MATHEMATICAL ANALAYSIS 2

Lecture 5

Prepared by

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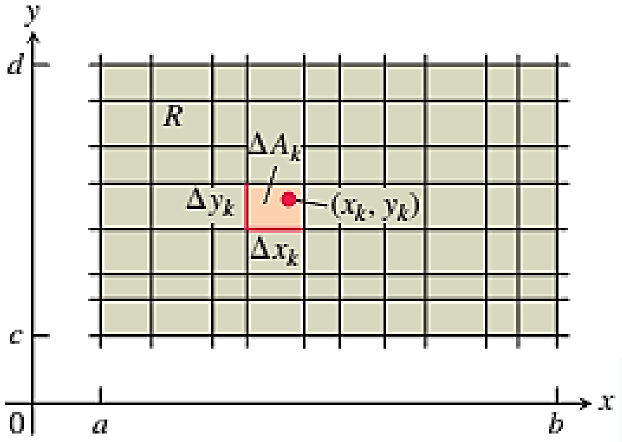
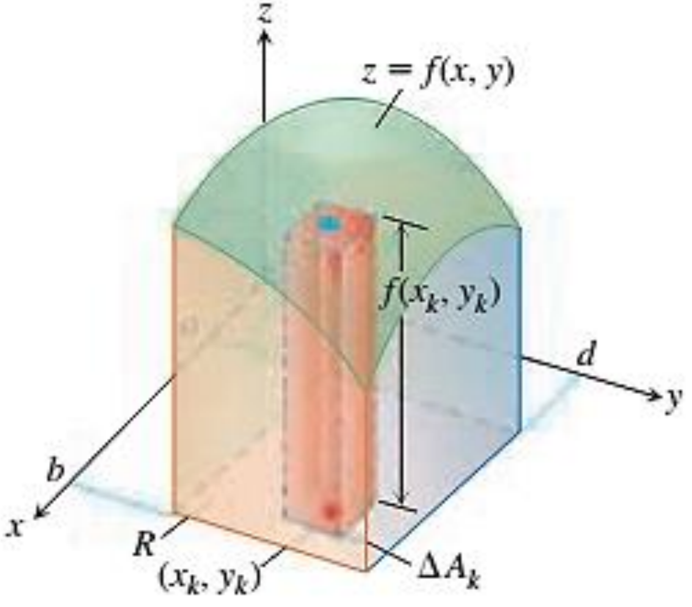
- Double and Iterated Integrals over Rectangles
- Double Integrals over General Regions
- Area by Double Integration
- Double Integrals in Polar Form
- Triple Integrals in Rectangular Coordinates
- Applications
- Triple Integrals in Cylindrical and Spherical Coordinates
- Substitutions in Multiple Integrals

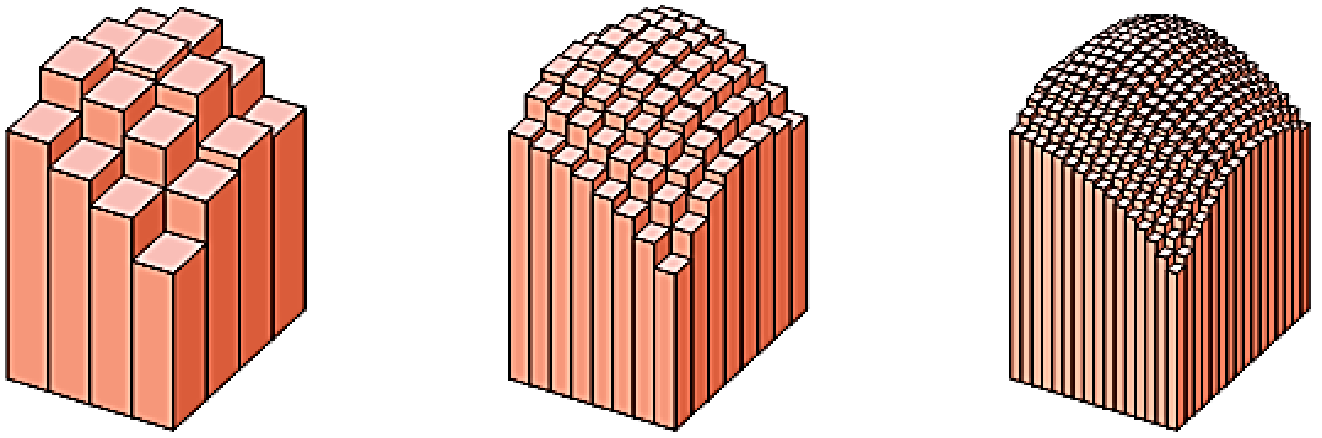
Double and Iterated Integrals over Rectangles

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k$$

$$\text{Volume} = \lim_{n \to \infty} S_n = \iint_R f(x, y)\, dA.$$







(a) $n = 16$ (b) $n = 64$ (c) $n = 256$

Double and Iterated Integrals over Rectangles

THEOREM 1—Fubini's Theorem (First Form)

If $f(x, y)$ is continuous throughout the rectangular region $R: a \le x \le b$, $c \le y \le d$, then

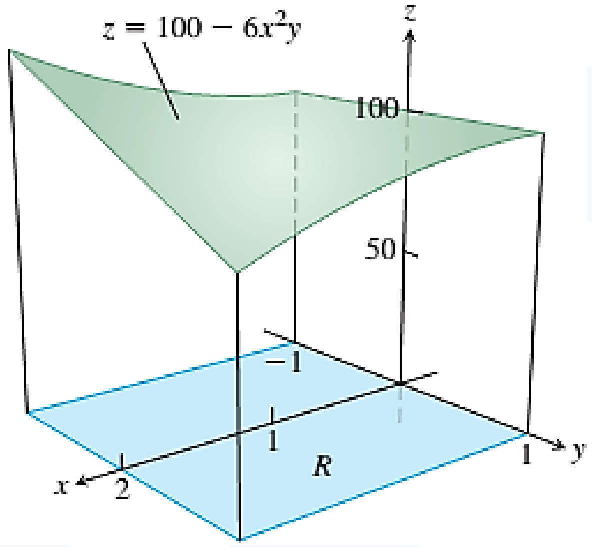
$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx.$$

EXAMPLE 1 Calculate $\iint_R f(x, y)\, dA$ for

$$f(x, y) = 100 - 6x^2y \quad \text{and} \quad R: \quad 0 \le x \le 2, \quad -1 \le y \le 1.$$

$$\iint_R f(x, y)\, dA = \int_{-1}^{1} \int_0^2 (100 - 6x^2y)\, dx\, dy = \int_{-1}^{1} \Big[100x - 2x^3y\Big]_{x=0}^{x=2} dy$$

$$= \int_{-1}^{1} (200 - 16y)\, dy = \Big[200y - 8y^2\Big]_{-1}^{1} = 400.$$



Exercises

- $$\iint_R y\sin(x+y)\,dA, \qquad R:\ -\pi \le x \le 0,\quad 0 \le y \le \pi$$ $=4$

- $$\iint_R \frac{y}{x^2y^2+1}\,dA, \qquad R:\ 0 \le x \le 1,\quad 0 \le y \le 1$$ $=\frac{\pi}{4}-\frac{1}{2}\ln 2$

- Evaluate $\int_{-1}^{1}\int_{0}^{\pi/2} x\sin\sqrt{y}\,dy\,dx.$ $=0$

- Find the volume of the region bounded above by the surface $z = 4 - y^2$ and below by the rectangle $R: 0 \le x \le 1$ $0 \le y \le 2.$

$=\frac{16}{3}$

Double Integrals over General Regions

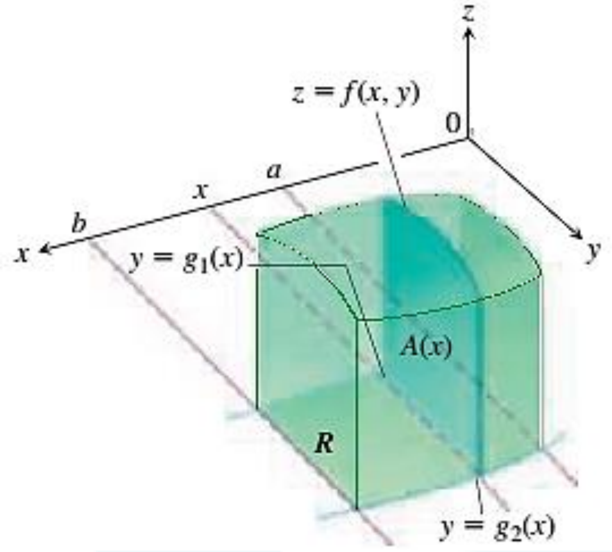
THEOREM 2—Fubini's Theorem (Stronger Form)
Let $f(x, y)$ be continuous on a region R.

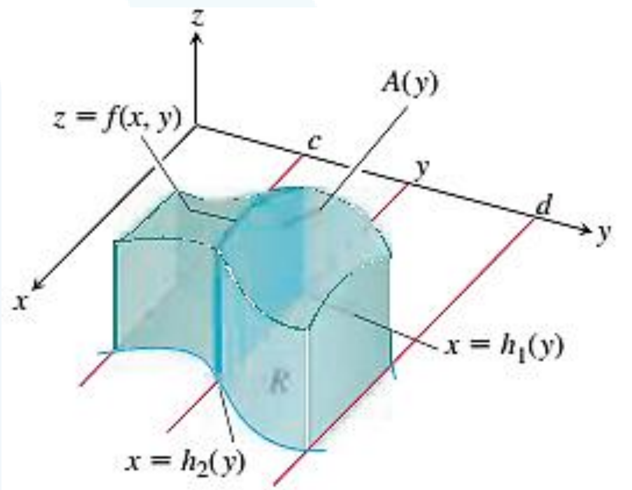
1. If R is defined by $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then

$$\iint\limits_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

2. If R is defined by $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then

$$\iint\limits_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

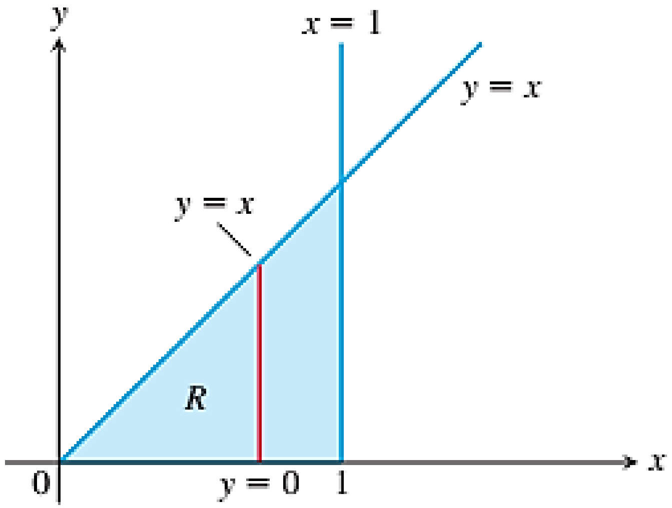




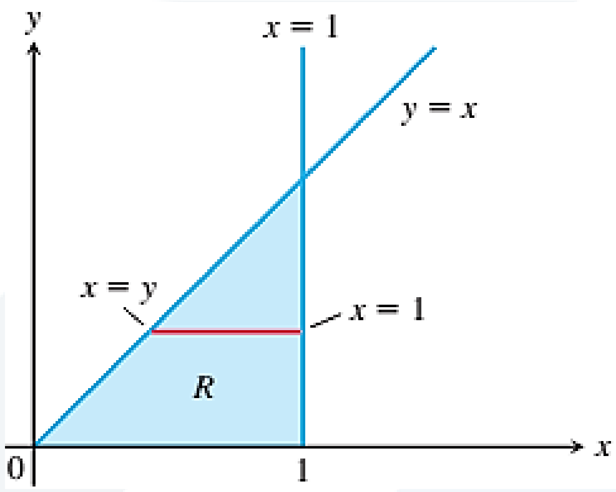
Double Integrals over General Regions

EXAMPLE 1 Find the volume of the prism whose base is the triangle in the xy-plane bounded by the x-axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane

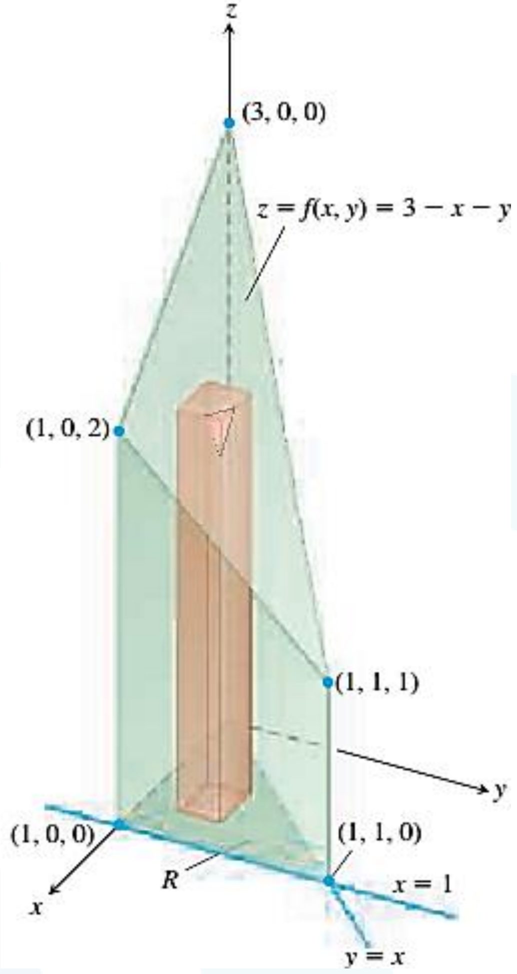
$$z = f(x, y) = 3 - x - y.$$



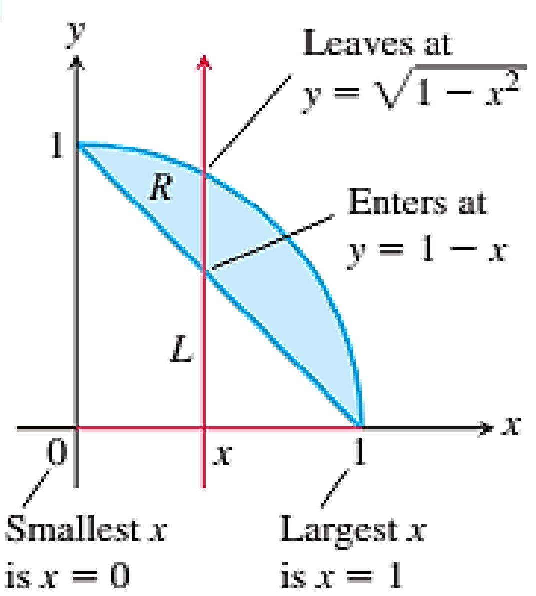
$$V = \int_0^1 \int_0^x (3 - x - y)\, dy\, dx$$

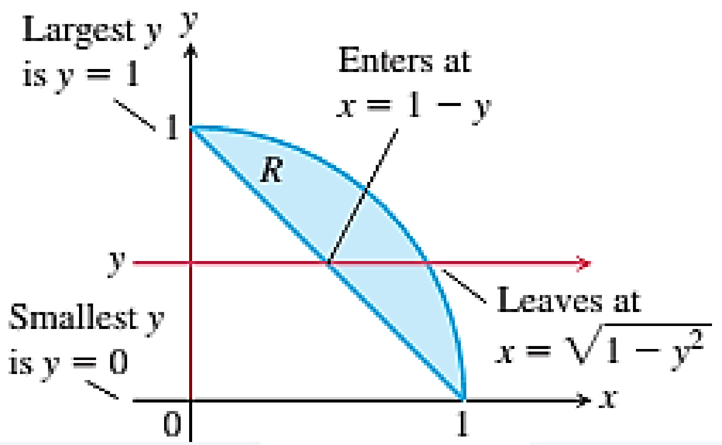


$$V = \int_0^1 \int_y^1 (3 - x - y)\, dx\, dy$$



Double Integrals over General Regions

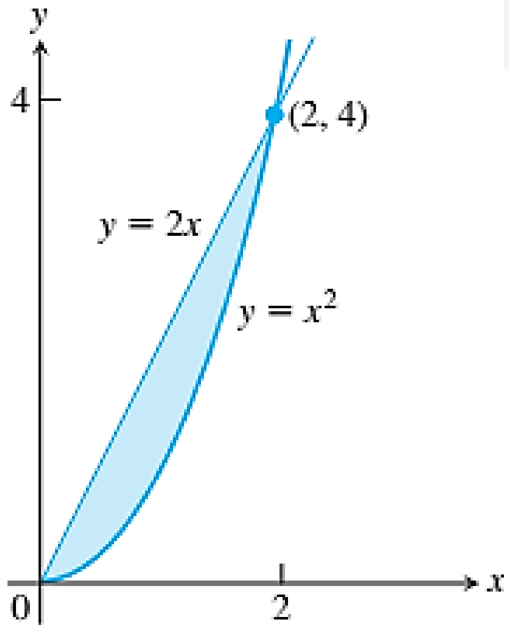




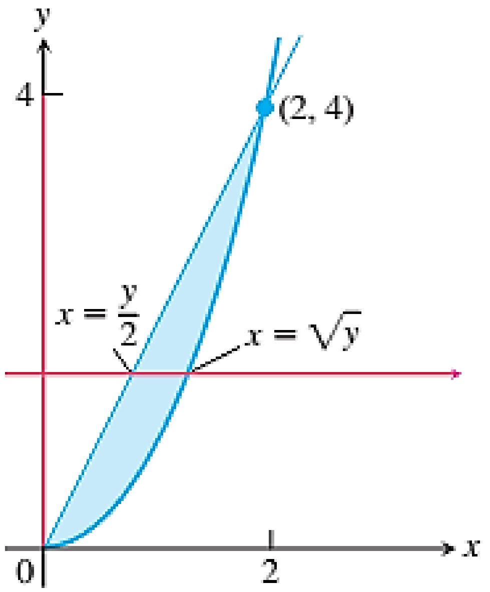
$$\iint_R f(x, y)\, dA = \int_{x=0}^{x=1} \int_{y=1-x}^{y=\sqrt{1-x^2}} f(x, y)\, dy\, dx$$

$$\iint_R f(x, y)\, dA = \int_0^1 \int_{1-y}^{\sqrt{1-y^2}} f(x, y)\, dx\, dy$$

Double Integrals over General Regions

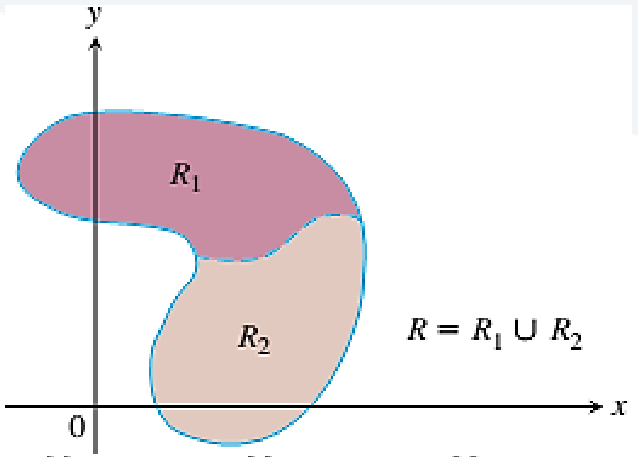


$$\int_0^2 \int_{x^2}^{2x} (4x + 2)\, dy\, dx$$



$$\int_0^4 \int_{y/2}^{\sqrt{y}} (4x + 2)\, dx\, dy.$$

Properties of Double Integrals



$$\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA$$

If $f(x, y)$ and $g(x, y)$ are continuous on the bounded region R, then the following properties hold.

1. *Constant Multiple:* $\displaystyle\iint_R c f(x, y)\, dA = c \iint_R f(x, y)\, dA$ (any number c)

2. *Sum and Difference:*

$$\iint_R \big(f(x, y) \pm g(x, y)\big)\, dA = \iint_R f(x, y)\, dA \pm \iint_R g(x, y)\, dA$$

3. Domination:

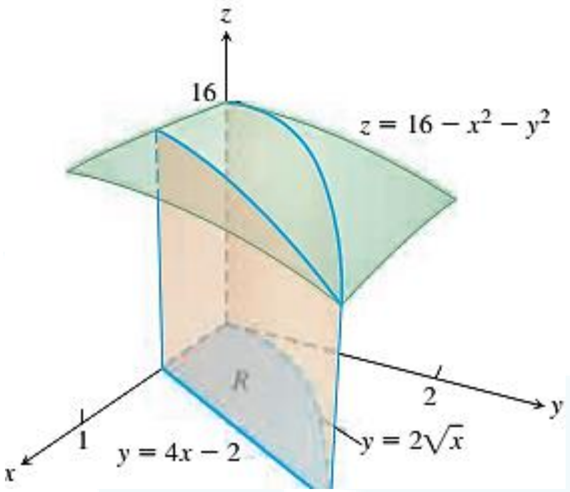
(a) $\displaystyle\iint_R f(x, y)\, dA \geq 0$ if $f(x, y) \geq 0$ on R

(b) $\displaystyle\iint_R f(x, y)\, dA \geq \iint_R g(x, y)\, dA$ if $f(x, y) \geq g(x, y)$ on R

4. *Additivity:* If R is the union of two nonoverlapping regions R_1 and R_2, then

$$\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA$$

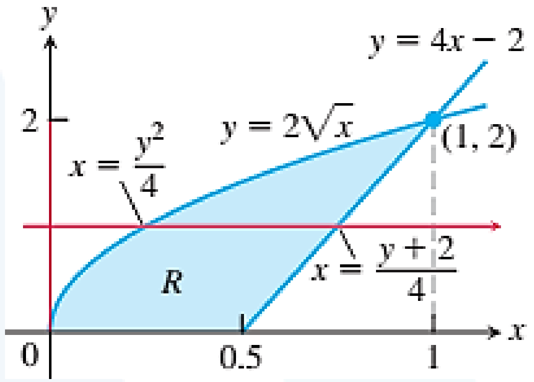
EXAMPLE 4 Find the volume of the wedgelike solid that lies beneath the surface $z = 16 - x^2 - y^2$ and above the region R bounded by the curve $y = 2\sqrt{x}$, the line $y = 4x - 2$, and the x-axis.



$$\iint_R (16 - x^2 - y^2)\,dA = \int_0^2 \int_{y^2/4}^{(y+2)/4} (16 - x^2 - y^2)\,dx\,dy = \int_0^2 \left[16x - \frac{x^3}{3} - xy^2\right]_{x=y^2/4}^{x=(y+2)/4} dx$$

$$= \int_0^2 \left[4(y+2) - \frac{(y+2)^3}{3\cdot 64} - \frac{(y+2)y^2}{4} - 4y^2 + \frac{y^6}{3\cdot 64} + \frac{y^4}{4}\right] dy$$

$$= \left[\frac{191y}{24} + \frac{63y^2}{32} - \frac{145y^3}{96} - \frac{49y^4}{768} + \frac{y^5}{20} + \frac{y^7}{1344}\right]_0^2 = \frac{20803}{1680} \approx 12.4.$$



Exercises

integrate f over the given region.

- $f(x, y) = x^2 + y^2$ over the triangular region with vertices (0, 0), (1, 0), and (0, 1) $\int_0^1 \int_0^{1-x} \left(x^2 + y^2\right) dy\, dx \quad = \frac{1}{6}$
- $f(s, t) = e^s \ln t$ over the region in the first quadrant of the st-plane that lies above the curve $s = \ln t$ from $t = 1$ to $t = 2$

$$\int_1^2 \int_0^{\ln t} e^s \ln t \, ds\, dt \quad = \frac{1}{4}$$

- $\iint_R (y - 2x^2)\, dA$ where R is the region bounded by the square $|x| + |y| = 1$ $\quad = -\frac{2}{3}$

- Find the volume of the solid whose base is the region in the xy-plane that is bounded by the parabola $y = 4 - x^2$ and the line $y = 3x$, while the top of the solid is bounded by the plane $z = x + 4$.

$$V = \int_{-4}^{1} \int_{3x}^{4-x^2} (x+4)\, dy\, dx \quad = \frac{625}{12}$$

Area by Double Integration

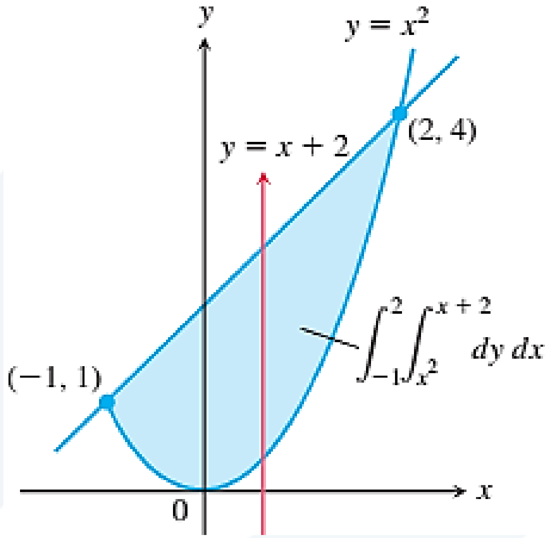
DEFINITION The **area** of a closed, bounded plane region R is

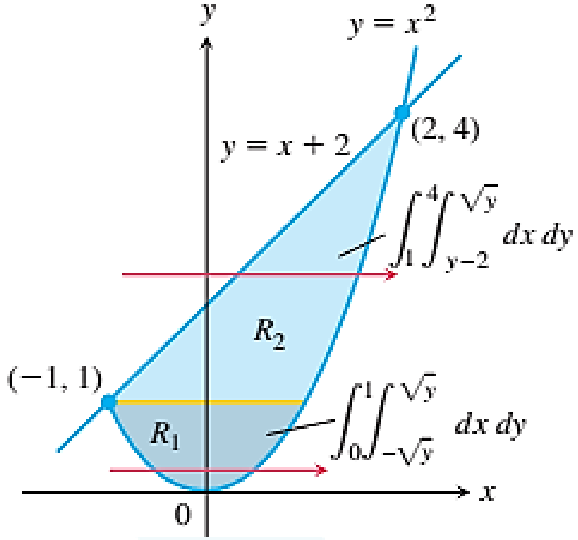
$$A = \iint_R dA.$$

EXAMPLE 2 Find the area of the region R enclosed by the parabola $y = x^2$ and the line $y = x + 2$.

$$A = \int_{-1}^{2} \Big[y \Big]_{x^2}^{x+2} dx = \int_{-1}^{2} (x + 2 - x^2)\, dx$$

$$= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^{2} = \frac{9}{2}$$

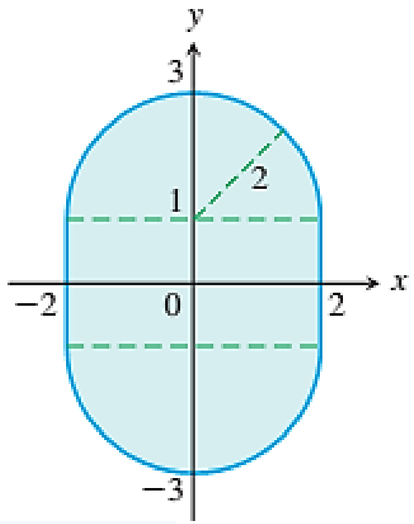




$$A = \iint_{R_1} dA + \iint_{R_2} dA = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx\, dy + \int_1^4 \int_{y-2}^{\sqrt{y}} dx\, dy.$$

Area by Double Integration

EXAMPLE 3 Find the area of the playing field described by $R: -2 \le x \le 2, -1 - \sqrt{4 - x^2} \le y \le 1 + \sqrt{4 - x^2}$, using **(a)** Fubini's Theorem **(b)** Simple geometry.



(a) $$A = \iint_R dA = 4\int_0^2 \int_0^{1+\sqrt{4-x^2}} dy\, dx$$

$$= 4\int_0^2 \left(1 + \sqrt{4 - x^2}\right) dx$$

$$= 4\left[x + \frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2 = 8 + 4\pi$$

(b) $$A = 8 + \pi 2^2 = 8 + 4\pi.$$

Area by Double Integration

Average Value

$$\text{Average value of } f \text{ over } R = \frac{1}{\text{area of } R} \iint_R f \, dA. \qquad (3)$$

EXAMPLE 4 Find the average value of $f(x, y) = x \cos xy$ over the rectangle $R: 0 \leq x \leq \pi,\ 0 \leq y \leq 1$.

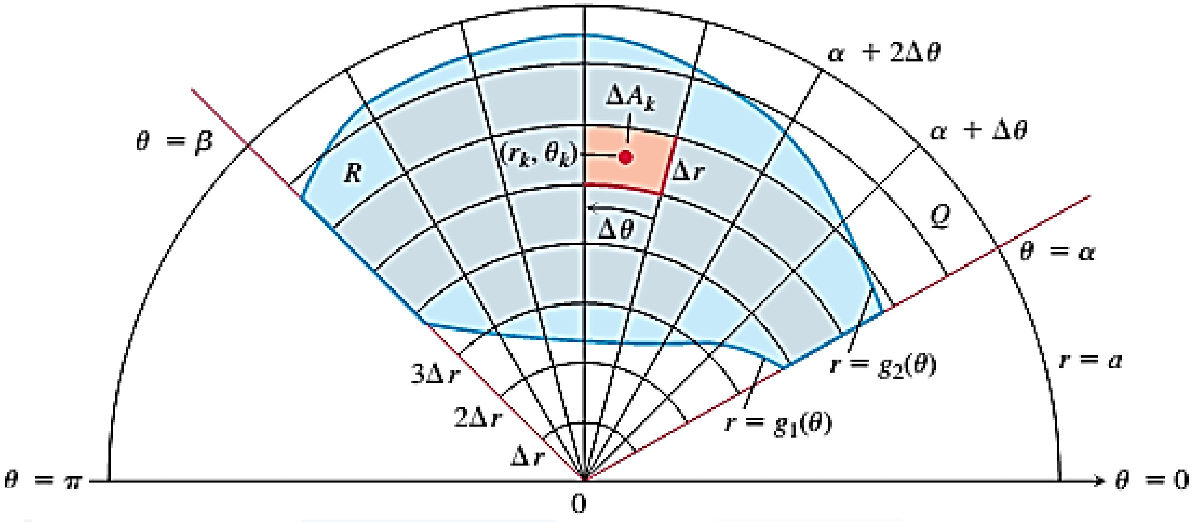
$$\int_0^{\pi} \int_0^1 x \cos xy \, dy \, dx = \int_0^{\pi} \left[\sin xy \right]_{y=0}^{y=1} dx = \int_0^{\pi} (\sin x - 0) \, dx = -\cos x \Big]_0^{\pi} = 2.$$

The area of R is π. The average value of f over R is $2/\pi$.

Double Integrals in Polar Form

$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k)\,\Delta A_k.$$

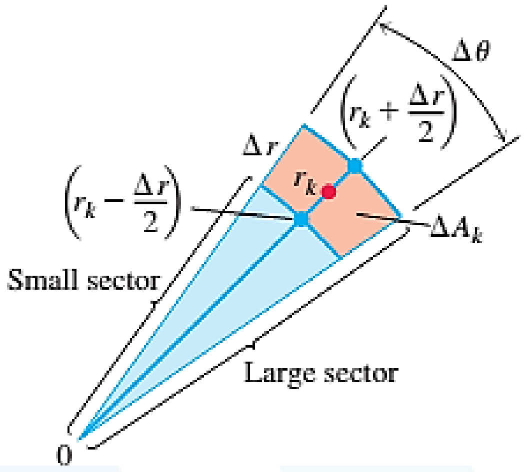
$$\lim_{n\to\infty} S_n = \iint_R f(r, \theta)\, dA.$$



ΔA_k = area of large sector − area of small sector

$$= \frac{\Delta\theta}{2}\left[\left(r_k + \frac{\Delta r}{2}\right)^2 - \left(r_k - \frac{\Delta r}{2}\right)^2\right] = \frac{\Delta\theta}{2}(2r_k\,\Delta r) = r_k\,\Delta r\,\Delta\theta$$

$$\Rightarrow \quad S_n = \sum_{k=1}^{n} f(r_k, \theta_k)\, r_k\,\Delta r\,\Delta\theta \quad \Rightarrow \quad \lim_{n\to\infty} S_n = \iint_R f(r, \theta)\, r\, dr\, d\theta.$$



$$\iint_R f(r, \theta)\, dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r, \theta)\, r\, dr\, d\theta.$$

Double Integrals in Polar Form

Area in Polar Coordinates

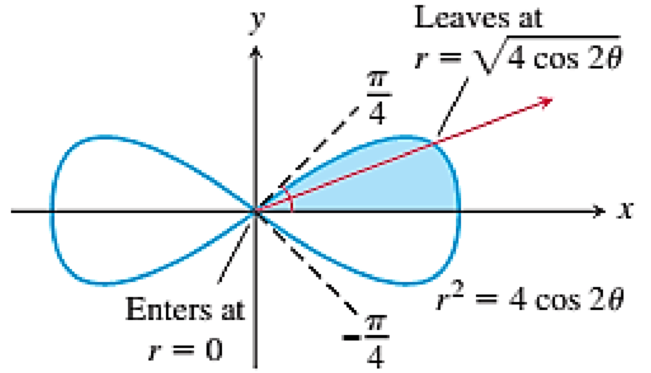
The area of a closed and bounded region R in the polar coordinate plane is

$$A = \iint_R r\,dr\,d\theta.$$

EXAMPLE 2 Find the area enclosed by the lemniscate $r^2 = 4\cos 2\theta$.

$$A = 4\int_0^{\pi/4}\int_0^{\sqrt{4\cos 2\theta}} r\,dr\,d\theta = 4\int_0^{\pi/4}\left[\frac{r^2}{2}\right]_{r=0}^{r=\sqrt{4\cos 2\theta}} d\theta$$

$$= 4\int_0^{\pi/4} 2\cos 2\theta\,d\theta = 4\sin 2\theta\Big]_0^{\pi/4} = 4$$



Double Integrals in Polar Form

Changing Cartesian Integrals into Polar Integrals

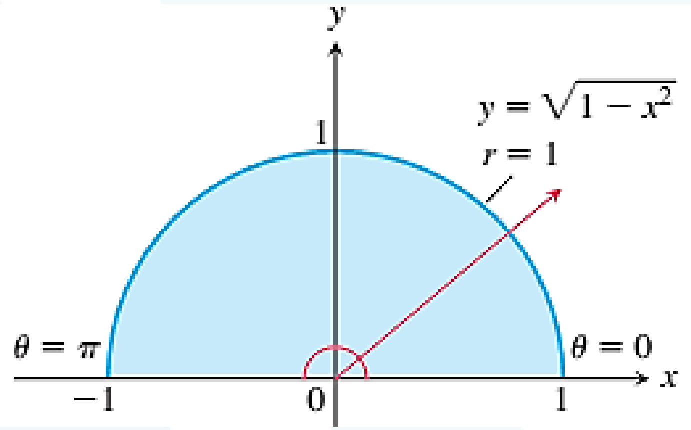
$$\iint_R f(x, y)\, dx\, dy = \iint_G f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

EXAMPLE 3 Evaluate $\displaystyle\iint_R e^{x^2+y^2}\, dy\, dx$

where R is the semicircular region bounded by the x-axis and the curve $y = \sqrt{1 - x^2}$

$$\iint_R e^{x^2+y^2}\, dy\, dx = \int_0^{\pi}\int_0^1 e^{r^2}\, r\, dr\, d\theta = \int_0^{\pi}\left[\frac{1}{2}e^{r^2}\right]_0^1 d\theta$$

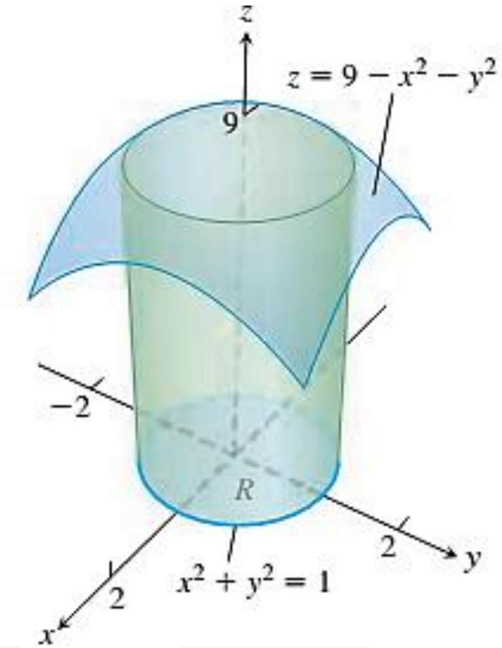
$$= \int_0^{\pi}\frac{1}{2}(e - 1)\, d\theta = \frac{\pi}{2}(e - 1).$$



Double Integrals in Polar Form

Changing Cartesian Integrals into Polar Integrals

EXAMPLE 5 Find the volume of the solid region bounded above by the paraboloid $z = 9 - x^2 - y^2$ and below by the unit circle in the xy-plane.



$$\iint_R (9 - x^2 - y^2)\, dA = \int_0^{2\pi}\int_0^1 (9 - r^2)\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^1 (9r - r^3)\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{9}{2}r^2 - \frac{1}{4}r^4\right]_{r=0}^{r=1} d\theta$$

$$= \frac{17}{4}\int_0^{2\pi} d\theta = \frac{17\pi}{2}$$

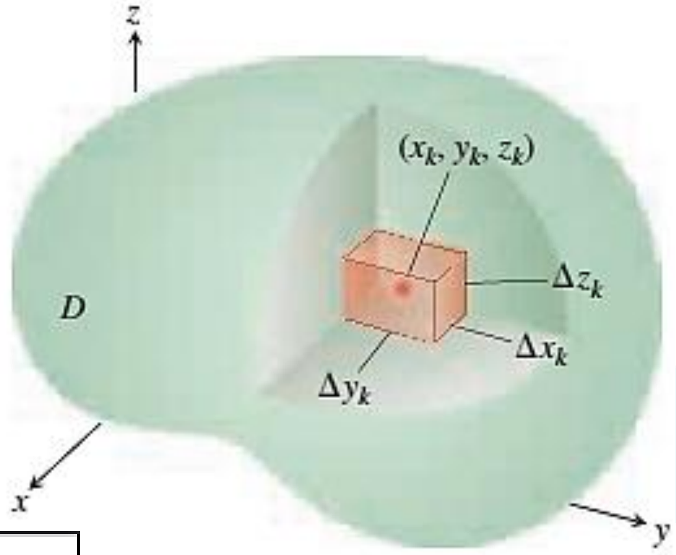
Exercises

- change the Cartesian integral into an equivalent polar integral. Then evaluate the polar integral
- $$\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} \frac{2}{1+\sqrt{x^2+y^2}}\,dy\,dx \qquad (1-\ln 2)\pi$$
- $$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln\left(x^2+y^2+1\right)dx\,dy \qquad \pi(\ln 4-1)$$
- **One leaf of a rose** Find the area enclosed by one leaf of the rose $r = 12\cos 3\theta$ $\qquad 12\pi$
- Integrate $f(x, y) = \left[\ln\left(x^2+y^2\right)\right]/\sqrt{x^2+y^2}$ over the region $1 \le x^2+y^2 \le e$. $\qquad 2\pi\left(2-\sqrt{e}\right)$

Triple Integrals in Rectangular Coordinates

$$S_n = \sum_{k=1}^{n} F(x_k, y_k, z_k)\,\Delta V_k$$

$$\lim_{n\to\infty} S_n = \iiint_D F(x, y, z)\,dV \qquad \text{or} \qquad \lim_{\|P\|\to 0} S_n = \iiint_D F(x, y, z)\,dx\,dy\,dz.$$

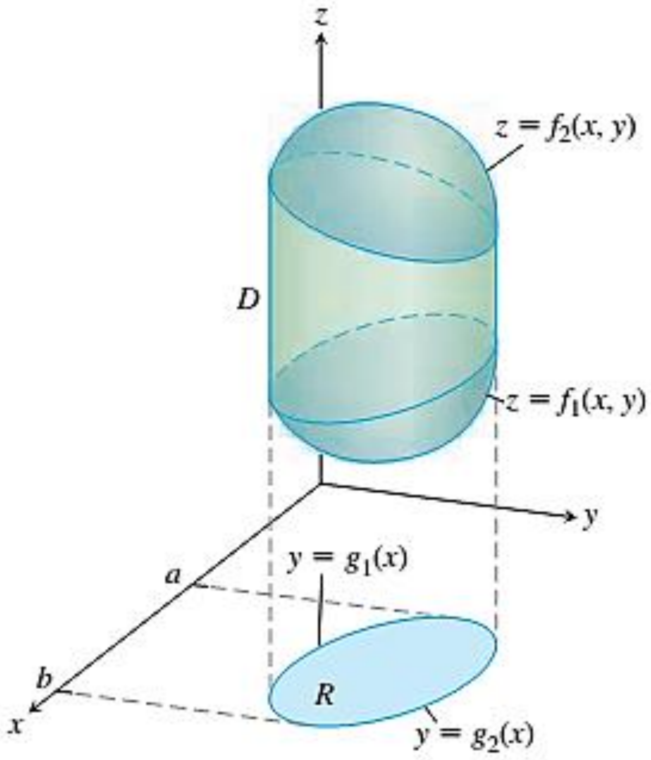


DEFINITION The **volume** of a closed, bounded region D in space is

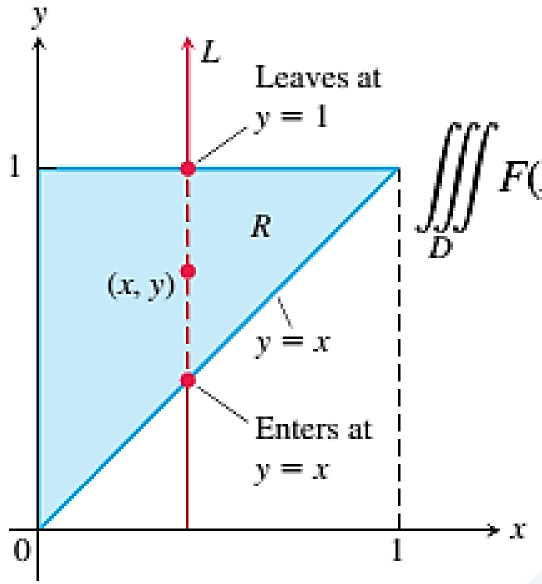
$$V = \iiint_D dV.$$

Triple Integrals in Rectangular Coordinates

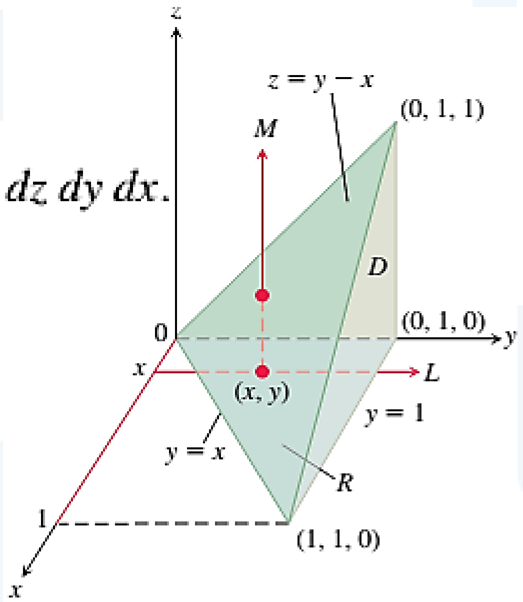
$$\int_{x=a}^{x=b}\int_{y=g_1(x)}^{y=g_2(x)}\int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x, y, z)\, dz\, dy\, dx.$$



EXAMPLE 2 Set up the limits of integration for evaluating the triple integral of a function $F(x, y, z)$ over the tetrahedron D whose vertices are (0, 0, 0), (1, 1, 0), (0, 1, 0), and (0, 1, 1). Use the order of integration $dz\,dy\,dx$.



$$\iiint_D F(x, y, z)\, dz\, dy\, dx = \int_0^1\int_x^1\int_0^{y-x} F(x, y, z)\, dz\, dy\, dx.$$

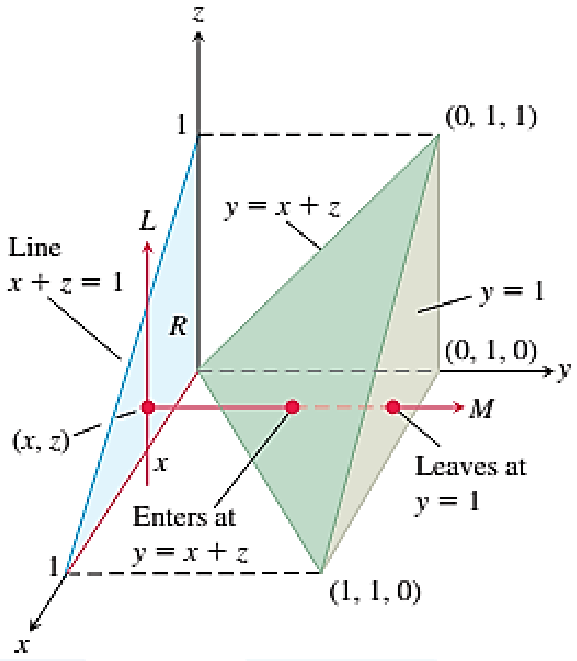


Triple Integrals in Rectangular Coordinates

EXAMPLE 3 Find the volume of the tetrahedron D from Example 2 by integrating $F(x, y, z) = 1$ over the region using the order $dz\,dy\,dx$. Then do the same calculation using the order $dy\,dz\,dx$.

$$V = \int_0^1 \int_x^1 \int_0^{y-x} dz\,dy\,dx = \frac{1}{6}.$$

$$V = \int_0^1 \int_0^{1-x} \int_{x+z}^{1} dy\,dz\,dx = \frac{1}{6}.$$



Triple Integrals in Rectangular Coordinates

EXAMPLE 4 Find the volume of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

$$V = \iiint_D dz\,dy\,dx,$$

$$x^2 + 3y^2 \le z \le 8 - x^2 - y^2$$

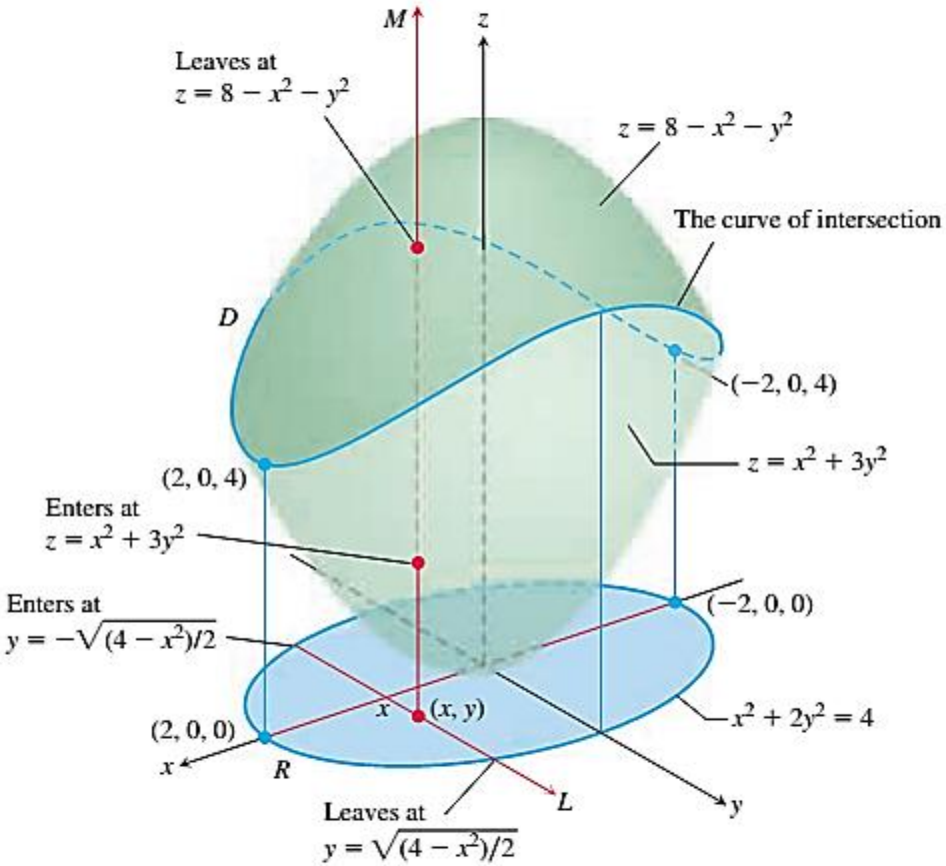
$$x^2 + 3y^2 \cap 8 - x^2 - y^2$$

$$x^2 + 3y^2 = 8 - x^2 - y^2 \quad \Rightarrow \quad x^2 + 2y^2 = 4\ ,\ z > 0$$

$$-\sqrt{\left(4 - x^2\right)/2} \le y \le \sqrt{\left(4 - x^2\right)/2}$$

$$-2 \le x \le 2$$

$$V = \iiint_D dz\,dy\,dx = \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx = 8\pi\sqrt{2}.$$

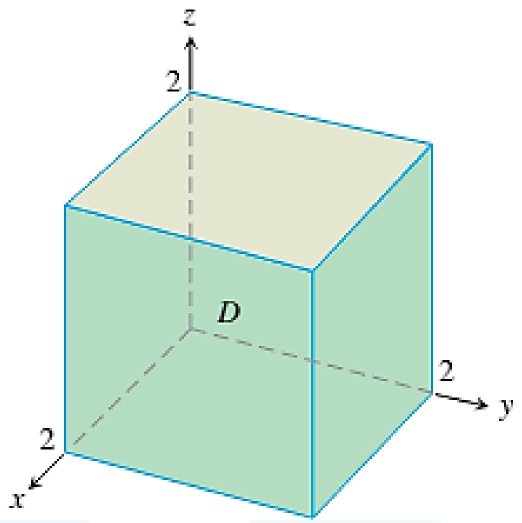


Triple Integrals in Rectangular Coordinates

Average Value of a Function in Space

$$\textbf{Average value of } F \text{ over } D = \frac{1}{\text{volume of } D}\iiint_D F\,dV.$$

EXAMPLE 5 Find the average value of $F(x, y, z) = xyz$ throughout the cubical region D bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$ in the first octant.



The volume of the region D is $(2)(2)(2) = 8$.

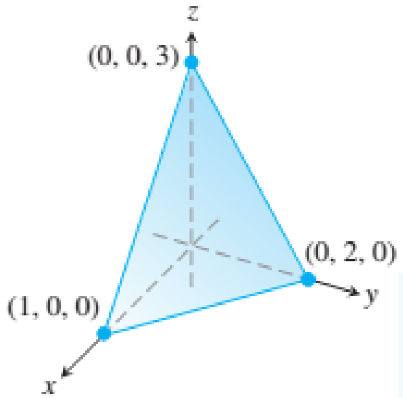
$$\int_0^2\int_0^2\int_0^2 xyz\,dx\,dy\,dz = 8$$

$$\begin{array}{l}\text{Average value of}\\ xyz \text{ over the cube}\end{array} = \frac{1}{\text{volume}}\iiint_{\text{cube}} xyz\,dV = \left(\frac{1}{8}\right)(8) = 1.$$

Exercises

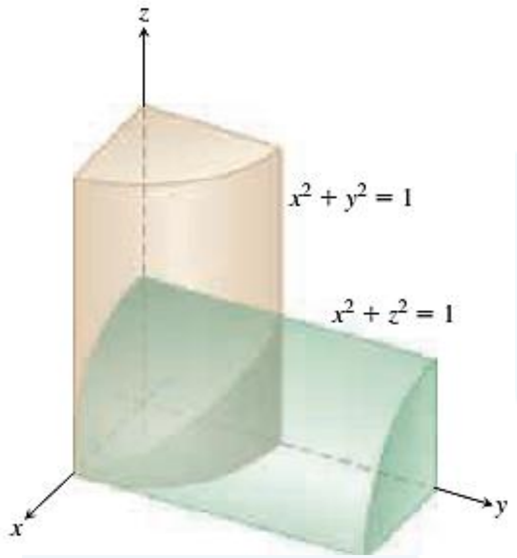
Find the volumes of the regions

- The tetrahedron in the first octant bounded by the coordinate planes and the plane passing through (1, 0, 0), (0, 2, 0), and (0, 0, 3) $=1$



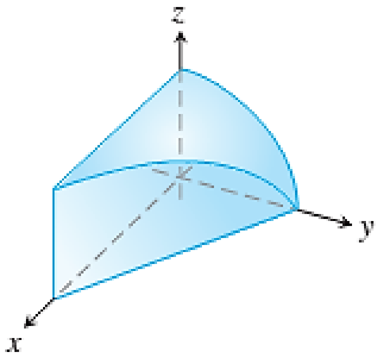
- The region common to the interiors of the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$, one-eighth of which is shown in the accompanying figure

$\frac{16}{3}$



- The region in the first octant bounded by the coordinate planes, the plane $x + y = 4$, and the cylinder $y^2 + 4z^2 = 16$

$8\pi - \frac{32}{3}$



Applications

Masses and First Moments

TWO-DIMENSIONAL PLATE

Mass: $M = \iint_R \delta \, dA$ $\quad$ $\delta = \delta(x, y)$ is the density at (x, y).

First moments: $M_y = \iint_R x\,\delta\,dA, \qquad M_x = \iint_R y\,\delta\,dA$

Center of mass: $\bar{x} = \dfrac{M_y}{M}, \qquad \bar{y} = \dfrac{M_x}{M}$

THREE-DIMENSIONAL SOLID

Mass: $M = \iiint_D \delta \, dV$ $\quad$ $\delta = \delta(x, y, z)$ is the density at (x, y, z).

First moments about the coordinate planes:

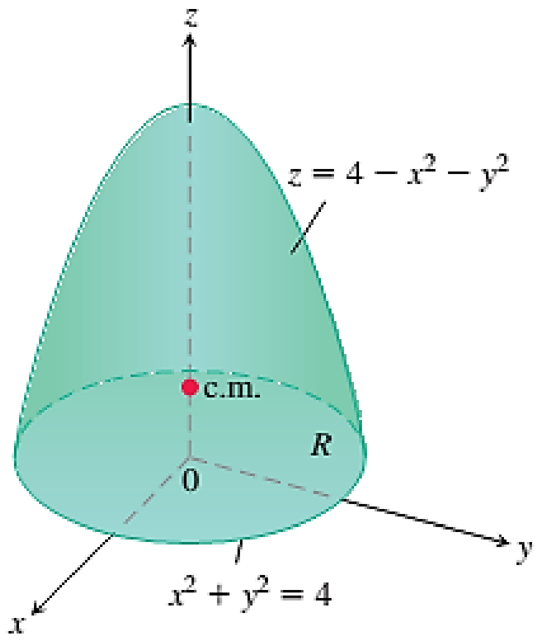
$$M_{yz} = \iiint_D x\,\delta\,dV, \qquad M_{xz} = \iiint_D y\,\delta\,dV, \qquad M_{xy} = \iiint_D z\,\delta\,dV$$

Center of mass: $\bar{x} = \dfrac{M_{yz}}{M}, \qquad \bar{y} = \dfrac{M_{xz}}{M}, \qquad \bar{z} = \dfrac{M_{xy}}{M}$

Applications

Masses and First Moments

EXAMPLE 1 Find the center of mass of a solid of constant density δ bounded below by the disk $R: x^2 + y^2 \le 4$ in the plane $z = 0$ and above by the paraboloid $z = 4 - x^2 - y^2$ (Figure 15.38).



By symmetry $\bar{x} = \bar{y} = 0$. To find $\bar{z}$,

$$M_{xy} = \iint_R \int_{z=0}^{z=4-x^2-y^2} z\,\delta\,dz\,dy\,dx = \iint_R \left[\frac{z^2}{2}\right]_{z=0}^{z=4-x^2-y^2} \delta\,dy\,dx = \frac{\delta}{2}\iint_R (4 - x^2 - y^2)^2\,dy\,dx$$

$$= \frac{\delta}{2}\int_0^{2\pi}\int_0^2 (4 - r^2)^2\,r\,dr\,d\theta = \frac{32\pi\delta}{3}$$

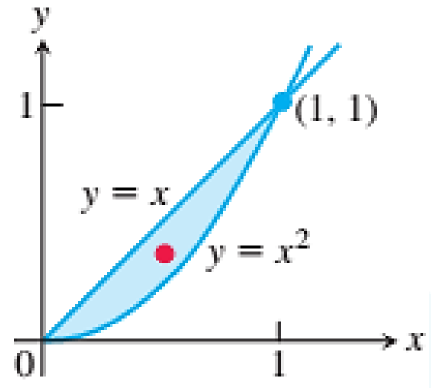
$$M = \iint_R \int_0^{4-x^2-y^2} \delta\,dz\,dy\,dx = 8\pi\delta.$$

Therefore $\bar{z} = (M_{xy}/M) = 4/3$ and the center of mass is $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, 4/3)$

Applications

Masses and First Moments

EXAMPLE 2 Find the centroid of the region in the first quadrant that is bounded above by the line $y = x$ and below by the parabola $y = x^2$.



$$M = \int_0^1 \int_{x^2}^{x} 1\, dy\, dx = \frac{1}{6}$$

$$M_x = \int_0^1 \int_{x^2}^{x} y\, dy\, dx = \frac{1}{15} \qquad M_y = \int_0^1 \int_{x^2}^{x} x\, dy\, dx = \frac{1}{12}$$

$$\bar{x} = \frac{M_y}{M} = \frac{1/12}{1/6} = \frac{1}{2} \quad \text{and} \quad \bar{y} = \frac{M_x}{M} = \frac{1/15}{1/6} = \frac{2}{5}.$$

The centroid is the point $(1/2, 2/5)$.

Applications

Moments of Inertia

TWO-DIMENSIONAL PLATE

About the x-axis:	$I_x = \iint y^2 \, \delta \, dA$	$\delta = \delta(x, y)$
About the y-axis:	$I_y = \iint x^2 \, \delta \, dA$	
About a line L:	$I_L = \iint r^2(x, y) \, \delta \, dA$	$r(x, y) =$ distance from (x, y) to L
About the origin (polar moment):	$I_0 = \iint (x^2 + y^2) \, \delta \, dA = I_x + I_y$	

Applications

Moments of Inertia

THREE-DIMENSIONAL SOLID

About the x-axis:	$I_x = \iiint (y^2 + z^2)\,\delta\,dV$	$\delta = \delta(x, y, z)$
About the y-axis:	$I_y = \iiint (x^2 + z^2)\,\delta\,dV$	
About the z-axis:	$I_z = \iiint (x^2 + y^2)\,\delta\,dV$	
About a line L:	$I_L = \iiint r^2(x, y, z)\,\delta\,dV$	$r(x, y, z) =$ distance from the point (x, y, z) to line L

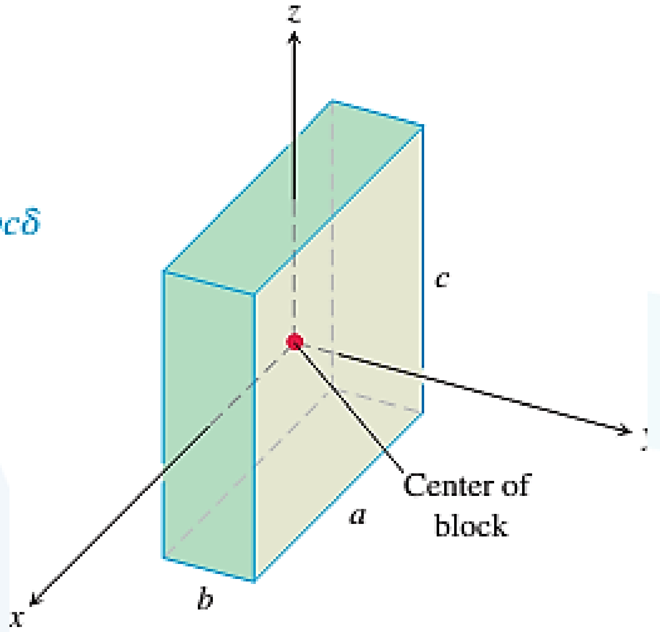
Applications

Moments of Inertia

EXAMPLE 3 Find I_x, I_y, I_z for the rectangular solid of constant density δ

$$I_x = \int_{-c/2}^{c/2}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2}(y^2 + z^2)\,\delta\,dx\,dy\,dz = \frac{abc\delta}{12}(b^2 + c^2) = \frac{M}{12}(b^2 + c^2).$$

$M = abc\delta$

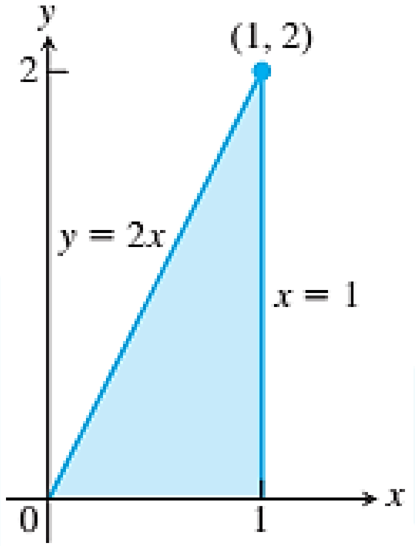


$$I_y = \frac{M}{12}(a^2 + c^2) \qquad \text{and} \qquad I_z = \frac{M}{12}(a^2 + b^2).$$

Applications

Moments of Inertia

EXAMPLE 4 A thin plate covers the triangular region bounded by the x-axis and the lines $x = 1$ and $y = 2x$ in the first quadrant. The plate's density at the point (x, y) is $\delta(x, y) = 6x + 6y + 6$. Find the plate's moments of inertia about the coordinate axes and the origin.



$$I_x = \int_0^1 \int_0^{2x} y^2 \delta(x, y)\, dy\, dx = \int_0^1 \int_0^{2x} (6xy^2 + 6y^3 + 6y^2)\, dy\, dx = 12$$

$$I_y = \int_0^1 \int_0^{2x} x^2 \delta(x, y)\, dy\, dx = \frac{39}{5}.$$

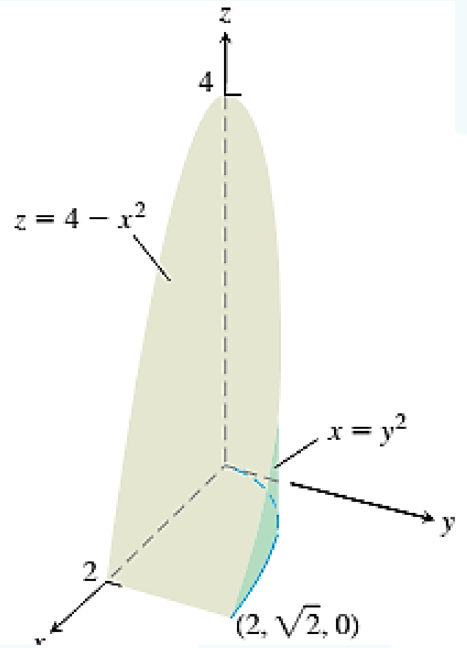
$$I_0 = 12 + \frac{39}{5} = \frac{60 + 39}{5} = \frac{99}{5}$$

Exercises

- Find the moment of inertia about the x-axis of a thin plate bounded by the parabola $x = y - y^2$ and the line $x + y = 0$ if $\delta(x, y) = x + y$.

$\frac{64}{105}$

- A solid in the first octant is bounded by the planes $y = 0$ and $z = 0$ and by the surfaces $z = 4 - x^2$ and $x = y^2$ (see the accompany-ing figure). Its density function is $\delta(x, y, z) = kxy$, k a constant.



a. the mass of the solid. **b.** the center of mass.

$M = \frac{32k}{15}$ $M_{yz} = \frac{8k}{3}$ $\bar{x} = \frac{5}{4}$ $M_{xz} = \frac{256\sqrt{2}k}{231}$ $\bar{y} = \frac{40\sqrt{2}}{77}$ $M_{xy} = \frac{256k}{105}$ $\bar{z} = \frac{8}{7}$

- Find the mass of the solid region bounded by the parabolic surfaces $z = 16 - 2x^2 - 2y^2$ and $z = 2x^2 + 2y^2$ if the density of the solid is $\delta(x, y, z) = \sqrt{x^2 + y^2}$.

$\frac{512\pi}{15}$