جامعة المنارة

MANARA UNIVERSITY

MATHEMATICAL ANALAYSIS 2

Lecture 4

Prepared by

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- Functions of Several Variables
- Limits and Continuity in Higher Dimensions
- Partial Derivatives
- The Chain Rule
- Directional Derivatives and Gradient Vectors
- Tangent Planes and Differentials
- Extreme Values and Saddle Points
- Lagrange Multipliers
- Taylor's Formula for Two Variables
- Partial Derivatives with Constrained Variables

Estimating Change in a Specific Direction

Estimating the Change in f in a Direction u

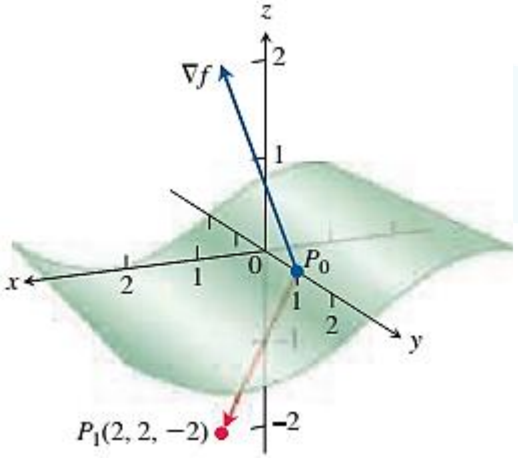
To estimate the change in the value of a differentiable function f when we move a small distance ds from a point P_0 in a particular direction $\mathbf{u}$, use the formula

$$df = \underbrace{(\nabla f|_{P_0} \cdot \mathbf{u})}_{\text{Directional derivative}} \underbrace{ds}_{\text{Distance increment}}$$

EXAMPLE 4 Estimate how much the value of

$$f(x, y, z) = y \sin x + 2yz$$

will change if the point $P(x, y, z)$ moves 0.1 unit from $P_0(0, 1, 0)$ straight toward $P_1(2, 2, -2)$.



$$\mathbf{u} = \frac{\overrightarrow{P_0P_1}}{|\overrightarrow{P_0P_1}|} = \frac{\overrightarrow{P_0P_1}}{3} = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}.$$

$$\nabla f|_{(0,1,0)} = ((y \cos x)\mathbf{i} + (\sin x + 2z)\mathbf{j} + 2y\mathbf{k})\Big|_{(0,1,0)} = \mathbf{i} + 2\mathbf{k}.$$

$$\nabla f|_{P_0} \cdot \mathbf{u} = (\mathbf{i} + 2\mathbf{k}) \cdot \left(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\right) = \frac{2}{3} - \frac{4}{3} = -\frac{2}{3}$$

$$\Rightarrow \quad df = (\nabla f|_{P_0} \cdot \mathbf{u})(ds) = \left(-\frac{2}{3}\right)(0.1) \approx -0.067 \text{ unit.}$$

How to Linearize a Function of Two Variables

DEFINITIONS The **linearization** of a function $f(x, y)$ at a point (x_0, y_0) where f is differentiable is the function

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

The approximation

$$f(x, y) \approx L(x, y)$$

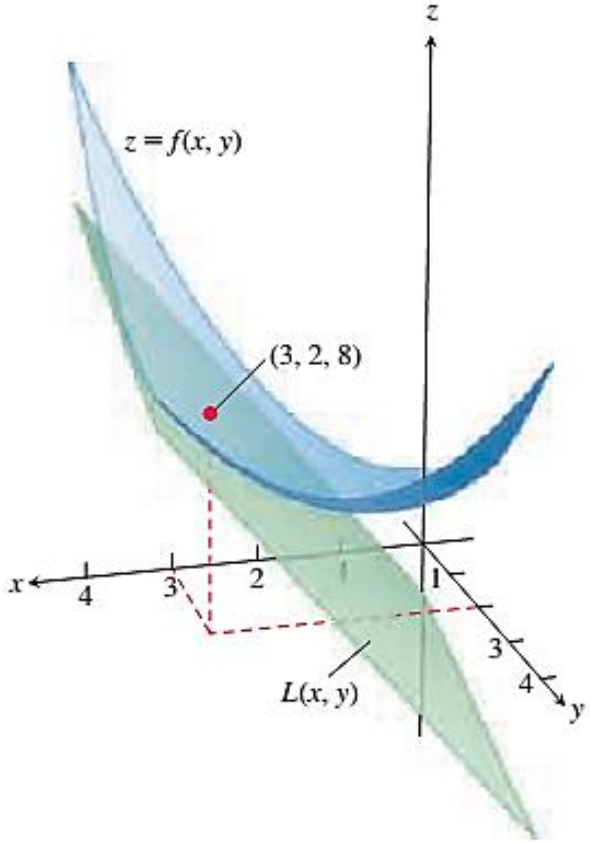
is the **standard linear approximation** of f at (x_0, y_0).

EXAMPLE 5 Find the linearization of

$$f(x, y) = x^2 - xy + \frac{1}{2}y^2 + 3$$ at the point (3, 2).

$f(3, 2) = 8$ $\quad f_x(3, 2) = 4$ $\quad f_y(3, 2) = -1$

The linearization of f at (3, 2) is $L(x, y) = 4x - y - 2$



The Error in the Standard Linear Approximation

If f has continuous first and second partial derivatives throughout an open set containing a rectangle R centered at (x_0, y_0) and if M is any upper bound for the values of $|f_{xx}|$, $|f_{yy}|$, and $|f_{xy}|$ on R, then the error $E(x, y)$ incurred in replacing $f(x, y)$ on R by its linearization

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

satisfies the inequality

$$|E(x, y)| \leq \frac{1}{2} M\left(|x - x_0| + |y - y_0|\right)^2.$$

Differentials

DEFINITION If we move from (x_0, y_0) to a point $(x_0 + dx, y_0 + dy)$ nearby, the resulting change

$$df = f_x(x_0, y_0)\, dx + f_y(x_0, y_0)\, dy$$

in the linearization of f is called the **total differential of f**.

EXAMPLE 6 Suppose that a cylindrical can is designed to have a radius of 1 in. and a height of 5 in., but that the radius and height are off by the amounts $dr = +0.03$ and $dh = -0.1$. Estimate the resulting absolute change in the volume of the can.

$V = \pi r^2 h,$ → $$\Delta V \approx dV = V_r(r_0, h_0)\,dr + V_h(r_0, h_0)\,dh$$

$$dV = 2\pi r_0 h_0\,dr + \pi r_0{}^2\,dh = 2\pi(1)(5)(0.03) + \pi(1)^2(-0.1)$$

$$= 0.3\pi - 0.1\pi = 0.2\pi \approx 0.63 \text{ in}^3$$

Functions of More Than Two Variables

1. The **linearization** of $f(x, y, z)$ at a point $P_0(x_0, y_0, z_0)$ is

$$L(x, y, z) = f(P_0) + f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0).$$

2. Suppose that R is a closed rectangular solid centered at P_0 and lying in an open region on which the second partial derivatives of f are continuous. Suppose also that $|f_{xx}|, |f_{yy}|, |f_{zz}|, |f_{xy}|, |f_{xz}|$, and $|f_{yz}|$ are all less than or equal to M throughout R. Then the **error** $E(x, y, z) = f(x, y, z) - L(x, y, z)$ in the approximation of f by L is bounded throughout R by the inequality

$$|E| \leq \frac{1}{2} M\left(|x - x_0| + |y - y_0| + |z - z_0|\right)^2.$$

3. If the second partial derivatives of f are continuous and if x, y, and z change from x_0, y_0, and z_0 by small amounts dx, dy, and dz, the **total differential**

$$df = f_x(P_0)\,dx + f_y(P_0)\,dy + f_z(P_0)\,dz$$

gives a good approximation of the resulting change in f.

EXAMPLE 8 Find the linearization $L(x, y, z)$ of

$$f(x, y, z) = x^2 - xy + 3\sin z$$

at the point $(x_0, y_0, z_0) = (2, 1, 0)$. Find an upper bound for the error incurred in replacing f by L on the rectangular region

$$R: \quad |x - 2| \le 0.01, \qquad |y - 1| \le 0.02, \qquad |z| \le 0.01.$$

$$f(2, 1, 0) = 2, \qquad f_x(2, 1, 0) = 3, \qquad f_y(2, 1, 0) = -2, \qquad f_z(2, 1, 0) = 3.$$

$$L(x, y, z) = 2 + 3(x - 2) + (-2)(y - 1) + 3(z - 0) = 3x - 2y + 3z - 2.$$

$$f_{xx} = 2, \qquad f_{yy} = 0, \qquad f_{zz} = -3\sin z, \qquad f_{xy} = -1, \qquad f_{xz} = 0, \qquad f_{yz} = 0, \quad \Rightarrow \quad M = 2$$

$$|E| \le \frac{1}{2}(2)(0.01 + 0.02 + 0.01)^2 = 0.0016.$$

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Exercises

- find equations for the (a) tangent plane and (b) normal line at the point P_0 on the given surface.

$$x^2 + y^2 + z^2 = 3, \quad P_0(1, 1, 1)$$

$x + y + z = 3$ $\qquad$ $x = 1 + 2t,\ y = 1 + 2t,\ z = 1 + 2t$

- find parametric equations for the line tangent to the curve of intersection of the surfaces at the given point.

Surfaces: $x + y^2 + 2z = 4, \quad x = 1$ $\qquad$ Point: $(1, 1, 1)$

Tangent line: $x = 1,\ y = 1 + 2t,\ z = 1 - 2t$

- By about how much will $f(x, y, z) = \ln\sqrt{x^2 + y^2 + z^2}$ change if the point $P(x, y, z)$ moves from $P_0(3, 4, 12)$ a distance of $ds = 0.1$ unit in the direction of $3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$?

$$df = (\nabla f \cdot \mathbf{u})\, ds = \left(\frac{9}{1183}\right)(0.1) \approx 0.0008$$

- find the linearization $L(x, y, z)$ of the function $f(x, y, z)$ at P_0. Then find an upper bound for the magnitude of the error E in the approximation $f(x, y, z) \approx L(x, y, z)$ over the region R.

$$f(x, y, z) = xz - 3yz + 2 \quad \text{at} \quad P_0(1, 1, 2),$$
$$R: \quad |x - 1| \le 0.01, \quad |y - 1| \le 0.01, \quad |z - 2| \le 0.02$$

$2x - 6y - 2z + 6$ $\qquad$ $M = 3$

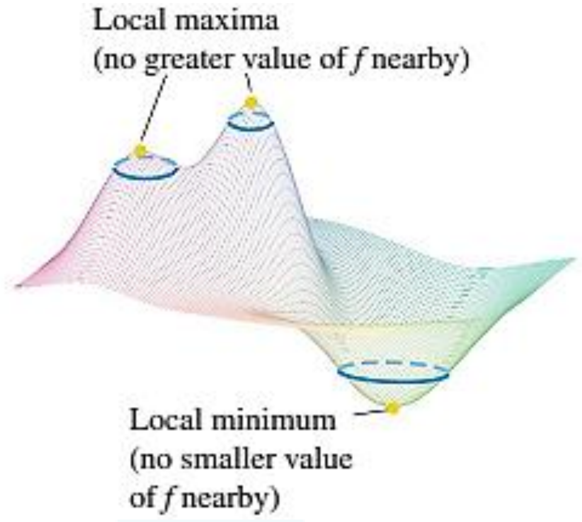
$|E(x, y, z)| \le 0.0024$

Extreme Values and Saddle Points

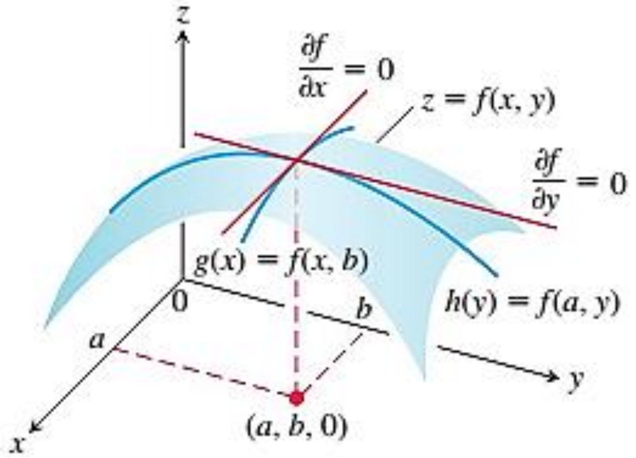
Derivative Tests for Local Extreme Values

DEFINITIONS Let $f(x, y)$ be defined on a region R containing the point (a, b). Then

1. $f(a, b)$ is a **local maximum** value of f if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).
2. $f(a, b)$ is a **local minimum** value of f if $f(a, b) \leq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).



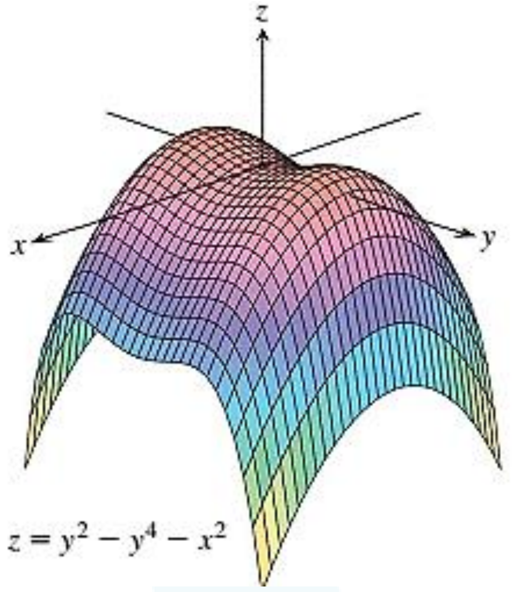
THEOREM 10—First Derivative Test for Local Extreme Values
If $f(x, y)$ has a local maximum or minimum value at an interior point (a, b) of its domain and if the first partial derivatives exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.



Extreme Values and Saddle Points

DEFINITION An interior point of the domain of a function $f(x, y)$ where both f_x and f_y are zero or where one or both of f_x and f_y do not exist is a **critical point** of f.

DEFINITION A differentiable function $f(x, y)$ has a **saddle point** at a critical point (a, b) if in every open disk centered at (a, b) there are domain points (x, y) where $f(x, y) > f(a, b)$ and domain points (x, y) where $f(x, y) < f(a, b)$. The corresponding point $(a, b, f(a, b))$ on the surface $z = f(x, y)$ is called a saddle point of the surface (Figure 14.45).

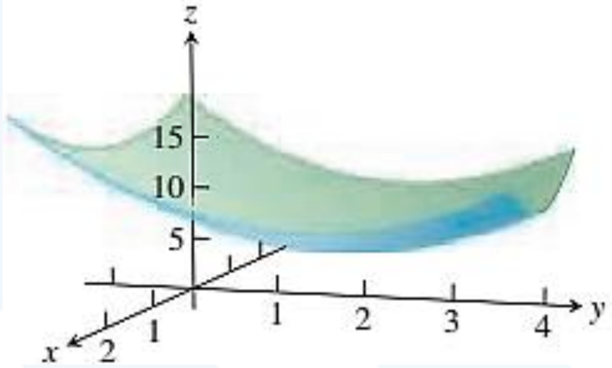


EXAMPLE 1 Find the local extreme values of $f(x, y) = x^2 + y^2 - 4y + 9$.

$f_x = 2x = 0$ and $f_y = 2y - 4 = 0.$ → the critical point (0, 2)

$$f(x, y) = x^2 + (y-2)^2 + 5 \geq 5$$

→ The critical point (0, 2) gives a local minimum



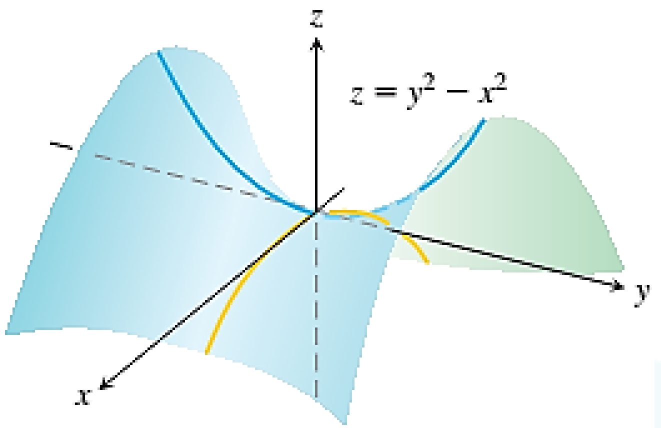
EXAMPLE 2 Find the local extreme values (if any) of $f(x, y) = y^2 - x^2$.

$f_x = -2x = 0$ and $f_y = 2y = 0$ → The critical point $(0,0)$

Along the positive x-axis $f(x,0) = -x^2 \le 0$

Along the positive y-axis $f(0,y) = y^2 \ge 0$

The function has a saddle point at the origin and no local extreme values



THEOREM 11—Second Derivative Test for Local Extreme Values
Suppose that $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$. Then

i) f has a **local maximum** at (a, b) if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).

ii) f has a **local minimum** at (a, b) if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).

iii) f has a **saddle point** at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b).

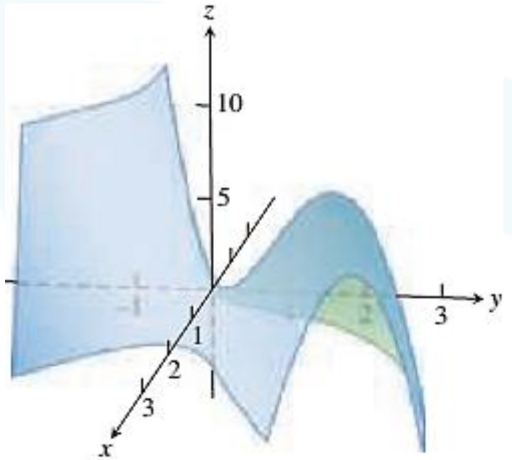
iv) **the test is inconclusive** at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a, b). In this case, we must find some other way to determine the behavior of f at (a, b).

Extreme Values and Saddle Points

The expression $f_{xx}f_{yy} - f_{xy}^2$ is called the **discriminant** or **Hessian** of f. It is sometimes easier to remember it in determinant form,

$$f_{xx}f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}.$$

EXAMPLE 4 Find the local extreme values of $f(x, y) = 3y^2 - 2y^3 - 3x^2 + 6xy$.



$f_x = 6y - 6x = 0$ and $f_y = 6y - 6y^2 + 6x = 0.$

The critical points $(0,0)$, $(2,2)$

$f_{xx} = -6, \quad f_{yy} = 6 - 12y, \quad f_{xy} = 6.$

$D = f_{xx}f_{yy} - f_{xy}^2 = 72(y-1)$

- $D(0,0) = -72 < 0$ the function has a saddle point at the origin
- $D(2,2) = 72 > 0$, $f_{xx}(0,0) = -6 < 0$ the function has a local maximum value $f(2,2) = 8$

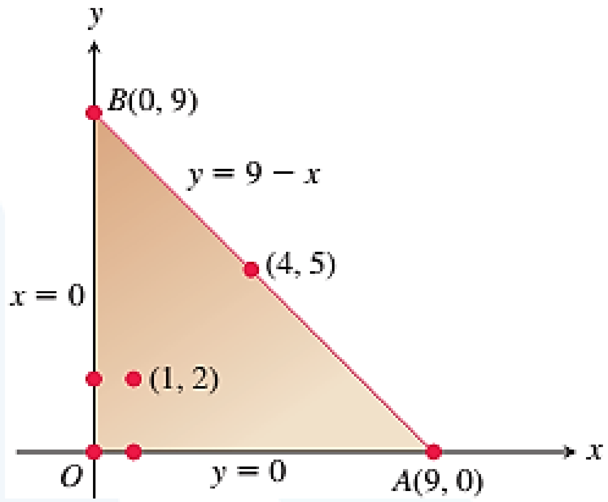
Absolute Maxima and Minima on Closed Bounded Regions

1. *List the interior points of R* where f may have local maxima and minima and evaluate f at these points. These are the critical points of f.
2. *List the boundary points of R* where f has local maxima and minima and evaluate f at these points. We show how to do this in the next example.
3. *Look through the lists* for the maximum and minimum values of f. These will be the absolute maximum and minimum values of f on R.

EXAMPLE 6 Find the absolute maximum and minimum values of

$$f(x, y) = 2 + 2x + 4y - x^2 - y^2$$

on the triangular region in the first quadrant bounded by the lines $x = 0$, $y = 0$, and $y = 9 - x$.



(a) Interior points.

$$f_x = 2 - 2x = 0, \qquad f_y = 4 - 2y = 0,$$ → The critical point $(1, 2)$

$$f(1, 2) = 7.$$

Extreme Values and Saddle Points

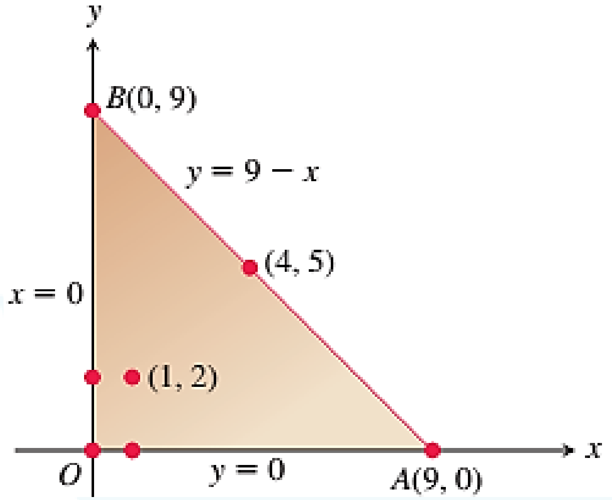
(b) Boundary points.

i) On the segment OA, $y = 0$. $f(x, y) = f(x, 0) = 2 + 2x - x^2 \qquad 0 \leq x \leq 9$

$f(0, 0) = 2 \qquad f(9, 0) = 2 + 18 - 81 = -61$

At the interior points $f'(x, 0) = 2 - 2x = 0.$ ⟹ $x = 1$

$$f(x, 0) = f(1, 0) = 3$$



ii) On the segment OB, $x = 0$ $f(x, y) = f(0, y) = 2 + 4y - y^2$

$$f(0, 0) = 2, \qquad f(0, 9) = -43$$

$f'(0, y) = 4 - 2y = 0$ ⟹ $y = 2$ ⟹ $f(0, 2) = 6.$

Extreme Values and Saddle Points

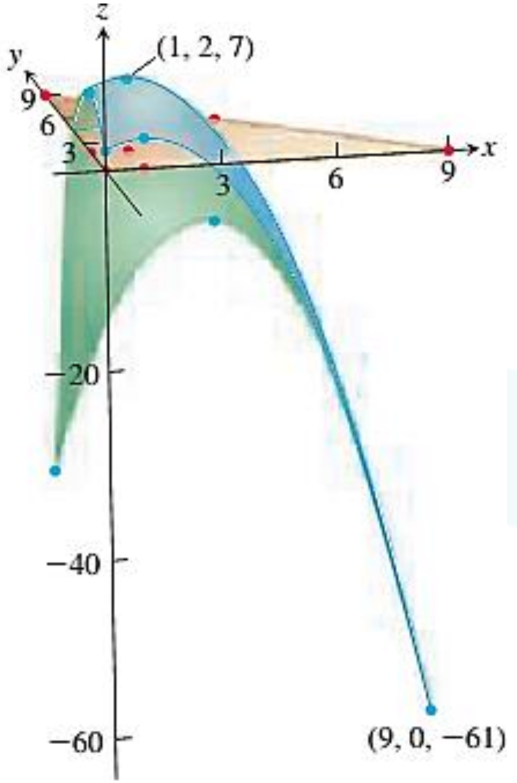
iii) We have already accounted for the values of f at the endpoints of AB.

at the interior points of the line segment AB. With $y = 9 - x$

$$f(x, y) = 2 + 2x + 4(9 - x) - x^2 - (9 - x)^2 = -43 + 16x - 2x^2.$$

$\Rightarrow f'(x, 9 - x) = 16 - 4x = 0 \quad \Rightarrow \quad x = 4.$

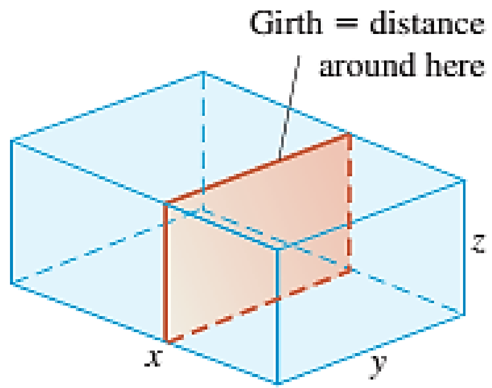
$$f(x, y) = f(4, 5) = -11.$$



$(1,2)$ the function has a maximum value $f\,(1,2) = 7$

$(9,0)$ the function has a minimum value $f\,(9,0) = -61$

EXAMPLE 7 A delivery company accepts only rectangular boxes the sum of whose length and girth (perimeter of a cross-section) does not exceed 108 in. Find the dimensions of an acceptable box of largest volume.



Let x, y, and z represent the length, width, and height of the rectangular box

$$V = xyz \qquad x + 2y + 2z = 108$$

$$V(y, z) = (108 - 2y - 2z)yz = 108yz - 2y^2z - 2yz^2$$

$$V_y(y, z) = 108z - 4yz - 2z^2 = (108 - 4y - 2z)z = 0$$

$$V_z(y, z) = 108y - 2y^2 - 4yz = (108 - 2y - 4z)y = 0$$

the critical points (0, 0), (0, 54), (54, 0), and (18, 18)

$$V_{yy} = -4z, \qquad V_{zz} = -4y, \qquad V_{yz} = 108 - 4y - 4z \qquad V_{yy}V_{zz} - V_{yz}^{\;2} = 16yz - 16(27 - y - z)^2$$

$$V_{yy}(18, 18) = -4(18) < 0 \;\Rightarrow\; \left(V_{yy}V_{zz} - V_{yz}^{\;2}\right)\Big|_{(18,18)} = 16(18)(18) - 16(-9)^2 > 0$$

$\Rightarrow$ (18, 18) gives a maximum volume

$$x = 108 - 2(18) - 2(18) = 36 \text{ in.},\; y = 18 \text{ in., and } z = 18$$

$$\Rightarrow \quad V = (36)(18)(18) = 11{,}664 \text{ in}^3, \text{ or } 6.75 \text{ ft}^3$$

Exercises

- Find all the local maxima, local minima, and saddle points of the functions

$f(x, y) = x^2 + xy + y^2 + 3x - 3y + 4$ (−3, 3) local minimum

$f(x, y) = x^2 + xy + 3x + 2y + 5$ (−2, 1) saddle point

- Find the absolute maxima and minima of the functions on the given domains

$f(x, y) = (4x - x^2)\cos y$ on the rectangular plate $1 \le x \le 3$ $-\pi/4 \le y \le \pi/4$

Therefore the absolute maximum is 4 at $(2, 0)$ and the absolute minimum is $\frac{3\sqrt{2}}{2}$ at $\left(3, -\frac{\pi}{4}\right), \left(3, \frac{\pi}{4}\right), \left(1, -\frac{\pi}{4}\right)$, and $\left(1, \frac{\pi}{4}\right)$.

- Find two numbers a and b with $a \le b$ such that

$$\int_a^b (6 - x - x^2)\, dx$$

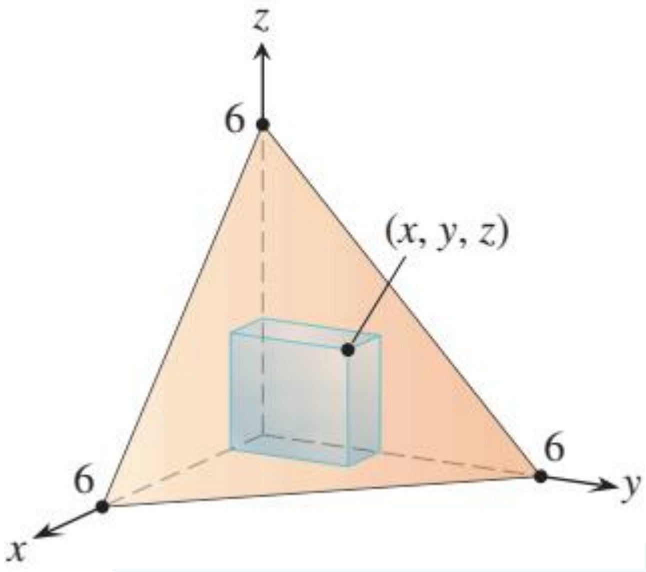
has its largest value.

$a = -3$ and $b = 2$.

- Find the point on the plane $3x + 2y + z = 6$ that is nearest the origin.

local minimum of $d\left(\frac{9}{7}, \frac{6}{7}, \frac{3}{7}\right) = \frac{3\sqrt{14}}{7}$

- A rectangular box is inscribed in the region in the first octant bounded above by the plane with x-intercept 6, y-intercept 6, and z-intercept 6.



local maximum of $V(2, 2, 2) = 8$

Lagrange Multipliers

The Method of Lagrange Multipliers

THEOREM 12—The Orthogonal Gradient Theorem
Suppose that $f(x, y, z)$ is differentiable in a region whose interior contains a smooth curve

$$C: \quad \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}.$$

If P_0 is a point on C where f has a local maximum or minimum relative to its values on C, then ∇f is orthogonal to C at P_0.

COROLLARY At the points on a smooth curve $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ where a differentiable function $f(x, y)$ takes on its local maxima and minima relative to its values on the curve, $\nabla f \cdot \mathbf{r}' = 0$.

Lagrange Multipliers

The Method of Lagrange Multipliers

The Method of Lagrange Multipliers

Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable and $\nabla g \neq \mathbf{0}$ when $g(x, y, z) = 0$. To find the local maximum and minimum values of f subject to the constraint $g(x, y, z) = 0$ (if these exist), find the values of x, y, z, and λ that simultaneously satisfy the equations

$$\nabla f = \lambda \nabla g \quad \text{and} \quad g(x, y, z) = 0. \tag{1}$$

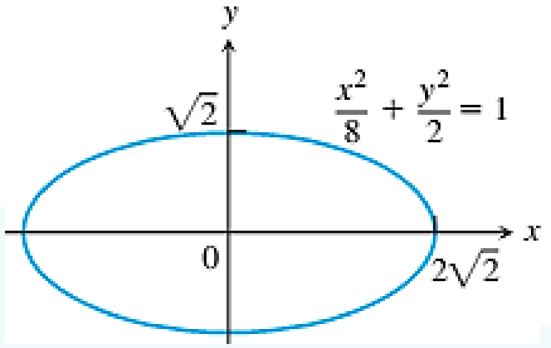
For functions of two independent variables, the condition is similar, but without the variable z.

Lagrange Multipliers

EXAMPLE 3 Find the greatest and smallest values that the function

$$f(x, y) = xy$$

takes on the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$



We want to find the extreme values of $f(x, y) = xy$ subject to the constraint $g(x, y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0.$

$\nabla f = \lambda \nabla g$ and $g(x, y) = 0.$ → $y\mathbf{i} + x\mathbf{j} = \frac{\lambda}{4}x\mathbf{i} + \lambda y\mathbf{j},$

$y = \frac{\lambda}{4}x,$ $x = \lambda y,$ and $y = \frac{\lambda}{4}(\lambda y) = \frac{\lambda^2}{4}y,$ → $y = 0$ or $\lambda = \pm 2$

Case 1: If $y = 0$, then $x = y = 0$. But $(0, 0)$ is not on the ellipse. Hence, $y \neq 0$.

Case 2: If $y \neq 0$, then $\lambda = \pm 2$ and $x = \pm 2y$.

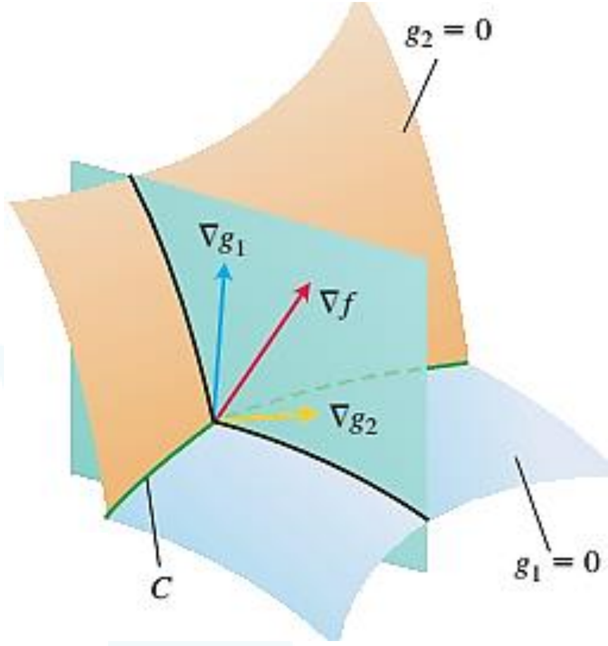
→ $g(x, y) = 0$ → $\frac{(\pm 2y)^2}{8} + \frac{y^2}{2} = 1$ → $y = \pm 1$

The function $f(x, y) = xy$ therefore takes on its extreme values on the ellipse at the four points $(\pm 2, 1), (\pm 2, -1)$. The extreme values are $xy = 2$ and $xy = -2$.

Lagrange Multipliers

Lagrange Multipliers with Two Constraints

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2, \qquad g_1(x, y, z) = 0, \qquad g_2(x, y, z) = 0 \qquad (2)$$



EXAMPLE 5 The plane $x + y + z = 1$ cuts the cylinder $x^2 + y^2 = 1$ in an ellipse (Figure 14.59). Find the points on the ellipse that lie closest to and farthest from the origin.

$$f(x, y, z) = x^2 + y^2 + z^2$$

$$g_1(x, y, z) = x^2 + y^2 - 1 = 0$$

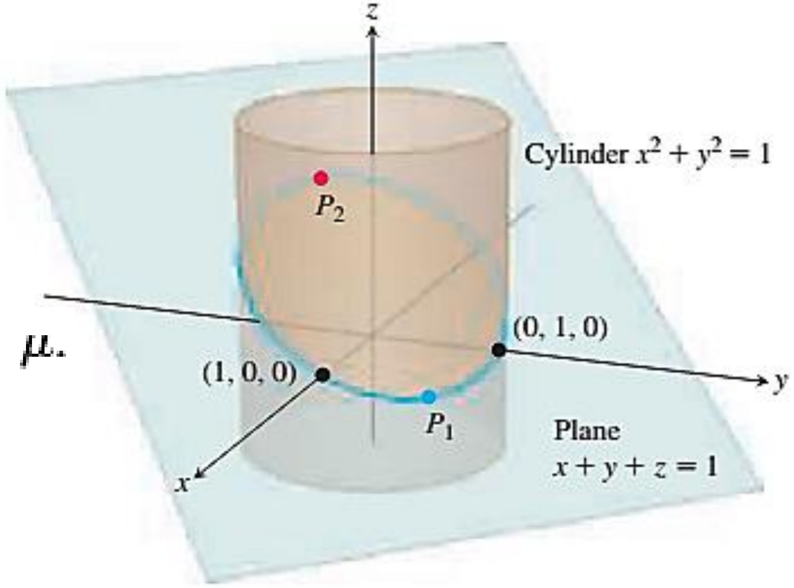
$$g_2(x, y, z) = x + y + z - 1 = 0.$$

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$$

$$2x = 2\lambda x + \mu, \qquad 2y = 2\lambda y + \mu, \qquad 2z = \mu.$$

$$2x = 2\lambda x + 2z \Rightarrow (1 - \lambda)x = z,$$

$$2y = 2\lambda y + 2z \Rightarrow (1 - \lambda)y = z.$$



Lagrange Multipliers

$2x = 2\lambda x + 2z \Rightarrow (1-\lambda)x = z,$

$2y = 2\lambda y + 2z \Rightarrow (1-\lambda)y = z.$

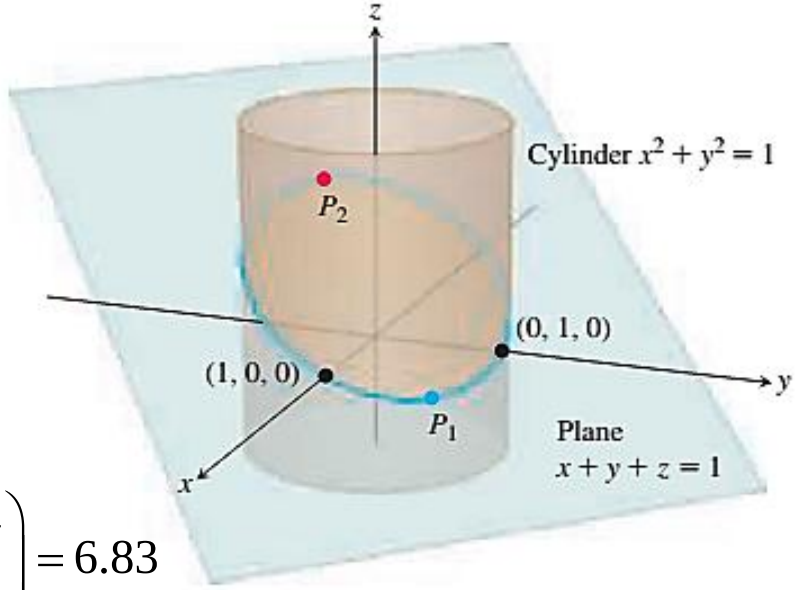
→ $\lambda = 1$ and $z = 0$

→ $\lambda \neq 1$ and $x = y = z/(1-\lambda).$

If $z = 0$ → $(1, 0, 0)$ and $(0, 1, 0)$

If $x = y$, → $x^2 + y^2 - 1 = 0$ → $x = \pm\frac{\sqrt{2}}{2}$

$x + x + z - 1 = 0$ → $z = 1 \mp \sqrt{2}.$



$$P_1 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1 - \sqrt{2}\right) \quad \text{and} \quad P_2 = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1 + \sqrt{2}\right)$$

$$f(1,0,0) = 1,\ f(0,1,0) = 1,\ f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1-\sqrt{2}\right) = 1.172,\ f\left(\frac{-\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}, 1+\sqrt{2}\right) = 6.83$$

The points on the ellipse closest to the origin are (1, 0, 0) and (0, 1, 0)

The point on the ellipse farthest from the origin is P_2.

Exercises

- Find the dimensions of the closed right circular cylindrical can of smallest surface area whose volume is 16π cm^3

$r = 2$ $h = 4$ 24π cm^2

- Find the extreme values of the function $f(x, y, z) = xy + z^2$ on the circle in which the plane $y - x = 0$ intersects the sphere $x^2 + y^2 + z^2 = 4$.

Therefore the maximum value of f is 4 at $(0, 0, \pm 2)$ and the minimum value of f is 2 at $f\left(\pm\sqrt{2}, \pm\sqrt{2}, 0\right)$

- A space probe in the shape of the ellipsoid $4x^2 + y^2 + 4z^2 = 16$ enters Earth's atmosphere and its surface begins to heat. After 1 hour, the temperature at the point (x, y, z) on the probe's surface is $T(x, y, z) = 8x^2 + 4yz - 16z + 600$. Find the hottest point on the probe's surface.

$\left(\pm\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}\right)$

Exact Equations

Definition 2.4.1 Exact Equation

A differential expression $M(x, y)\,dx + N(x, y)\,dy$ is an **exact differential** in a region R of the xy-plane if it corresponds to the differential of some function $f(x, y)$. A first-order differential equation of the form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is said to be an **exact equation** if the expression on the left side is an exact differential.

Theorem 2.4.1 Criterion for an Exact Differential

Let $M(x, y)$ and $N(x, y)$ be continuous and have continuous first partial derivatives in a rectangular region R defined by $a < x < b$, $c < y < d$. Then a necessary and sufficient condition that $M(x, y)\,dx + N(x, y)\,dy$ be an exact differential is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \qquad (4)$$

Method of Solution

$$df(x,y)=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=M(x,y)dx+N(x,y)dy$$

$$\frac{\partial f}{\partial x}=M(x,y)\ ,\ \frac{\partial f}{\partial y}=N(x,y)$$

Integration x

$$f(x,y)=\int M(x,y)dx+g(y)$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\int M(x,y)dx+g'(y)=N(x,y)$$

Integration y

$$g(y)$$

$$f(x,y)=C$$

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EXAMPLE

$$2xydx+(x^2-1)dy=0$$

$$M(x,y)=2xy\ ,\ N(x,y)=x^2-1$$

$$\frac{\partial M}{\partial y}=\frac{\partial(2xy)}{\partial y}=2x$$

$$\frac{\partial N}{\partial x}=\frac{\partial(x^2-1)}{\partial x}=2x$$

$$\Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Exact Equation

$$\frac{\partial f}{\partial x}=2xy\ ,\ \frac{\partial f}{\partial y}=x^2-1$$

Integration x

$$f(x,y)=x^2y+g(y)$$

$$\frac{\partial f}{\partial y}=x^2+g'(y)=x^2-1$$

$$g'(y)=-1 \Rightarrow g(y)=-y$$

$$f(x,y)=x^2y-y \Rightarrow x^2y-y=c \Rightarrow y=\frac{c}{x^2-1}$$

EXAMPLE

$$\frac{dy}{dx}=\frac{xy^2-\cos x\sin x}{y(1-x^2)},\quad y(0)=2$$

An initial value problem

$$\left(\cos x\sin x-xy^2\right)dx+y\left(1-x^2\right)dy=0$$

$$M(x,y)=\left(\cos x\sin x-xy^2\right),\quad N(x,y)=y\left(1-x^2\right)$$

$$\frac{\partial M}{\partial y}=\frac{\partial\left(\cos x\sin x-xy^2\right)}{\partial y}=-2xy$$

$$\frac{\partial N}{\partial x}=\frac{\partial\left(y\left(1-x^2\right)\right)}{\partial x}=-2xy$$

$$\Rightarrow\quad \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Exact Equation

$$\frac{\partial f}{\partial x} = \cos x \sin x - xy^2 \quad , \quad \frac{\partial f}{\partial y} = y(1-x^2)$$

Integration y

$$f(x,y) = \frac{y^2}{2}(1-x^2) + h(x)$$

$$\frac{\partial f}{\partial x} = -xy^2 + h'(x) = \cos x \sin x - xy^2 \Rightarrow h'(x) = \cos x \sin x$$

$$\Rightarrow h(x) = \int \cos x \sin x \, dx = -\frac{1}{2}\cos^2 x$$

$$\Rightarrow f(x,y) = \frac{y^2}{2}(1-x^2) - \frac{1}{2}\cos^2 x \Rightarrow y^2(1-x^2) - \cos^2 x = C$$

$y(0) = 2$

$$(2)^2(1-(0)^2) - \cos^2 0 = C \Rightarrow C = 3$$

$$\Rightarrow y^2(1-x^2) - \cos^2 x = 3$$

Nonexact Equations and Integrating Factor

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \longrightarrow$$ **Nonexact Equation**

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$$ **Exact Equation**

$$\longrightarrow \quad \mu M_y + \mu_y M = \mu N_x + \mu_x N$$

Finding an Integrating Factor

1. $$\frac{\left(M_y - N_x\right)}{N} = \varphi(x) \Longrightarrow \mu(x) = e^{\int \frac{\left(M_y - N_x\right)}{N} dx}$$

2. $$\frac{\left(N_x - M_y\right)}{M} = \psi(y) \Longrightarrow \mu(y) = e^{\int \frac{\left(N_x - M_y\right)}{M} dy}$$

EXAMPLE

$$xydx + \left(2x^2 + 3y^2 - 20\right)dy = 0$$

$$M(x,y) = xy \;,\; N(x,y) = 2x^2 + 3y^2 - 20$$

$$\frac{\partial M}{\partial y} = \frac{\partial (xy)}{\partial y} = x$$

$$\frac{\partial N}{\partial x} = \frac{\partial \left(2x^2 + 3y^2 - 20\right)}{\partial x} = 4x$$

$$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Nonexact Equation

$$\frac{\left(M_y - N_x\right)}{N} = \frac{x - 4x}{2x^2 + 3y^2 - 20} = \frac{-3x}{2x^2 + 3y^2 - 20} = \varphi(x,y)$$

X

$$\frac{\left(N_x - M_y\right)}{M} = \frac{4x - x}{xy} = \frac{3x}{xy} = \frac{3}{y} = \psi(y)$$

Ok

$$\mu(y)=e^{\int\frac{(N_x-M_y)}{M}dy}=e^{\int\frac{3}{y}dy}=e^{3\ln y}=e^{\ln y^3}=y^3$$

Integration Factor

$$xydx+\left(2x^2+3y^2-20\right)dy=0$$

$$\times\mu(y)=y^3$$

$$xy^4dx+\left(2x^2y^3+3y^5-20y^3\right)dy=0$$

$$M(x,y)=xy^4\ ,\ N(x,y)=2x^2y^3+3y^5-20y^3$$

$$\frac{\partial M}{\partial y}=\frac{\partial\left(xy^4\right)}{\partial y}=4xy^3$$

$$\frac{\partial N}{\partial x}=\frac{\partial\left(2x^2y^3+3y^5-20y^3\right)}{\partial x}=4xy^3$$

$$\Rightarrow\ \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Exact Equation

$$\frac{\partial f}{\partial x} = xy^4 \quad , \quad \frac{\partial f}{\partial y} = 2x^2y^3 + 3y^5 - 20y^3$$

Integration x

$$f(x,y) = \frac{1}{2}x^2y^4 + g(y)$$

$$\frac{\partial f}{\partial y} = 2x^2y^3 + g'(y) = 2x^2y^3 + 3y^5 - 20y^3$$

$$g'(y) = 3y^5 - 20y^3 \longrightarrow g(y) = \frac{1}{2}y^6 - 5y^4$$

$$f(x,y) = \frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 \longrightarrow \frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 = c$$

Exercises

- Determine whether the given differential equation is exact. If it is exact, solve it.

$(\sin y - y\sin x)\,dx + (\cos x + x\cos y - y)\,dy = 0$ $\quad x\sin y + y\cos x - \frac{1}{2}y^2 = c.$

$\left(1 + \ln x + \frac{y}{x}\right)dx = (1 - \ln x)\,dy$ $\quad -y + y\ln x + x\ln x = c.$

- Solve the given initial-value problem.

$\left(\frac{3y^2 - t^2}{y^5}\right)\frac{dy}{dt} + \frac{t}{2y^4} = 0, \quad y(1) = 1$ $\quad \frac{t^2}{4y^4} - \frac{3}{2y^2} = -\frac{5}{4}$

$\left(\frac{1}{1 + y^2} + \cos x - 2xy\right)\frac{dy}{dx} = y(y + \sin x), \quad y(0) = 1$ $\quad xy^2 - y\cos x - \tan^{-1} y = -1 - \frac{\pi}{4}.$

- Solve the given differential equation by finding an appropriate integrating factor.

$\cos x\,dx + \left(1 + \frac{2}{y}\right)\sin x\,dy = 0$ $\quad \csc x$ $\quad \ln(\sin x) + y + \ln y^2 = c.$

Exercises

$(y^2 + xy^3)\,dx + (5y^2 - xy + y^3 \sin y)\,dy = 0$ $1/y^3.$ $x/y + \frac{1}{2}x^2 + 5\ln|y| - \cos y = c.$

$(-xy \sin x + 2y \cos x)\,dx + 2x \cos x\,dy = 0; \quad \mu(x, y) = xy$ $x^2y^2 \cos x = c.$