\mathbf{j} + 2\mathbf{k}$. We also know that initially (at time $t = 0$) the glider departed from the point $(4, 0, 0)$ with velocity $\mathbf{v}(0) = 3\mathbf{j}$. Find the glider's position as a function of t.



$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = -(3\cos t)\mathbf{i} - (3\sin t)\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{v}(0) = 3\mathbf{j} \quad \text{and} \quad \mathbf{r}(0) = 4\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}.$$

Integrating

$$\longrightarrow \mathbf{v}(t) = -(3\sin t)\mathbf{i} + (3\cos t)\mathbf{j} + 2t\mathbf{k} + \mathbf{C}_1. \xrightarrow{\mathbf{v}(0) = 3\mathbf{j}} \mathbf{C}_1 = \mathbf{0}.$$

$$\longrightarrow \frac{d\mathbf{r}}{dt} = \mathbf{v}(t) = -(3\sin t)\mathbf{i} + (3\cos t)\mathbf{j} + 2t\mathbf{k}.$$

Integrating

$$\longrightarrow \mathbf{r}(t) = (3\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + t^2\mathbf{k} + \mathbf{C}_2. \xrightarrow{\mathbf{r}(0) = 4\mathbf{i}} \mathbf{C}_2 = \mathbf{i}.$$

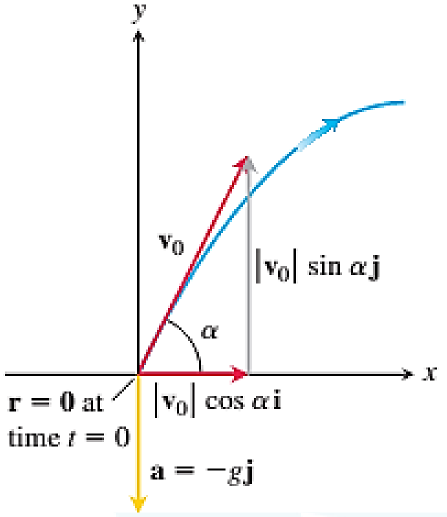
$$\mathbf{r}(t) = (1 + 3\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + t^2\mathbf{k}.$$

The Vector and Parametric Equations for Ideal Projectile Motion

$$\mathbf{v}_0 = (|\mathbf{v}_0|\cos\alpha)\mathbf{i} + (|\mathbf{v}_0|\sin\alpha)\mathbf{j}.$$

$$\mathbf{v}_0 = (v_0\cos\alpha)\mathbf{i} + (v_0\sin\alpha)\mathbf{j}.$$

$$\mathbf{r}_0 = 0\mathbf{i} + 0\mathbf{j} = \mathbf{0}.$$



Newton's second law of motion

$$m\frac{d^2\mathbf{r}}{dt^2} = -mg\mathbf{j} \Longrightarrow \frac{d^2\mathbf{r}}{dt^2} = -g\mathbf{j},$$

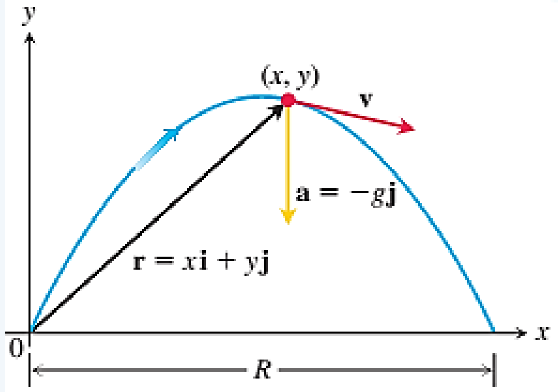
$$\mathbf{r} = \mathbf{r}_0 \quad \text{and} \quad \frac{d\mathbf{r}}{dt} = \mathbf{v}_0 \quad \text{when } t = 0$$

The first integration gives $\quad \dfrac{d\mathbf{r}}{dt} = -(gt)\mathbf{j} + \mathbf{v}_0.$

A second integration gives $\quad \mathbf{r} = -\dfrac{1}{2}gt^2\mathbf{j} + \mathbf{v}_0 t + \mathbf{r}_0.$

Ideal Projectile Motion Equation

$$\mathbf{r} = (v_0\cos\alpha)t\mathbf{i} + \left((v_0\sin\alpha)t - \frac{1}{2}gt^2\right)\mathbf{j}. \qquad (5)$$



Horizontal range

Parametric Equations $\quad x = (v_0\cos\alpha)t \quad \text{and} \quad y = (v_0\sin\alpha)t - \dfrac{1}{2}gt^2,$

The Vector and Parametric Equations for Ideal Projectile Motion

$t = x/(v_0 \cos \alpha)$ → $$y = -\left(\frac{g}{2{v_0}^2 \cos^2 \alpha}\right)x^2 + (\tan \alpha)x.$$

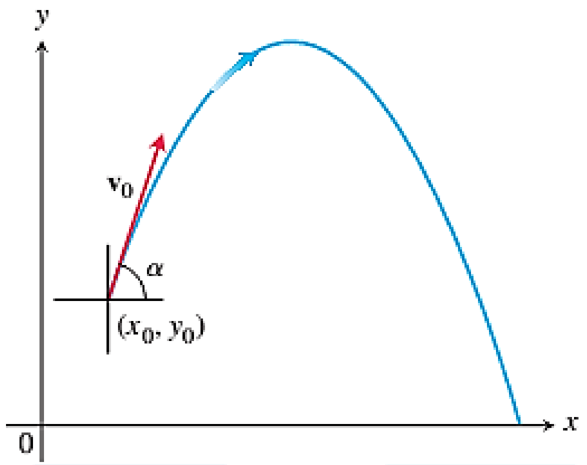
Height, Flight Time, and Range for Ideal Projectile Motion

For ideal projectile motion when an object is launched from the origin over a horizontal surface with initial speed v_0 and launch angle α:

Maximum height: $$y_{\max} = \frac{(v_0 \sin \alpha)^2}{2g}$$

Flight time: $$t = \frac{2v_0 \sin \alpha}{g}$$

Range: $$R = \frac{{v_0}^2}{g} \sin 2\alpha.$$



$$\mathbf{r} = (x_0 + (v_0 \cos \alpha)t)\mathbf{i} + \left(y_0 + (v_0 \sin \alpha)t - \frac{1}{2}gt^2\right)\mathbf{j},$$

The Vector and Parametric Equations for Ideal Projectile Motion

EXAMPLE 4 A projectile is fired from the origin over horizontal ground at an initial speed of 500 m/sec and a launch angle of 60°. Where will the projectile be 10 sec later?

$v_0 = 500$, $\alpha = 60°$, $g = 9.8$, and $t = 10$

$$\mathbf{r} = (v_0 \cos \alpha)t\mathbf{i} + \left((v_0 \sin \alpha)t - \frac{1}{2}gt^2\right)\mathbf{j}$$

$$= (500)\left(\frac{1}{2}\right)(10)\mathbf{i} + \left((500)\left(\frac{\sqrt{3}}{2}\right)10 - \left(\frac{1}{2}\right)(9.8)(100)\right)\mathbf{j}$$

$$\approx 2500\mathbf{i} + 3840\mathbf{j}$$

Projectile Motion with Wind Gusts

EXAMPLE 5 A baseball is hit when it is 3 ft above the ground. It leaves the bat with initial speed of 152 ft/sec, making an angle of 20° with the horizontal. At the instant the ball is hit, an instantaneous gust of wind blows in the horizontal direction directly opposite the direction the ball is taking toward the outfield, adding a component of $-8.8\mathbf{i}$ (ft/sec) to the ball's initial velocity (8.8 ft/sec = 6 mph).

(a) Find a vector equation (position vector) for the path of the baseball.

(b) How high does the baseball go, and when does it reach maximum height?

(c) Assuming that the ball is not caught, find its range and flight time.

$$\mathbf{v}_0 = (v_0 \cos \alpha)\mathbf{i} + (v_0 \sin \alpha)\mathbf{j} - 8.8\mathbf{i}$$

$$= (152 \cos 20° - 8.8)\mathbf{i} + (152 \sin 20°)\mathbf{j}$$

Projectile Motion with Wind Gusts

The first integration gives $\dfrac{d\mathbf{r}}{dt} = -(gt)\mathbf{j} + \mathbf{v}_0.$

A second integration gives $\mathbf{r} = -\dfrac{1}{2}gt^2\mathbf{j} + \mathbf{v}_0 t + \mathbf{r}_0.$

The initial position is $\mathbf{r}_0 = 0\mathbf{i} + 3\mathbf{j}.$ $\longrightarrow$ $\mathbf{r} = (152\cos 20° - 8.8)t\mathbf{i} + \left(3 + (152\sin 20°)t - 16t^2\right)\mathbf{j}.$

(b) The baseball reaches its highest point when the vertical component of velocity is zero, or

$$\frac{dy}{dt} = 152\sin 20° - 32t = 0. \longrightarrow t = \frac{152\sin 20°}{32} \approx 1.62 \text{ sec.}$$

$$y_{\max} = 3 + (152\sin 20°)(1.62) - 16(1.62)^2 \approx 45.2 \text{ ft.}$$

(c) To find when the baseball lands, we set the vertical component for **r** equal to 0 and solve for t:

$$3 + (152\sin 20°)t - 16t^2 = 0 \longrightarrow t = 3.3 \text{ sec and } t = -0.06 \text{ sec.} \longrightarrow R = (152\cos 20° - 8.8)(3.3) \approx 442 \text{ ft.}$$

Exercises

Solve the initial value problems

$$\frac{d\mathbf{r}}{dt} = \frac{3}{2}(t+1)^{1/2}\mathbf{i} + e^{-t}\mathbf{j} + \frac{1}{t+1}\mathbf{k} \qquad \mathbf{r}(0) = \mathbf{k}$$

$$\mathbf{r} = \left[(t+1)^{3/2} - 1\right]\mathbf{i} + \left(1 - e^{-t}\right)\mathbf{j} + \left[1 + \ln(t+1)\right]\mathbf{k}$$

$$\frac{d^2\mathbf{r}}{dt^2} = -(\mathbf{i} + \mathbf{j} + \mathbf{k}) \qquad \mathbf{r}(0) = 10\mathbf{i} + 10\mathbf{j} + 10\mathbf{k} \text{ and } \left.\frac{d\mathbf{r}}{dt}\right|_{t=0} = \mathbf{0}$$

$$\mathbf{r} = \left(-\frac{t^2}{2} + 10\right)\mathbf{i} + \left(-\frac{t^2}{2} + 10\right)\mathbf{j} + \left(-\frac{t^2}{2} + 10\right)\mathbf{k}$$

Flight time and height A projectile is fired with an initial speed of 500 m/sec at an angle of elevation of 45°.

a. When and how far away will the projectile strike? 72.2 seconds 25,510.2 m

b. How high overhead will the projectile be when it is 5 km downrange? 4020 m

c. What is the greatest height reached by the projectile? 6378 m

Arc Length in Space

Arc Length Along a Space Curve

DEFINITION The **length** of a smooth curve $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, $a \le t \le b$, that is traced exactly once as t increases from $t = a$ to $t = b$, is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt. \qquad (1)$$

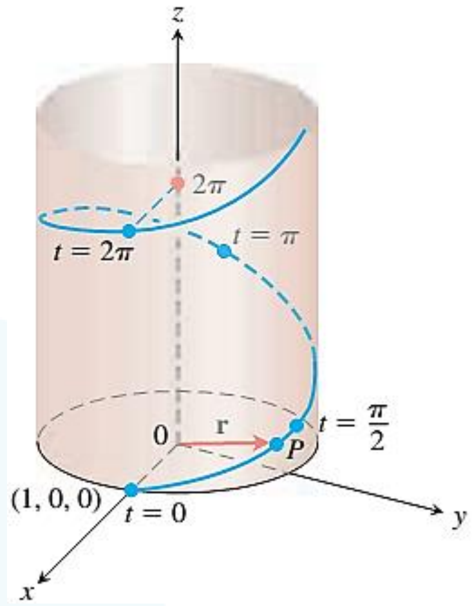
Arc Length Formula

$$L = \int_a^b |\mathbf{v}|\, dt \qquad (2)$$

Arc Length in Space

Arc Length Along a Space Curve

EXAMPLE 1 A glider is soaring upward along the helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$. How long is the glider's path from $t = 0$ to $t = 2\pi$?

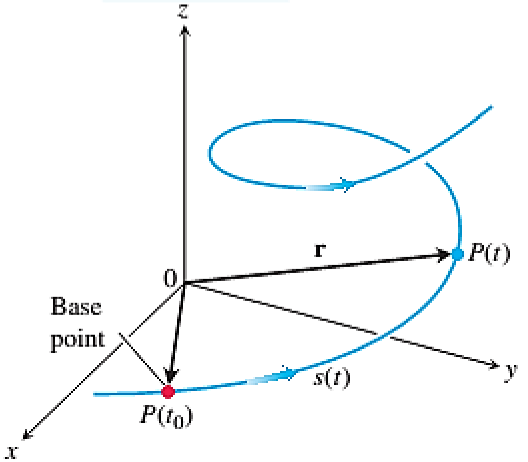


$$L = \int_a^b |\mathbf{v}|\, dt = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{2}\, dt = 2\pi\sqrt{2} \text{ units of length}$$

Arc Length Parameter with Base Point $P(t_0)$ 'directed distance'

$$s(t) = \int_{t_0}^{t} \sqrt{[x'(\tau)]^2 + [y'(\tau)]^2 + [z'(\tau)]^2}\, d\tau = \int_{t_0}^{t} |\mathbf{v}(\tau)|\, d\tau \qquad (3)$$



Arc Length in Space

Arc Length Along a Space Curve

EXAMPLE 2 This is an example for which we can actually find the arc length parametrization of a curve. If $t_0 = 0$, the arc length parameter along the helix

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$$

$$s(t) = \int_{t_0}^{t} |\mathbf{v}(\tau)|\, d\tau = \sqrt{2}\, t.$$

$$\xrightarrow{t = s/\sqrt{2}} \quad \mathbf{r}(t(s)) = \left(\cos \frac{s}{\sqrt{2}}\right)\mathbf{i} + \left(\sin \frac{s}{\sqrt{2}}\right)\mathbf{j} + \frac{s}{\sqrt{2}}\mathbf{k}.$$

Arc Length in Space

Speed on a Smooth Curve

$$\frac{ds}{dt} = |\mathbf{v}(t)|$$

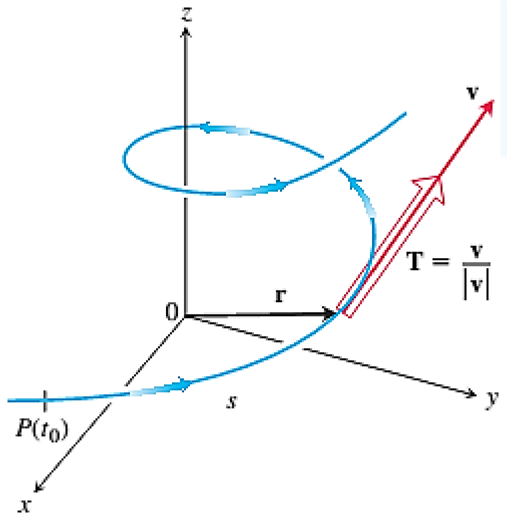
Notice that $ds/dt > 0$ since, by definition, $|\mathbf{v}|$ is never zero for a smooth curve. We see once again that s is an increasing function of t.

Unit Tangent Vector

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

If the position vector change with respect to arc length

$$\frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{dt}\frac{dt}{ds} = \mathbf{v}\frac{1}{|\mathbf{v}|} = \frac{\mathbf{v}}{|\mathbf{v}|} = \mathbf{T}$$



EXAMPLE 3 Find the unit tangent vector of the curve

$$\mathbf{r}(t) = (1 + 3\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + t^2\mathbf{k}$$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -(3\sin t)\mathbf{i} + (3\cos t)\mathbf{j} + 2t\mathbf{k} \longrightarrow |\mathbf{v}| = \sqrt{9 + 4t^2}.$$

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = -\frac{3\sin t}{\sqrt{9 + 4t^2}}\mathbf{i} + \frac{3\cos t}{\sqrt{9 + 4t^2}}\mathbf{j} + \frac{2t}{\sqrt{9 + 4t^2}}\mathbf{k}.$$

Exercises

- Find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve.

$$\mathbf{r}(t) = (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} + \left(2\sqrt{2}/3\right)t^{3/2}\mathbf{k}, \quad 0 \le t \le \pi$$

$\left(\frac{\cos t - t\sin t}{t+1}\right)\mathbf{i} + \left(\frac{\sin t + t\cos t}{t+1}\right)\mathbf{j} + \left(\frac{\sqrt{2}t^{1/2}}{t+1}\right)\mathbf{k}$ $\quad \frac{\pi^2}{2} + \pi$

- Find the point on the curve

$$\mathbf{r}(t) = (5\sin t)\mathbf{i} + (5\cos t)\mathbf{j} + 12t\mathbf{k}$$

at a distance 26π units along the curve from the point $(0, 5, 0)$ in the direction of increasing arc length. $\quad (0, 5, 24\pi)$

- find the arc length parameter along the curve from the point where $t = 0$ by evaluating the integral

$$s = \int_0^t |\mathbf{v}(\tau)|\, d\tau$$

Then find the length of the indicated portion of the curve.

$$\mathbf{r}(t) = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j}, \quad \pi/2 \le t \le \pi$$

$\frac{t^2}{2} \quad \frac{3\pi^2}{8}$

Curvature of a Plane Curve

DEFINITION If $\mathbf{T}$ is the unit vector of a smooth curve, the **curvature** function of the curve is

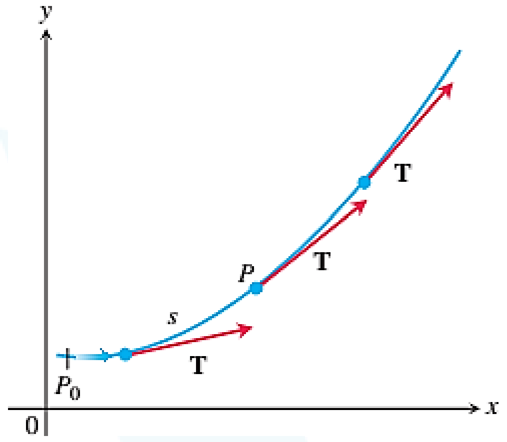
$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right|.$$

Formula for Calculating Curvature

If $\mathbf{r}(t)$ is a smooth curve, then the curvature is the scalar function

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right|, \qquad (1)$$

where $\mathbf{T} = \mathbf{v}/|\mathbf{v}|$ is the unit tangent vector.



Curvature and Normal Vectors of a Curve

Curvature of a Plane Curve

EXAMPLE 2 Here we find the curvature of a circle. We begin with the parametrization

$$\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}$$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -(a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} \longrightarrow |\mathbf{v}| = \sqrt{(-a\sin t)^2 + (a\cos t)^2} = \sqrt{a^2} = |a| = a.$$

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = -(\sin t)\mathbf{i} + (\cos t)\mathbf{j} \longrightarrow \frac{d\mathbf{T}}{dt} = -(\cos t)\mathbf{i} - (\sin t)\mathbf{j} \longrightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{\cos^2 t + \sin^2 t} = 1$$

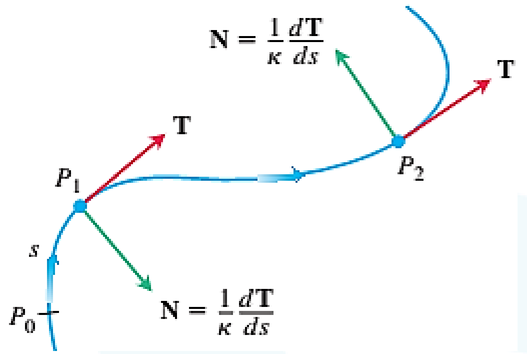
$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{1}{a}(1) = \frac{1}{a} = \frac{1}{\text{radius}}.$$

Curvature and Normal Vectors of a Curve

Curvature of a Plane Curve

DEFINITION At a point where $\kappa \neq 0$, the **principal unit normal** vector for a smooth curve in the plane is

$$\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}.$$



Formula for Calculating *N*

If $\mathbf{r}(t)$ is a smooth curve, then the principal unit normal is

$$\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}, \qquad (2)$$

where $\mathbf{T} = \mathbf{v}/|\mathbf{v}|$ is the unit tangent vector.

Curvature and Normal Vectors of a Curve

Curvature of a Plane Curve

EXAMPLE 3 Find **T** and **N** for the circular motion

$$\mathbf{r}(t) = (\cos 2t)\mathbf{i} + (\sin 2t)\mathbf{j}.$$

$$\mathbf{v} = -(2\sin 2t)\mathbf{i} + (2\cos 2t)\mathbf{j} \longrightarrow |\mathbf{v}| = \sqrt{4\sin^2 2t + 4\cos^2 2t} = 2$$

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = -(\sin 2t)\mathbf{i} + (\cos 2t)\mathbf{j} \longrightarrow \frac{d\mathbf{T}}{dt} = -(2\cos 2t)\mathbf{i} - (2\sin 2t)\mathbf{j}$$

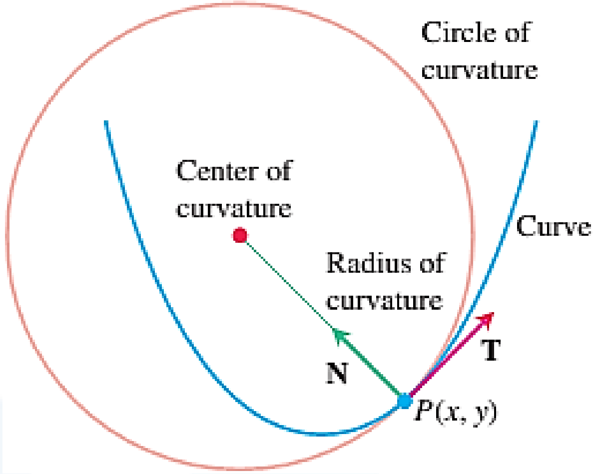
$$\longrightarrow \left|\frac{d\mathbf{T}}{dt}\right| = \sqrt{4\cos^2 2t + 4\sin^2 2t} = 2 \longrightarrow \mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} = -(\cos 2t)\mathbf{i} - (\sin 2t)\mathbf{j}$$

Curvature and Normal Vectors of a Curve

Circle of Curvature for Plane Curves

The **circle of curvature** or **osculating circle** at a point P on a plane curve where $\kappa \neq 0$ is the circle in the plane of the curve that

1. is tangent to the curve at P (has the same tangent line the curve has)
2. has the same curvature the curve has at P
3. has center that lies toward the concave or inner side of the curve (as in Figure 13.20).



The **radius of curvature** of the curve at P is the radius of the circle of curvature,

$$\text{Radius of curvature} = \rho = \frac{1}{\kappa}$$

$$C = \mathbf{r}(t_0) + \rho(t_0)\mathbf{N}(t_0) \quad \text{Center of Curvature}$$

The center of curvature of the curve at P is the center of the circle of curvature.

Curvature and Normal Vectors of a Curve

Circle of Curvature for Plane Curves

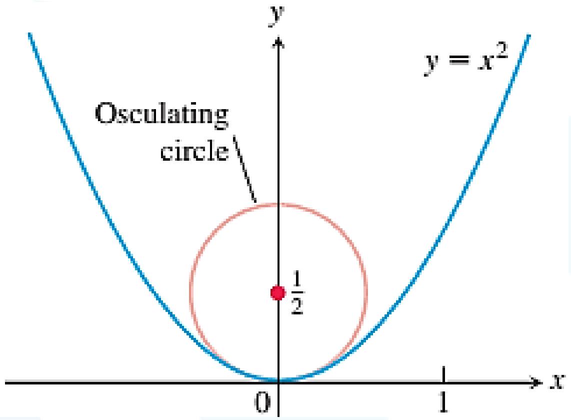
EXAMPLE 4 Find and graph the osculating circle of the parabola $y = x^2$ at the origin.

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \longrightarrow |\mathbf{v}| = \sqrt{1 + 4t^2}$$

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = (1 + 4t^2)^{-1/2}\mathbf{i} + 2t(1 + 4t^2)^{-1/2}\mathbf{j}$$

$$\frac{d\mathbf{T}}{dt} = -4t(1 + 4t^2)^{-3/2}\mathbf{i} + \left[2(1 + 4t^2)^{-1/2} - 8t^2(1 + 4t^2)^{-3/2}\right]\mathbf{j}$$



$$\kappa(0) = \frac{1}{|\mathbf{v}(0)|}\left|\frac{d\mathbf{T}}{dt}(0)\right| = \frac{1}{\sqrt{1}}|0\mathbf{i} + 2\mathbf{j}| = (1)\sqrt{0^2 + 2^2} = 2$$

$\longrightarrow$ the radius of curvature is $1/\kappa = 1/2$ $\longrightarrow$ Thus the center of the circle is $(0, 1/2)$

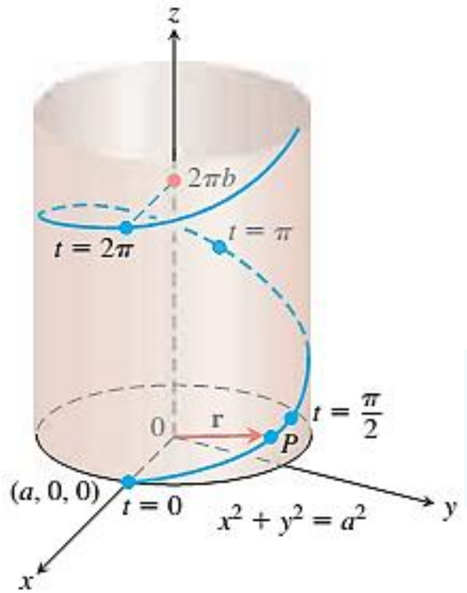
$$(x - 0)^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$$

Curvature and Normal Vectors of a Curve

Curvature and Normal Vectors for Space Curves

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right|$$

$$\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$$



EXAMPLE 5 Find the curvature for the helix (Figure 13.22)

$$\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + bt\mathbf{k}, \qquad a, b \geq 0, \qquad a^2 + b^2 \neq 0.$$

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{a^2+b^2}}\left[-(a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + b\mathbf{k}\right]$$

$$\kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| = \frac{a}{a^2+b^2}$$

$$\mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} = -(\cos t)\mathbf{i} - (\sin t)\mathbf{j}.$$

If $b = 0$, the helix reduces to a circle of radius a and its curvature reduces to $1/a$

If $a = 0$, the helix becomes the z-axis, and its curvature reduces to 0,

Thus, $\mathbf{N}$ is parallel to the xy-plane and always points toward the z-axis.

Exercises

- Find $\mathbf{T}$, $\mathbf{N}$, and κ for the plane curves

  $\mathbf{r}(t) = (\ln \sec t)\mathbf{i} + t\mathbf{j}, \quad -\pi/2 < t < \pi/2$

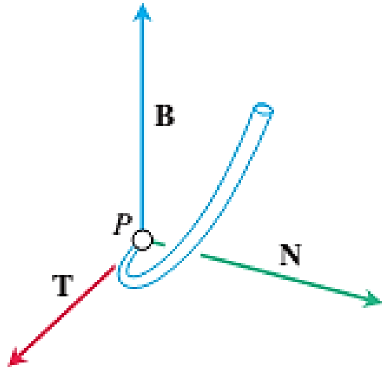
  $(\sin t)\mathbf{i} + (\cos t)\mathbf{j}$ $(\cos t)\mathbf{i} - (\sin t)\mathbf{j}$ $\cos t.$

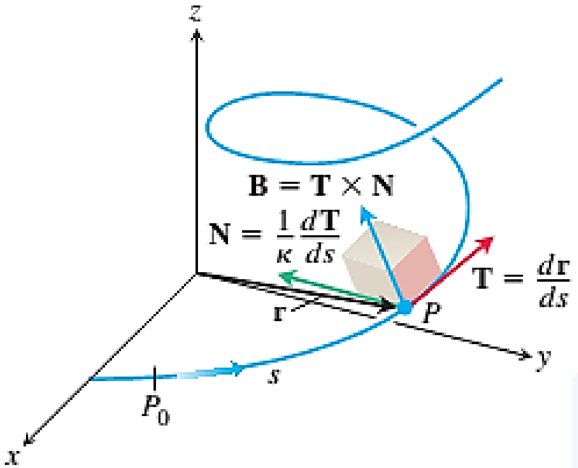
- Find an equation for the circle of curvature of the curve $\mathbf{r}(t) = t\mathbf{i} + (\sin t)\mathbf{j}$ at the point $(\pi/2, 1)$. (The curve parametrizes the graph of $y = \sin x$ in the xy-plane.)

  $\left(x - \frac{\pi}{2}\right)^2 + y^2 = 1$

Tangential and Normal Components of Acceleration

The TNB Frame





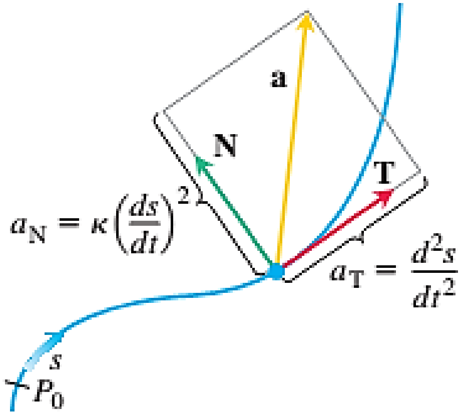
DEFINITION If the acceleration vector is written as

$$\mathbf{a} = a_{\mathrm{T}}\mathbf{T} + a_{\mathrm{N}}\mathbf{N}, \tag{1}$$

then

$$a_{\mathrm{T}} = \frac{d^2s}{dt^2} = \frac{d}{dt}|\mathbf{v}| \quad \text{and} \quad a_{\mathrm{N}} = \kappa\left(\frac{ds}{dt}\right)^2 = \kappa|\mathbf{v}|^2 \tag{2}$$

are the **tangential** and **normal** scalar components of acceleration.



Tangential and Normal Components of Acceleration

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \mathbf{T}\frac{ds}{dt}.$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(\mathbf{T}\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\frac{d\mathbf{T}}{dt} = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\left(\frac{d\mathbf{T}}{ds}\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\left(\kappa\mathbf{N}\frac{ds}{dt}\right) \qquad \frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$$

Formula for Calculating the Normal Component of Acceleration

$$a_{\mathrm{N}} = \sqrt{|\mathbf{a}|^2 - a_{\mathrm{T}}{}^2} \qquad (3)$$

Tangential and Normal Components of Acceleration

EXAMPLE 1 Without finding **T** and **N**, write the acceleration of the motion

$$\mathbf{r}(t) = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j}, \qquad t > 0$$

in the form $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$.

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j}$$

$$|\mathbf{v}| = \sqrt{t^2\cos^2 t + t^2\sin^2 t} = \sqrt{t^2} = |t| = t$$

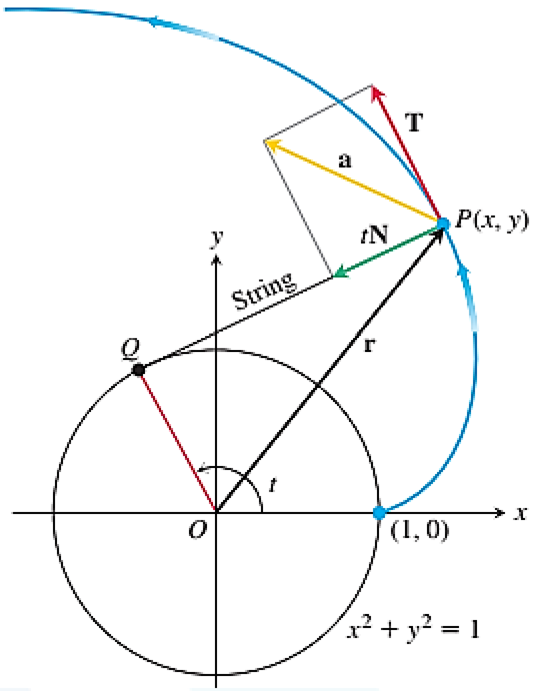
$$a_T = \frac{d}{dt}|\mathbf{v}| = \frac{d}{dt}(t) = 1.$$

$$\mathbf{a} = (\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j}$$

$$|\mathbf{a}|^2 = t^2 + 1$$

$$a_N = \sqrt{|\mathbf{a}|^2 - a_T^{\,2}} = t.$$

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N} = (1)\mathbf{T} + (t)\mathbf{N} = \mathbf{T} + t\mathbf{N}.$$



Torsion

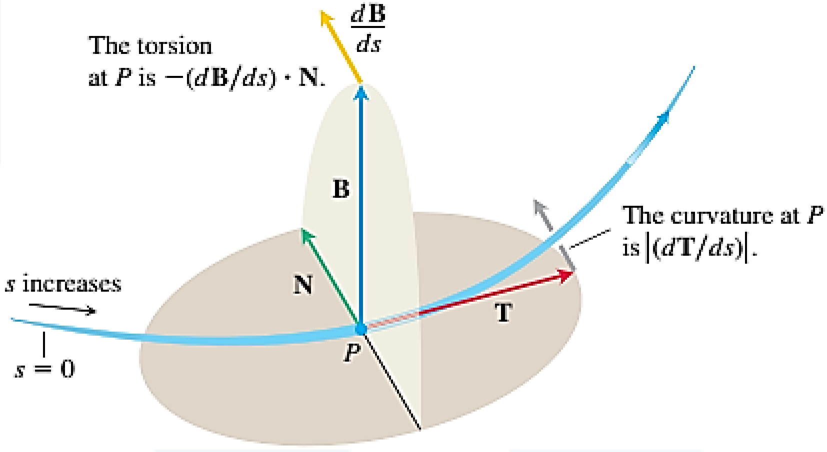
$$\frac{d\mathbf{B}}{ds} = \frac{d(\mathbf{T}\times\mathbf{N})}{ds} = \frac{d\mathbf{T}}{ds}\times\mathbf{N} + \mathbf{T}\times\frac{d\mathbf{N}}{ds} \xrightarrow{(d\mathbf{T}/ds)\times\mathbf{N} = \mathbf{0}} \frac{d\mathbf{B}}{ds} = \mathbf{T}\times\frac{d\mathbf{N}}{ds} \longrightarrow \frac{d\boldsymbol{B}}{ds}\perp\boldsymbol{T}$$

$$\frac{d\boldsymbol{B}}{ds}.\boldsymbol{B} = 0 \longrightarrow \frac{d\boldsymbol{B}}{ds}\perp\boldsymbol{B} \longrightarrow \frac{d\boldsymbol{B}}{ds}\perp(\boldsymbol{T},\boldsymbol{B}) \longrightarrow$$ $d\mathbf{B}/ds$ is parallel to $\mathbf{N}$,

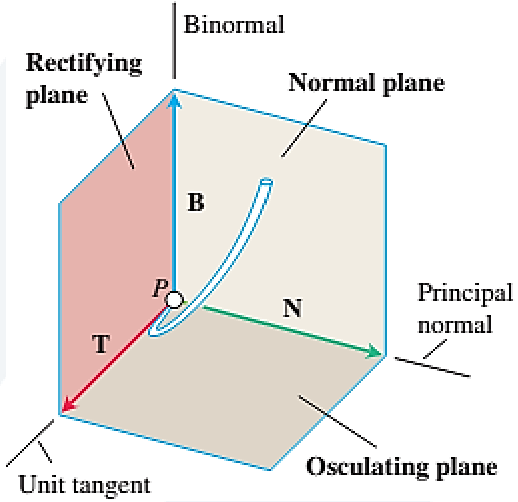
$$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}.$$

DEFINITION Let $\mathbf{B} = \mathbf{T}\times\mathbf{N}$. The **torsion** function of a smooth curve is

$$\tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N}. \qquad (4)$$



The three planes determined by **T**, **N**, and **B** are named and shown in Figure 13.28. The curvature $\kappa = |d\mathbf{T}/ds|$ can be thought of as the rate at which the normal plane turns as the point P moves along its path. Similarly, the torsion $\tau = -(d\mathbf{B}/ds) \cdot \mathbf{N}$ is the rate at which the osculating plane turns about **T** as P moves along the curve. Torsion measures how the curve twists.



Formulas for Computing Curvature and Torsion

$$\mathbf{v} \times \mathbf{a} = \left(\frac{ds}{dt}\mathbf{T}\right) \times \left[\frac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}\right] = \left(\frac{ds}{dt}\frac{d^2s}{dt^2}\right)(\mathbf{T} \times \mathbf{T}) + \kappa\left(\frac{ds}{dt}\right)^3(\mathbf{T} \times \mathbf{N}) = \kappa\left(\frac{ds}{dt}\right)^3\mathbf{B}.$$

$$|\mathbf{v} \times \mathbf{a}| = \kappa\left|\frac{ds}{dt}\right|^3|\mathbf{B}| = \kappa|\mathbf{v}|^3.$$

Vector Formula for Curvature

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} \tag{5}$$

Formula for Torsion

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2} \qquad (\text{if } \mathbf{v} \times \mathbf{a} \neq \mathbf{0}) \tag{6}$$

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EXAMPLE 2 Use Equations (5) and (6) to find the curvature κ and torsion τ for the helix

$$\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + bt\mathbf{k}, \qquad a, b \geq 0, \qquad a^2 + b^2 \neq 0.$$

$$\mathbf{v} = -(a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + b\mathbf{k}$$

$$\mathbf{a} = -(a\cos t)\mathbf{i} - (a\sin t)\mathbf{j}$$

$$\mathbf{v}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = (ab\sin t)\mathbf{i} - (ab\cos t)\mathbf{j} + a^2\mathbf{k}$$

$$\kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{\sqrt{a^2b^2 + a^4}}{(a^2+b^2)^{3/2}} = \frac{a\sqrt{a^2+b^2}}{(a^2+b^2)^{3/2}} = \frac{a}{a^2+b^2}$$

$$\dot{\mathbf{a}} = \frac{d\mathbf{a}}{dt} = (a\sin t)\mathbf{i} - (a\cos t)\mathbf{j}.$$

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\mathbf{v}\times\mathbf{a}|^2} = \frac{\begin{vmatrix} -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \\ a\sin t & -a\cos t & 0 \end{vmatrix}}{\left(a\sqrt{a^2+b^2}\right)^2} = \frac{b}{a^2+b^2}$$

Computation Formulas for Curves in Space

Unit tangent vector: $\mathbf{T} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$

Principal unit normal vector: $\mathbf{N} = \dfrac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$

Binormal vector: $\mathbf{B} = \mathbf{T} \times \mathbf{N}$

Curvature: $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right| = \dfrac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$

Torsion: $\tau = -\dfrac{d\mathbf{B}}{ds} \cdot \mathbf{N} = \dfrac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|\mathbf{v} \times \mathbf{a}|^2}$

Tangential and normal scalar components of acceleration:

$$\mathbf{a} = a_{\mathrm{T}}\mathbf{T} + a_{\mathrm{N}}\mathbf{N}$$

$$a_{\mathrm{T}} = \frac{d}{dt}|\mathbf{v}|$$

$$a_{\mathrm{N}} = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_{\mathrm{T}}^2}$$

Exercises

- write $\mathbf{a}$ in the form $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$ without finding $\mathbf{T}$ and $\mathbf{N}$

$$\mathbf{r}(t) = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j} + \sqrt{2}e^t\mathbf{k}, \quad t = 0$$

$\mathbf{a}(0) = 2\mathbf{T} + \sqrt{2}\mathbf{N}$

- find $\mathbf{r}$, $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ at the given value of t. Then find equations for the osculating, normal, and rectifying planes at that value of t.

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} - \mathbf{k}, \quad t = \pi/4$$

$(-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$ $\quad$ $(-\cos t)\mathbf{i} - (\sin t)\mathbf{j}$ $\quad$ $\mathbf{k}$ $\quad$ $z = -1$ $\quad$ $-x + y = 0$ $\quad$ $x + y = \sqrt{2}$

- find $\mathbf{B}$ and τ for these space curves.

$$\mathbf{r}(t) = (6 \sin 2t)\mathbf{i} + (6 \cos 2t)\mathbf{j} + 5t\mathbf{k}$$

$\left(\frac{5}{13}\cos 2t\right)\mathbf{i} - \left(\frac{5}{13}\sin 2t\right)\mathbf{j} - \frac{12}{13}\mathbf{k}$ $\qquad$ $-\frac{10}{169}$