



Calculus 1

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Calculus 1

Lecture 4

Derivatives



Chapter 3

Derivatives

3.1 Tangents and the Derivative at a Point

3.2 The Derivative as a Function

3.3 Differentiation Rules

3.4 Derivatives of Trigonometric Functions



Tangent Lines and the Derivative at a Point

Finding a Tangent Line to the Graph of a Function

$$y = f(a) + f'(a)(x - a)$$

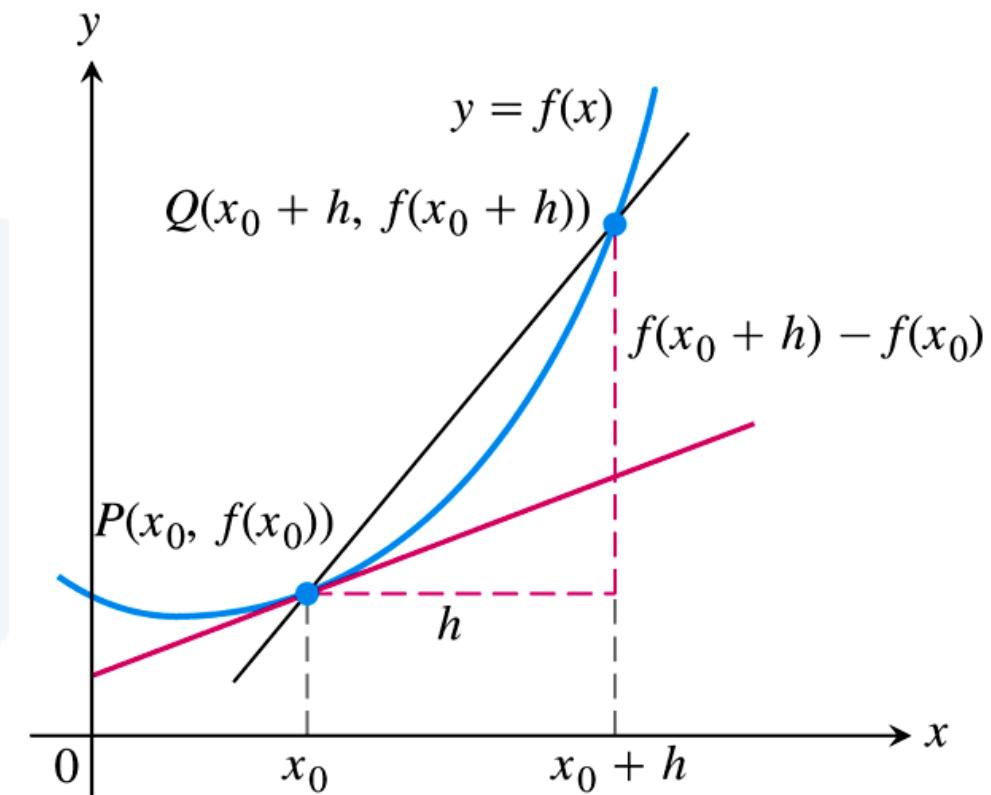


FIGURE 3.1 The slope of the tangent line at P is $\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$.

Rates of Change: Derivative at a Point

DEFINITIONS The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} \quad (\text{provided the limit exists}).$$

The **tangent line** to the curve at P is the line through P with this slope.

DEFINITION The **derivative of a function f at a point x_0** , denoted $f'(x_0)$, is

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided this limit exists.

Rates of Change: Derivative at a Point

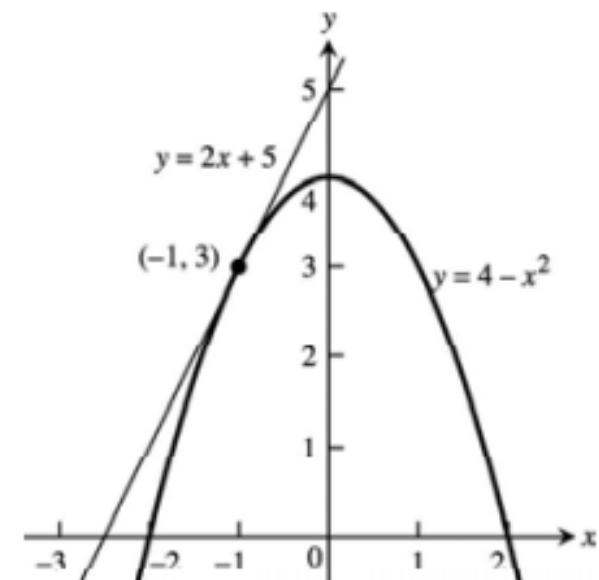
Example

find an equation for the tangent to the curve at the given point. Then sketch the curve and tangent together

$$y = 4 - x^2, \quad (-1, 3)$$

Solution

$$\begin{aligned} m &= \lim_{h \rightarrow 0} \frac{[4 - (-1+h)^2] - (4 - (-1)^2)}{h} = \lim_{h \rightarrow 0} \frac{-(1-2h+h^2)+1}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(2-h)}{h} = 2; \text{ at } (-1, 3): y = 3 + 2(x - (-1)) \\ \Rightarrow y &= 2x + 5, \text{ tangent line} \end{aligned}$$





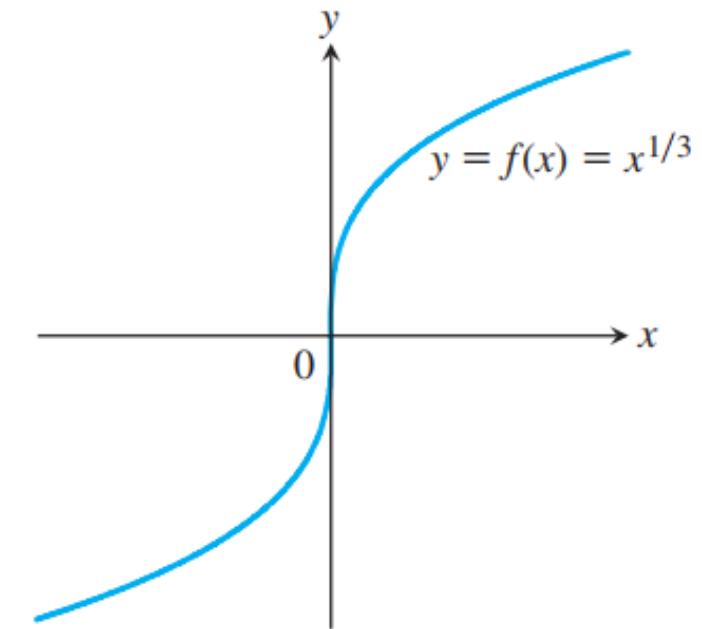
Vertical Tangents

We say that a continuous curve $y = f(x)$ has a **vertical tangent** at the point where $x = x_0$ if the limit of the difference quotient is ∞ or $-\infty$.

Example

For example, $y = x^{1/3}$ has a vertical tangent at $x = 0$ (see accompanying figure):

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{f(0 + h) - f(0)}{h} &= \lim_{h \rightarrow 0} \frac{h^{1/3} - 0}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h^{2/3}} = \infty.\end{aligned}$$



VERTICAL TANGENT AT ORIGIN

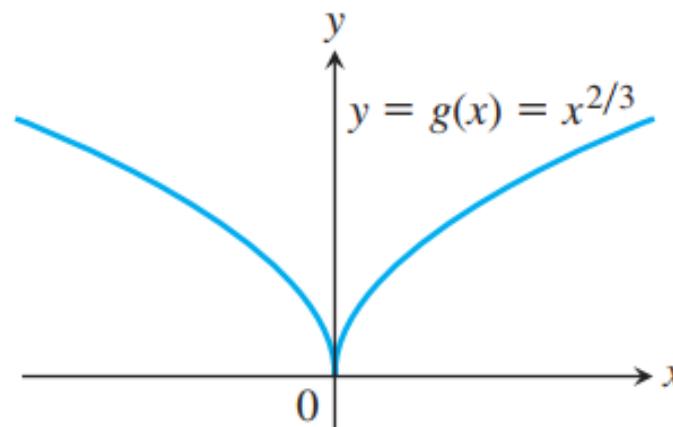


Vertical Tangents

However, $y = x^{2/3}$ has *no* vertical tangent at $x = 0$ (see next figure):

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{g(0 + h) - g(0)}{h} &= \lim_{h \rightarrow 0} \frac{h^{2/3} - 0}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h^{1/3}}\end{aligned}$$

does not exist, because the limit is ∞ from the right and $-\infty$ from the left.



NO VERTICAL TANGENT AT ORIGIN



The Derivative as a Function

In the last section we defined the derivative of $y = f(x)$ at the point $x = x_0$ to be the limit

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

We now investigate the derivative as a *function* derived from f by considering the limit at each point x in the domain of f .

DEFINITION The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h},$$

provided the limit exists.



The Derivative as a Function

If we write $z = x + h$, then $h = z - x$ and h approaches 0 if and only if z approaches x .

Alternative Formula for the Derivative

$$f'(x) = \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x}$$

Notations

There are many ways to denote the derivative of a function $y = f(x)$, where the independent variable is x and the dependent variable is y . Some common alternative notations for the derivative are

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = D(f)(x) = D_xf(x).$$



Calculating Derivatives from the Definition

EXAMPLE 1 Differentiate $f(x) = \frac{x}{x - 1}$.

Solution We use the definition of derivative, which requires us to calculate $f(x + h)$ and then subtract $f(x)$ to obtain the numerator in the difference quotient. We have

$$f(x) = \frac{x}{x - 1} \quad \text{and} \quad f(x + h) = \frac{(x + h)}{(x + h) - 1}, \text{ so}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} \qquad \text{Definition}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{x + h}{x + h - 1} - \frac{x}{x - 1}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \cdot \frac{(x + h)(x - 1) - x(x + h - 1)}{(x + h - 1)(x - 1)} \qquad \frac{a}{b} - \frac{c}{d} = \frac{ad - cb}{bd}$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \cdot \frac{-h}{(x + h - 1)(x - 1)} \qquad \text{Simplify.}$$

$$= \lim_{h \rightarrow 0} \frac{-1}{(x + h - 1)(x - 1)} = \frac{-1}{(x - 1)^2}. \qquad \text{Cancel } h \neq 0. \quad \blacksquare$$



Calculating Derivatives from the Definition

EXAMPLE 2

- (a) Find the derivative of $f(x) = \sqrt{x}$ for $x > 0$.
- (b) Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.

Solution

- (a) We use the alternative formula to calculate f' :

$$\begin{aligned}f'(x) &= \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x} \\&= \lim_{z \rightarrow x} \frac{\sqrt{z} - \sqrt{x}}{z - x} \\&= \lim_{z \rightarrow x} \frac{\sqrt{z} - \sqrt{x}}{(\sqrt{z} - \sqrt{x})(\sqrt{z} + \sqrt{x})} \\&= \lim_{z \rightarrow x} \frac{1}{\sqrt{z} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.\end{aligned}$$



Calculating Derivatives from the Definition

(b) The slope of the curve at $x = 4$ is

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

The tangent is the line through the point $(4, 2)$ with slope $1/4$ (Figure 3).

$$y = 2 + \frac{1}{4}(x - 4)$$

$$y = \frac{1}{4}x + 1.$$

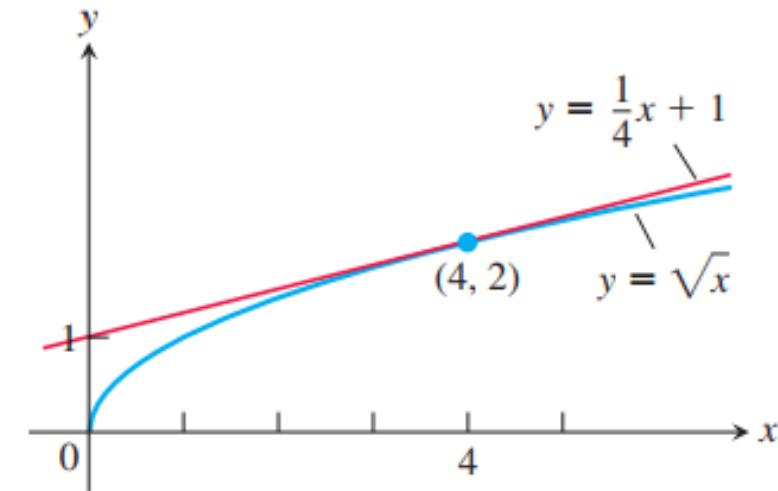


FIGURE 3.5 The curve $y = \sqrt{x}$ and its tangent at $(4, 2)$. The tangent's slope is found by evaluating the derivative at $x = 4$ (Example 2).



Calculating Derivatives from the Definition

To indicate the value of a derivative at a specified number $x = a$, we use the notation

$$f'(a) = \frac{dy}{dx} \Big|_{x=a} = \frac{df}{dx} \Big|_{x=a} = \frac{d}{dx} f(x) \Big|_{x=a}.$$

For instance, in Example 2

$$f'(4) = \frac{d}{dx} \sqrt{x} \Big|_{x=4} = \frac{1}{2\sqrt{x}} \Big|_{x=4} = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$



A function $y = f(x)$ is **differentiable on an open interval** (finite or infinite) if it has a derivative at each point of the interval. It is **differentiable on a closed interval** $[a, b]$ if it is differentiable on the interior (a, b) and if the limits

$$\lim_{h \rightarrow 0^+} \frac{f(a + h) - f(a)}{h}$$

Right-hand derivative at a

$$\lim_{h \rightarrow 0^-} \frac{f(b + h) - f(b)}{h}$$

Left-hand derivative at b

exist at the endpoints

EXAMPLE 4 Show that the function $y = |x|$ is differentiable on $(-\infty, 0)$ and $(0, \infty)$ but has no derivative at $x = 0$.

Solution From Section 3.1, the derivative of $y = mx + b$ is the slope m . Thus, to the right of the origin,

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(x) = \frac{d}{dx}(1 \cdot x) = 1. \quad \frac{d}{dx}(mx + b) = m, |x| = x$$



Differentiable on an Interval; One-Sided Derivatives

To the left,

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(-x) = \frac{d}{dx}(-1 \cdot x) = -1 \quad |x| = -x$$

(Figure 3.8). There is no derivative at the origin because the one-sided derivatives differ there:

$$\begin{aligned}\text{Right-hand derivative of } |x| \text{ at zero} &= \lim_{h \rightarrow 0^+} \frac{|0 + h| - |0|}{h} = \lim_{h \rightarrow 0^+} \frac{|h|}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{h}{h} \quad |h| = h \text{ when } h > 0 \\ &= \lim_{h \rightarrow 0^+} 1 = 1\end{aligned}$$

$$\begin{aligned}\text{Left-hand derivative of } |x| \text{ at zero} &= \lim_{h \rightarrow 0^-} \frac{|0 + h| - |0|}{h} = \lim_{h \rightarrow 0^-} \frac{|h|}{h} \\ &= \lim_{h \rightarrow 0^-} \frac{-h}{h} \quad |h| = -h \text{ when } h < 0 \\ &= \lim_{h \rightarrow 0^-} -1 = -1.\end{aligned}$$





Differentiable on an Interval; One-Sided Derivatives

EXAMPLE 5 In Example 2 we found that for $x > 0$,

$$\frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}.$$

We apply the definition to examine if the derivative exists at $x = 0$:

$$\lim_{h \rightarrow 0^+} \frac{\sqrt{0+h} - \sqrt{0}}{h} = \lim_{h \rightarrow 0^+} \frac{1}{\sqrt{h}} = \infty.$$

Since the (right-hand) limit is not finite, there is no derivative at $x = 0$. Since the slopes of the secant lines joining the origin to the points (h, \sqrt{h}) on a graph of $y = \sqrt{x}$ approach ∞ , the graph has a *vertical tangent* at the origin. (See Figure 1.17 on page 9.) ■



Differentiable Functions Are Continuous

THEOREM 1—Differentiability Implies Continuity If f has a derivative at $x = c$, then f is continuous at $x = c$.

Caution The converse of Theorem 1 is false. A function need not have a derivative at a point where it is continuous, as we saw with the absolute value function in Example 4.



Powers, Multiples, Sums, and Differences

Derivative of a Constant Function

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

Derivative of a Positive Integer Power

If n is a positive integer, then

$$\frac{d}{dx}x^n = nx^{n-1}.$$



Differentiation Rules

Power Rule (General Version)

If n is any real number, then

$$\frac{d}{dx}x^n = nx^{n-1},$$

for all x where the powers x^n and x^{n-1} are defined.

EXAMPLE 1 Differentiate the following powers of x .

- (a) x^3
- (b) $x^{2/3}$
- (c) $x^{\sqrt{2}}$
- (d) $\frac{1}{x^4}$
- (e) $x^{-4/3}$
- (f) $\sqrt{x^{2+\pi}}$



Differentiation Rules

Solution

$$(a) \frac{d}{dx}(x^3) = 3x^{3-1} = 3x^2$$

$$(b) \frac{d}{dx}(x^{2/3}) = \frac{2}{3}x^{(2/3)-1} = \frac{2}{3}x^{-1/3}$$

$$(c) \frac{d}{dx}(x^{\sqrt{2}}) = \sqrt{2}x^{\sqrt{2}-1}$$

$$(d) \frac{d}{dx}\left(\frac{1}{x^4}\right) = \frac{d}{dx}(x^{-4}) = -4x^{-4-1} = -4x^{-5} = -\frac{4}{x^5}$$

$$(e) \frac{d}{dx}(x^{-4/3}) = -\frac{4}{3}x^{-(4/3)-1} = -\frac{4}{3}x^{-7/3}$$

$$(f) \frac{d}{dx}(\sqrt{x^{2+\pi}}) = \frac{d}{dx}(x^{1+(\pi/2)}) = \left(1 + \frac{\pi}{2}\right)x^{1+(\pi/2)-1} = \frac{1}{2}(2 + \pi)\sqrt{x^\pi}$$

■



Derivative Constant Multiple Rule

If u is a differentiable function of x , and c is a constant, then

$$\frac{d}{dx}(cu) = c \frac{du}{dx}.$$

EXAMPLE 2

- (a) The derivative formula

$$\frac{d}{dx}(3x^2) = 3 \cdot 2x = 6x$$

$$\frac{d}{dx}(-u) = \frac{d}{dx}(-1 \cdot u) = -1 \cdot \frac{d}{dx}(u) = -\frac{du}{dx}.$$



Derivative Sum Rule

If u and v are differentiable functions of x , then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

For example, if $y = x^4 + 12x$, then y is the sum of $u(x) = x^4$ and $v(x) = 12x$. We then have

$$\frac{dy}{dx} = \frac{d}{dx}(x^4) + \frac{d}{dx}(12x) = 4x^3 + 12.$$



Differentiation Rules

EXAMPLE 3 Find the derivative of the polynomial $y = x^3 + \frac{4}{3}x^2 - 5x + 1$.

Solution $\frac{dy}{dx} = \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1)$ Sum and Difference Rules

$$= 3x^2 + \frac{4}{3} \cdot 2x - 5 + 0 = 3x^2 + \frac{8}{3}x - 5$$



Derivative Product Rule

If u and v are differentiable at x , then so is their product uv , and

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}.$$



Differentiation Rules

EXAMPLE 7 Find the derivative of $y = (x^2 + 1)(x^3 + 3)$.

Solution

(a) From the Product Rule with $u = x^2 + 1$ and $v = x^3 + 3$, we find

$$\begin{aligned}\frac{d}{dx}[(x^2 + 1)(x^3 + 3)] &= (x^2 + 1)(3x^2) + (x^3 + 3)(2x) \quad \frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx} \\ &= 3x^4 + 3x^2 + 2x^4 + 6x \\ &= 5x^4 + 3x^2 + 6x.\end{aligned}$$

(b) This particular product can be differentiated as well (perhaps better) by multiplying out the original expression for y and differentiating the resulting polynomial:

$$y = (x^2 + 1)(x^3 + 3) = x^5 + x^3 + 3x^2 + 3$$

$$\frac{dy}{dx} = 5x^4 + 3x^2 + 6x.$$

This is in agreement with our first calculation. ■



Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x , and

$$\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}.$$

In function notation,

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$



Differentiation Rules

EXAMPLE 8 Find the derivative of (a) $y = \frac{t^2 - 1}{t^3 + 1}$,

Solution

(a) We apply the Quotient Rule with $u = t^2 - 1$ and $v = t^3 + 1$:

$$\begin{aligned}\frac{dy}{dt} &= \frac{(t^3 + 1) \cdot 2t - (t^2 - 1) \cdot 3t^2}{(t^3 + 1)^2} & \frac{d}{dt}\left(\frac{u}{v}\right) &= \frac{v(du/dt) - u(dv/dt)}{v^2} \\ &= \frac{2t^4 + 2t - 3t^4 + 3t^2}{(t^3 + 1)^2} \\ &= \frac{-t^4 + 3t^2 + 2t}{(t^3 + 1)^2}.\end{aligned}$$

Differentiation Rules

EXAMPLE 9 Find the derivative of

$$y = \frac{(x - 1)(x^2 - 2x)}{x^4}.$$

Solution Using the Quotient Rule here will result in a complicated expression with many terms. Instead, use some algebra to simplify the expression. First expand the numerator and divide by x^4 :

$$y = \frac{(x - 1)(x^2 - 2x)}{x^4} = \frac{x^3 - 3x^2 + 2x}{x^4} = x^{-1} - 3x^{-2} + 2x^{-3}.$$

Then use the Sum and Power Rules:

$$\begin{aligned}\frac{dy}{dx} &= -x^{-2} - 3(-2)x^{-3} + 2(-3)x^{-4} \\ &= -\frac{1}{x^2} + \frac{6}{x^3} - \frac{6}{x^4}.\end{aligned}$$





Second- and Higher-Order Derivative

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y'' = D^2(f)(x) = D_x^2 f(x).$$

Example

The symbol D^2 means the operation of differentiation is performed twice.

If $y = x^6$, then $y' = 6x^5$ and we have

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}(6x^5) = 30x^4.$$

Thus $D^2(x^6) = 30x^4$.



Second- and Higher-Order Derivative

If y'' is differentiable, its derivative, $y''' = dy''/dx = d^3y/dx^3$, is the **third derivative** of y with respect to x . The names continue as you imagine, with

$$y^{(n)} = \frac{d}{dx}y^{(n-1)} = \frac{d^n y}{dx^n} = D^n y$$

denoting the **n th derivative** of y with respect to x for any positive integer n .

EXAMPLE 10 The first four derivatives of $y = x^3 - 3x^2 + 2$ are

First derivative: $y' = 3x^2 - 6x$

Second derivative: $y'' = 6x - 6$

Third derivative: $y''' = 6$

Fourth derivative: $y^{(4)} = 0.$

All polynomial functions have derivatives of all orders. In this example, the fifth and later derivatives are all zero. ■



Derivatives of Trigonometric Functions

Derivative of the Sine Function

To calculate the derivative of $f(x) = \sin x$, for x measured in radians, we combine the limits in Example 5a and Theorem 7 in Section 2.4 with the angle sum identity for the sine function:

$$\sin(x + h) = \sin x \cos h + \cos x \sin h.$$

If $f(x) = \sin x$, then

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sin(x + h) - \sin x}{h} && \text{Derivative definition} \\ &= \lim_{h \rightarrow 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\ &= \lim_{h \rightarrow 0} \left(\sin x \cdot \frac{\cos h - 1}{h} \right) + \lim_{h \rightarrow 0} \left(\cos x \cdot \frac{\sin h}{h} \right) \\ &= \sin x \cdot \underbrace{\lim_{h \rightarrow 0} \frac{\cos h - 1}{h}}_{\text{limit 0}} + \cos x \cdot \underbrace{\lim_{h \rightarrow 0} \frac{\sin h}{h}}_{\text{limit 1}} = \sin x \cdot 0 + \cos x \cdot 1 = \cos x. \end{aligned}$$

Example 5a and
Theorem 7, Section 2.4



Derivatives of Trigonometric Functions

The derivative of the sine function is the cosine function:

$$\frac{d}{dx}(\sin x) = \cos x.$$

EXAMPLE 1

We find derivatives of the sine function involving differences, products, and quotients.

(a) $y = x^2 - \sin x$: $\frac{dy}{dx} = 2x - \frac{d}{dx}(\sin x)$ Difference Rule
 $= 2x - \cos x$

(c) $y = \frac{\sin x}{x}$:
$$\frac{dy}{dx} = \frac{x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot 1}{x^2}$$
 Quotient Rule
 $= \frac{x \cos x - \sin x}{x^2}$



Derivatives of Trigonometric Functions

Derivative of the Cosine Function

With the help of the angle sum formula for the cosine function,

$$\cos(x + h) = \cos x \cos h - \sin x \sin h,$$

we can compute the limit of the difference quotient:

$$\begin{aligned} \frac{d}{dx}(\cos x) &= \lim_{h \rightarrow 0} \frac{\cos(x + h) - \cos x}{h} \\ &= \lim_{h \rightarrow 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h} \\ &= \lim_{h \rightarrow 0} \cos x \cdot \frac{\cos h - 1}{h} - \lim_{h \rightarrow 0} \sin x \cdot \frac{\sin h}{h} \\ &= \cos x \cdot \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} - \sin x \cdot \lim_{h \rightarrow 0} \frac{\sin h}{h} \\ &= \cos x \cdot 0 - \sin x \cdot 1 \\ &= -\sin x. \end{aligned}$$

Derivative definition

Cosine angle sum
identity

Example 5a and
Theorem 7, Section 2.4



Derivatives of Trigonometric Functions

The derivative of the cosine function is the negative of the sine function:

$$\frac{d}{dx} (\cos x) = -\sin x.$$

EXAMPLE 2

We find derivatives

(b) $y = \sin x \cos x$:

$$\begin{aligned}\frac{dy}{dx} &= \sin x \frac{d}{dx} (\cos x) + \cos x \frac{d}{dx} (\sin x) && \text{Product Rule} \\ &= \sin x (-\sin x) + \cos x (\cos x) \\ &= \cos^2 x - \sin^2 x\end{aligned}$$



Derivatives of Trigonometric Functions

(c) $y = \frac{\cos x}{1 - \sin x}$:

$$\frac{dy}{dx} = \frac{(1 - \sin x) \frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2}$$

Quotient Rule

$$= \frac{(1 - \sin x)(-\sin x) - \cos x(0 - \cos x)}{(1 - \sin x)^2}$$

$$= \frac{1 - \sin x}{(1 - \sin x)^2}$$

$$\sin^2 x + \cos^2 x = 1$$

$$= \frac{1}{1 - \sin x}$$



Derivatives of Trigonometric Functions

EXAMPLE

Find $d(\tan x)/dx$.

Solution We use the Derivative Quotient Rule to calculate the derivative:

$$\begin{aligned}\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x} && \text{Quotient Rule} \\ &= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} = \sec^2 x.\end{aligned}$$





Derivatives of Trigonometric Functions

EXAMPLE 6 Find y'' if $y = \sec x$.

Solution Finding the second derivative involves a combination of trigonometric derivatives.

$$y = \sec x$$

$$y' = \sec x \tan x$$

Derivative rule for secant function

$$y'' = \frac{d}{dx}(\sec x \tan x)$$

$$= \sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x) \quad \text{Derivative Product Rule}$$

$$= \sec x (\sec^2 x) + \tan x (\sec x \tan x) \quad \text{Derivative rules}$$

$$= \sec^3 x + \sec x \tan^2 x$$





Derivatives of Trigonometric Functions

The derivatives of the other trigonometric functions:

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

Exercises

Testing for Tangents

35. Does the graph of

$$f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

have a tangent at the origin? Give reasons for your answer.

36. Does the graph of

$$g(x) = \begin{cases} x \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

have a tangent at the origin? Give reasons for your answer.

Solution

35. Slope at origin = $\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{h^2 \sin\left(\frac{1}{h}\right)}{h} = \lim_{h \rightarrow 0} h \sin\left(\frac{1}{h}\right) = 0 \Rightarrow$ yes, $f(x)$ does have a tangent at the origin with slope 0.
36. $\lim_{h \rightarrow 0} \frac{g(0+h) - g(0)}{h} = \lim_{h \rightarrow 0} \frac{h \sin\left(\frac{1}{h}\right)}{h} = \lim_{h \rightarrow 0} \sin \frac{1}{h}$. Since $\lim_{h \rightarrow 0} \sin \frac{1}{h}$ does not exist, $f(x)$ has no tangent at the origin.

Exercises

37. Does the graph of

$$f(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

have a vertical tangent at the origin? Give reasons for your answer.

38. Does the graph of

$$U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

Solution

37. $\lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^-} \frac{-1-0}{h} = \infty$, and $\lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^+} \frac{1-0}{h} = \infty$. Therefore, $\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \infty$
 \Rightarrow yes, the graph of f has a vertical tangent at the origin.
38. $\lim_{h \rightarrow 0^-} \frac{U(0+h) - U(0)}{h} = \lim_{h \rightarrow 0^-} \frac{0-1}{h} = \infty$, and $\lim_{h \rightarrow 0^+} \frac{U(0+h) - U(0)}{h} = \lim_{h \rightarrow 0^+} \frac{1-1}{h} = 0 \Rightarrow$ no, the graph of f does not have a vertical tangent at $(0, 1)$ because the limit does not exist.

Exercises

Finding Derivative Functions and Values

Using the definition, calculate the derivatives of the functions in Exercises 1–6. Then find the values of the derivatives as specified.

1. $f(x) = 4 - x^2$; $f'(-3), f'(0), f'(1)$

2. $F(x) = (x - 1)^2 + 1$; $F'(-1), F'(0), F'(2)$

3. $g(t) = \frac{1}{t^2}$; $g'(-1), g'(2), g'(\sqrt{3})$

4. $k(z) = \frac{1 - z}{2z}$; $k'(-1), k'(1), k'(\sqrt{2})$

5. $p(\theta) = \sqrt{3\theta}$; $p'(1), p'(3), p'(2/3)$

6. $r(s) = \sqrt{2s + 1}$; $r'(0), r'(1), r'(1/2)$



1. Step 1: $f(x) = 4 - x^2$ and $f(x+h) = 4 - (x+h)^2$

Step 2: $\frac{f(x+h)-f(x)}{h} = \frac{[4-(x+h)^2] - (4-x^2)}{h} = \frac{(4-x^2-2xh-h^2) - 4+x^2}{h} = \frac{-2xh-h^2}{h} = \frac{h(-2x-h)}{h} = -2x-h$

Step 3: $f'(x) = \lim_{h \rightarrow 0} (-2x-h) = -2x; f'(-3) = 6, f'(0) = 0, f'(1) = -2$

2. $F(x) = (x-1)^2 + 1$ and $F(x+h) = (x+h-1)^2 + 1 \Rightarrow F'(x) = \lim_{h \rightarrow 0} \frac{[(x+h-1)^2+1] - [(x-1)^2+1]}{h}$

$$= \lim_{h \rightarrow 0} \frac{(x^2+2xh+h^2-2x-2h+1+1)-(x^2-2x+1+1)}{h} = \lim_{h \rightarrow 0} \frac{2xh+h^2-2h}{h} = \lim_{h \rightarrow 0} (2x+h-2) = 2(x-1);$$

$F'(-1) = -4, F'(0) = -2, F'(2) = 2$

3. Step 1: $g(t) = \frac{1}{t^2}$ and $g(t+h) = \frac{1}{(t+h)^2}$

Step 2: $\frac{g(t+h)-g(t)}{h} = \frac{\frac{1}{(t+h)^2} - \frac{1}{t^2}}{h} = \frac{\left(\frac{t^2-(t+h)^2}{(t+h)^2 \cdot t^2}\right)}{h} = \frac{t^2 - (t^2 + 2th + h^2)}{(t+h)^2 \cdot t^2 \cdot h} = \frac{-2th-h^2}{(t+h)^2 t^2 h} = \frac{h(-2t-h)}{(t+h)^2 t^2 h} = \frac{-2t-h}{(t+h)^2 t^2}$

Step 3: $g'(t) = \lim_{h \rightarrow 0} \frac{-2t-h}{(t+h)^2 t^2} = \frac{-2t}{t^2 \cdot t^2} = \frac{-2}{t^3}; g'(-1) = 2, g'(2) = -\frac{1}{4}, g'\left(\sqrt{3}\right) = -\frac{2}{3\sqrt{3}}$

4. $k(z) = \frac{1-z}{2z}$ and $k(z+h) = \frac{1-(z+h)}{2(z+h)} \Rightarrow k'(z) = \lim_{h \rightarrow 0} \frac{\left(\frac{1-(z+h)}{2(z+h)} - \frac{1-z}{2z}\right)}{h} = \lim_{h \rightarrow 0} \frac{(1-z-h)z - (1-z)(z+h)}{2(z+h)zh} = \lim_{h \rightarrow 0} \frac{z-z^2-zh-z-h+z^2+zh}{2(z+h)zh}$

$$= \lim_{h \rightarrow 0} \frac{-h}{2(z+h)zh} = \lim_{h \rightarrow 0} \frac{-1}{2(z+h)z} = \frac{-1}{2z^2}; k'(-1) = -\frac{1}{2}, k'(1) = -\frac{1}{2}, k'\left(\sqrt{2}\right) = -\frac{1}{4}$$

5. Step 1: $p(\theta) = \sqrt{3\theta}$ and $p(\theta + h) = \sqrt{3(\theta + h)}$

Step 2: $\frac{p(\theta+h)-p(\theta)}{h} = \frac{\sqrt{3(\theta+h)}-\sqrt{3\theta}}{h} = \frac{(\sqrt{3\theta+3h}-\sqrt{3\theta})}{h} \cdot \frac{(\sqrt{3\theta+3h}+\sqrt{3\theta})}{(\sqrt{3\theta+3h}+\sqrt{3\theta})} = \frac{(3\theta+3h)-3\theta}{h(\sqrt{3\theta+3h}+\sqrt{3\theta})}$
 $= \frac{3h}{h(\sqrt{3\theta+3h}+\sqrt{3\theta})} = \frac{3}{\sqrt{3\theta+3h}+\sqrt{3\theta}}$

Step 3: $p'(\theta) = \lim_{h \rightarrow 0} \frac{3}{\sqrt{3\theta+3h}+\sqrt{3\theta}} = \frac{3}{\sqrt{3\theta}+\sqrt{3\theta}} = \frac{3}{2\sqrt{3\theta}}$; $p'(1) = \frac{3}{2\sqrt{3}}$, $p'(3) = \frac{1}{2}$, $p'\left(\frac{2}{3}\right) = \frac{3}{2\sqrt{2}}$

6. $r(s) = \sqrt{2s+1}$ and $r(s+h) = \sqrt{2(s+h)+1} \Rightarrow r'(s) = \lim_{h \rightarrow 0} \frac{\sqrt{2s+2h+1}-\sqrt{2s+1}}{h}$

$$= \lim_{h \rightarrow 0} \frac{(\sqrt{2s+2h+1}-\sqrt{2s+1})}{h} \cdot \frac{(\sqrt{2s+2h+1}+\sqrt{2s+1})}{(\sqrt{2s+2h+1}+\sqrt{2s+1})} = \lim_{h \rightarrow 0} \frac{(2s+2h+1)-(2s+1)}{h(\sqrt{2s+2h+1}+\sqrt{2s+1})}$$

$$= \lim_{h \rightarrow 0} \frac{2h}{h(\sqrt{2s+2h+1}+\sqrt{2s+1})} = \lim_{h \rightarrow 0} \frac{2}{\sqrt{2s+2h+1}+\sqrt{2s+1}} = \frac{2}{\sqrt{2s+1}+\sqrt{2s+1}} = \frac{2}{2\sqrt{2s+1}} = \frac{1}{\sqrt{2s+1}};$$

$$r'(0) = 1, r'(1) = \frac{1}{\sqrt{3}}, r'\left(\frac{1}{2}\right) = \frac{1}{\sqrt{2}}$$

Exercises

Using the Alternative Formula for Derivatives

Use the formula

$$f'(x) = \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x}$$

to find the derivative of the functions in Exercises 23–26.

$$23. \quad f(x) = \frac{1}{x + 2}$$

$$24. \quad f(x) = x^2 - 3x + 4$$

$$25. \quad g(x) = \frac{x}{x - 1}$$

$$26. \quad g(x) = 1 + \sqrt{x}$$

Solution

$$23. \quad f'(x) = \lim_{z \rightarrow x} \frac{f(z)-f(x)}{z-x} = \lim_{z \rightarrow x} \frac{\frac{1}{z+2}-\frac{1}{x+2}}{z-x} = \lim_{z \rightarrow x} \frac{(x+2)-(z+2)}{(z-x)(z+2)(x+2)} = \lim_{z \rightarrow x} \frac{x-z}{(z-x)(z+2)(x+2)} = \lim_{z \rightarrow x} \frac{-1}{(z+2)(x+2)} = \frac{-1}{(x+2)^2}$$

$$24. \quad f'(x) = \lim_{z \rightarrow x} \frac{f(z)-f(x)}{z-x} = \lim_{z \rightarrow x} \frac{(z^2-3z+4)-(x^2-3x+4)}{z-x} = \lim_{z \rightarrow x} \frac{z^2-3z-x^2+3x}{z-x} = \lim_{z \rightarrow x} \frac{z^2-x^2-3z+3x}{z-x} = \lim_{z \rightarrow x} \frac{(z-x)(z+x)-3(z-x)}{z-x} \\ = \lim_{z \rightarrow x} \frac{(z-x)[(z+x)-3]}{z-x} = \lim_{z \rightarrow x} [(z+x)-3] = 2x-3$$

$$25. \quad g'(x) = \lim_{z \rightarrow x} \frac{g(z)-g(x)}{z-x} = \lim_{z \rightarrow x} \frac{\frac{z}{z-1}-\frac{x}{x-1}}{z-x} = \lim_{z \rightarrow x} \frac{z(x-1)-x(z-1)}{(z-x)(z-1)(x-1)} = \lim_{z \rightarrow x} \frac{-z+x}{(z-x)(z-1)(x-1)} = \lim_{z \rightarrow x} \frac{-1}{(z-1)(x-1)} = \frac{-1}{(x-1)^2}$$

$$26. \quad g'(x) = \lim_{z \rightarrow x} \frac{g(z)-g(x)}{z-x} = \lim_{z \rightarrow x} \frac{(1+\sqrt{z})-(1+\sqrt{x})}{z-x} = \lim_{z \rightarrow x} \frac{\sqrt{z}-\sqrt{x}}{z-x} \cdot \frac{\sqrt{z}+\sqrt{x}}{\sqrt{z}+\sqrt{x}} = \lim_{z \rightarrow x} \frac{z-x}{(z-x)(\sqrt{z}+\sqrt{x})} = \lim_{z \rightarrow x} \frac{1}{\sqrt{z}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Exercises

In Exercises 41 and 42, determine if the piecewise-defined function is differentiable at the origin.

$$41. \ f(x) = \begin{cases} 2x - 1, & x \geq 0 \\ x^2 + 2x + 7, & x < 0 \end{cases}$$

$$42. \ g(x) = \begin{cases} x^{2/3}, & x \geq 0 \\ x^{1/3}, & x < 0 \end{cases}$$

Solution

41. f is not continuous at $x = 0$ since $\lim_{x \rightarrow 0} f(x) =$ does not exist and $f(0) = -1$

42. Left-hand derivative: $\lim_{h \rightarrow 0^-} \frac{g(h)-g(0)}{h} = \lim_{h \rightarrow 0^-} \frac{h^{1/3}-0}{h} = \lim_{h \rightarrow 0^-} \frac{1}{h^{2/3}} = +\infty;$

Right-hand derivative: $\lim_{h \rightarrow 0^+} \frac{g(h)-g(0)}{h} = \lim_{h \rightarrow 0^+} \frac{h^{2/3}-0}{h} = \lim_{h \rightarrow 0^+} \frac{1}{h^{1/3}} = +\infty;$

Then $\lim_{h \rightarrow 0^-} \frac{g(h)-g(0)}{h} = \lim_{h \rightarrow 0^+} \frac{g(h)-g(0)}{h} = +\infty \Rightarrow$ the derivative $g'(0)$ does not exist.

Exercises

Derivative Calculations

In Exercises 1–12, find the first and second derivatives.

$$1. \ y = -x^2 + 3$$

$$2. \ y = x^2 + x + 8$$

$$3. \ s = 5t^3 - 3t^5$$

$$4. \ w = 3z^7 - 7z^3 + 21z^2$$

$$7. \ w = 3z^{-2} - \frac{1}{z}$$

$$8. \ s = -2t^{-1} + \frac{4}{t^2}$$

$$9. \ y = 6x^2 - 10x - 5x^{-2}$$

$$10. \ y = 4 - 2x - x^{-3}$$

$$11. \ r = \frac{1}{3s^2} - \frac{5}{2s}$$

$$12. \ r = \frac{12}{\theta} - \frac{4}{\theta^3} + \frac{1}{\theta^4}$$

$$1. \quad y = -x^2 + 3 \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(-x^2) + \frac{d}{dx}(3) = -2x + 0 = -2x \Rightarrow \frac{d^2y}{dx^2} = -2$$

$$2. \quad y = x^2 + x + 8 \Rightarrow \frac{dy}{dx} = 2x + 1 + 0 = 2x + 1 \Rightarrow \frac{d^2y}{dx^2} = 2$$

$$3. \quad s = 5t^3 - 3t^5 \Rightarrow \frac{ds}{dt} = \frac{d}{dt}(5t^3) - \frac{d}{dt}(3t^5) = 15t^2 - 15t^4 \Rightarrow \frac{d^2s}{dt^2} = \frac{d}{dt}(15t^2) - \frac{d}{dt}(15t^4) = 30t - 60t^3$$

$$4. \quad w = 3z^7 - 7z^3 + 21z^2 \Rightarrow \frac{dw}{dz} = 21z^6 - 21z^2 + 42z \Rightarrow \frac{d^2w}{dz^2} = 126z^5 - 42z + 42$$

$$7. \quad w = 3z^{-2} - z^{-1} \Rightarrow \frac{dw}{dz} = -6z^{-3} + z^{-2} = \frac{-6}{z^3} + \frac{1}{z^2} \Rightarrow \frac{d^2w}{dz^2} = 18z^{-4} - 2z^{-3} = \frac{18}{z^4} - \frac{2}{z^3}$$

$$8. \quad s = -2t^{-1} + 4t^{-2} \Rightarrow \frac{ds}{dt} = 2t^{-2} - 8t^{-3} = \frac{2}{t^2} - \frac{8}{t^3} \Rightarrow \frac{d^2s}{dt^2} = -4t^{-3} + 24t^{-4} = \frac{-4}{t^3} + \frac{24}{t^4}$$

$$9. \quad y = 6x^2 - 10x - 5x^{-2} \Rightarrow \frac{dy}{dx} = 12x - 10 + 10x^{-3} = 12x - 10 + \frac{10}{x^3} \Rightarrow \frac{d^2y}{dx^2} = 12 - 0 - 30x^{-4} = 12 - \frac{30}{x^4}$$

$$10. \quad y = 4 - 2x - x^{-3} \Rightarrow \frac{dy}{dx} = -2 + 3x^{-4} = -2 + \frac{3}{x^4} \Rightarrow \frac{d^2y}{dx^2} = 0 - 12x^{-5} = \frac{-12}{x^5}$$

$$11. \quad r = \frac{1}{3}s^{-2} - \frac{5}{2}s^{-1} \Rightarrow \frac{dr}{ds} = -\frac{2}{3}s^{-3} + \frac{5}{2}s^{-2} = \frac{-2}{3s^3} + \frac{5}{2s^2} \Rightarrow \frac{d^2r}{ds^2} = 2s^{-4} - 5s^{-3} = \frac{2}{s^4} - \frac{5}{s^3}$$

$$12. \quad r = 12\theta^{-1} - 4\theta^{-3} + \theta^{-4} \Rightarrow \frac{dr}{d\theta} = -12\theta^{-2} + 12\theta^{-4} - 4\theta^{-5} = \frac{-12}{\theta^2} + \frac{12}{\theta^4} - \frac{4}{\theta^5} \Rightarrow \frac{d^2r}{d\theta^2} = 24\theta^{-3} - 48\theta^{-5} + 20\theta^{-6}$$

$$= \frac{24}{\theta^3} - \frac{48}{\theta^5} + \frac{20}{\theta^6}$$

Find the derivatives of the functions in Exercises 17–40.

$$17. \ y = \frac{2x + 5}{3x - 2}$$

$$18. \ z = \frac{4 - 3x}{3x^2 + x}$$

$$19. \ g(x) = \frac{x^2 - 4}{x + 0.5}$$

$$20. \ f(t) = \frac{t^2 - 1}{t^2 + t - 2}$$

$$21. \ v = (1 - t)(1 + t^2)^{-1}$$

$$22. \ w = (2x - 7)^{-1}(x + 5)$$

$$23. \ f(s) = \frac{\sqrt{s} - 1}{\sqrt{s} + 1}$$

$$24. \ u = \frac{5x + 1}{2\sqrt{x}}$$

$$25. \ v = \frac{1 + x - 4\sqrt{x}}{x}$$

$$26. \ r = 2\left(\frac{1}{\sqrt{\theta}} + \sqrt{\theta}\right)$$

$$27. \ y = \frac{1}{(x^2 - 1)(x^2 + x + 1)} \quad 28. \ y = \frac{(x + 1)(x + 2)}{(x - 1)(x - 2)}$$

17. $y = \frac{2x+5}{3x-2}$; use the quotient rule: $u = 2x + 5$ and $v = 3x - 2 \Rightarrow u' = 2$ and $v' = 3 \Rightarrow y' = \frac{vu' - uv'}{v^2} = \frac{(3x-2)(2) - (2x+5)(3)}{(3x-2)^2}$

$$= \frac{6x-4-6x-15}{(3x-2)^2} = \frac{-19}{(3x-2)^2}$$

18. $y = \frac{4-3x}{3x^2+x}$; use the quotient rule: $u = 4 - 3x$ and $v = 3x^2 + x \Rightarrow u' = -3$ and $v' = 6x + 1 \Rightarrow y' = \frac{vu' - uv'}{v^2}$

$$= \frac{(3x^2+x)(-3) - (4-3x)(6x+1)}{(3x^2+x)^2} = \frac{-9x^2 - 3x + 18x^2 + 21x - 4}{(3x^2+x)^2} = \frac{9x^2 + 18x - 4}{(3x^2+x)^2}$$

19. $g(x) = \frac{x^2-4}{x+0.5}$; use the quotient rule: $u = x^2 - 4$ and $v = x + 0.5 \Rightarrow u' = 2x$ and $v' = 1 \Rightarrow g'(x) = \frac{vu' - uv'}{v^2}$

$$= \frac{(x+0.5)(2x) - (x^2-4)(1)}{(x+0.5)^2} = \frac{2x^2 + x - x^2 + 4}{(x+0.5)^2} = \frac{x^2 + x + 4}{(x+0.5)^2}$$

$$20. \quad f(t) = \frac{t^2 - 1}{t^2 + t - 2} = \frac{(t-1)(t+1)}{(t+2)(t-1)} = \frac{t+1}{t+2}, \quad t \neq 1 \Rightarrow f'(t) = \frac{(t+2)(1)-(t+1)(1)}{(t+2)^2} = \frac{t+2-t-1}{(t+2)^2} = \frac{1}{(t+2)^2}$$

$$21. \quad v = (1-t)(1+t^2)^{-1} = \frac{1-t}{1+t^2} \Rightarrow \frac{dv}{dt} = \frac{(1+t^2)(-1)-(1-t)(2t)}{(1+t^2)^2} = \frac{-1-t^2-2t+2t^2}{(1+t^2)^2} = \frac{t^2-2t-1}{(1+t^2)^2}$$

$$22. \quad w = \frac{x+5}{2x-7} \Rightarrow w' = \frac{(2x-7)(1)-(x+5)(2)}{(2x-7)^2} = \frac{2x-7-2x-10}{(2x-7)^2} = \frac{-17}{(2x-7)^2}$$

$$23. \quad f(s) = \frac{\sqrt{s}-1}{\sqrt{s}+1} \Rightarrow f'(s) = \frac{(\sqrt{s}+1)\left(\frac{1}{2\sqrt{s}}\right) - (\sqrt{s}-1)\left(\frac{1}{2\sqrt{s}}\right)}{(\sqrt{s}+1)^2} = \frac{(\sqrt{s}+1) - (\sqrt{s}-1)}{2\sqrt{s}(\sqrt{s}+1)^2} = \frac{1}{\sqrt{s}(\sqrt{s}+1)^2}$$

NOTE: $\frac{d}{ds}(\sqrt{s}) = \frac{1}{2\sqrt{s}}$ from Example 2 in Section 3.2

$$24. \quad u = \frac{5x+1}{2\sqrt{x}} \Rightarrow \frac{du}{dx} = \frac{(2\sqrt{x})(5)-(5x+1)\left(\frac{1}{\sqrt{x}}\right)}{4x} = \frac{5x-1}{4x^{3/2}}$$

$$25. \ v = \frac{1+x-4\sqrt{x}}{x} \Rightarrow v' = \frac{x\left(1-\frac{2}{\sqrt{x}}\right)-(1+x-4\sqrt{x})}{x^2} = \frac{2\sqrt{x}-1}{x^2}$$

$$26. \ r = 2\left(\frac{1}{\sqrt{\theta}} + \sqrt{\theta}\right) \Rightarrow r' = 2\left(\frac{\sqrt{\theta}(0)-1\left(\frac{1}{2\sqrt{\theta}}\right)}{\theta} + \frac{1}{2\sqrt{\theta}}\right) = -\frac{1}{\theta^{3/2}} + \frac{1}{\theta^{1/2}}$$

$$27. \ y = \frac{1}{(x^2-1)(x^2+x+1)}; \text{ use the quotient rule: } u = 1 \text{ and } v = (x^2-1)(x^2+x+1) \Rightarrow u' = 0 \text{ and}$$

$$v' = (x^2-1)(2x+1) + (x^2+x+1)(2x) = 2x^3 + x^2 - 2x - 1 + 2x^3 + 2x^2 + 2x = 4x^3 + 3x^2 - 1 \Rightarrow \frac{dy}{dx} = \frac{vu' - uv'}{v^2}$$

$$= \frac{0 - 1(4x^3 + 3x^2 - 1)}{(x^2-1)^2(x^2+x+1)^2} = \frac{-4x^3 - 3x^2 + 1}{(x^2-1)^2(x^2+x+1)^2}$$

$$28. \ y = \frac{(x+1)(x+2)}{(x-1)(x-2)} = \frac{x^2+3x+2}{x^2-3x+2} \Rightarrow y' = \frac{(x^2-3x+2)(2x+3) - (x^2+3x+2)(2x-3)}{(x-1)^2(x-2)^2} = \frac{-6x^2+12}{(x-1)^2(x-2)^2} = \frac{-6(x^2-2)}{(x-1)^2(x-2)^2}$$

Find the first and second derivatives of the functions in Exercises 45–52.

$$45. \ y = \frac{x^3 + 7}{x}$$

$$46. \ s = \frac{t^2 + 5t - 1}{t^2}$$

$$47. \ r = \frac{(\theta - 1)(\theta^2 + \theta + 1)}{\theta^3}$$

$$48. \ u = \frac{(x^2 + x)(x^2 - x + 1)}{x^4}$$

$$49. \ w = \left(\frac{1 + 3z}{3z} \right) (3 - z)$$

$$50. \ p = \frac{q^2 + 3}{(q - 1)^3 + (q + 1)^3}$$

$$45. \quad y = \frac{x^3 + 7}{x} = x^2 + 7x^{-1} \Rightarrow \frac{dy}{dx} = 2x - 7x^{-2} = 2x - \frac{7}{x^2} \Rightarrow \frac{d^2y}{dx^2} = 2 + 14x^{-3} = 2 + \frac{14}{x^3}$$

$$46. \quad s = \frac{t^2 + 5t - 1}{t^2} = 1 + \frac{5}{t} - \frac{1}{t^2} = 1 + 5t^{-1} - t^{-2} \Rightarrow \frac{ds}{dt} = 0 - 5t^{-2} + 2t^{-3} = -5t^{-2} + 2t^{-3} = \frac{-5}{t^2} + \frac{2}{t^3}$$

$$\Rightarrow \frac{d^2s}{dt^2} = 10t^{-3} - 6t^{-4} = \frac{10}{t^3} - \frac{6}{t^4}$$

$$47. \quad r = \frac{(\theta - 1)(\theta^2 + \theta + 1)}{\theta^3} = \frac{\theta^3 - 1}{\theta^3} = 1 - \frac{1}{\theta^3} = 1 - \theta^3 \Rightarrow \frac{dr}{d\theta} = 0 + 3\theta^{-4} = 3\theta^{-4} = \frac{3}{\theta^4} \Rightarrow \frac{d^2r}{d\theta^2} = -12\theta^{-5} = \frac{-12}{\theta^5}$$

$$48. \quad u = \frac{(x^2 + x)(x^2 - x + 1)}{x^4} = \frac{x(x+1)(x^2 - x + 1)}{x^4} = \frac{x(x^3 + 1)}{x^4} = \frac{x^4 + x}{x^4} = 1 + \frac{x}{x^4} = 1 + x^{-3}$$

$$\Rightarrow \frac{du}{dx} = 0 - 3x^{-4} = -3x^{-4} = \frac{-3}{x^4} \Rightarrow \frac{d^2u}{dx^2} = 12x^{-5} = \frac{12}{x^5}$$

$$49. \quad w = \left(\frac{1+3z}{3z} \right) (3 - z) = \left(\frac{1}{3}z^{-1} + 1 \right) (3 - z) = z^{-1} - \frac{1}{3} + 3 - z = z^{-1} + \frac{8}{3} - z \Rightarrow \frac{dw}{dz} = -z^{-2} + 0 - 1 = -z^{-2} - 1 = \frac{-1}{z^2} - 1$$

$$\Rightarrow \frac{d^2w}{dz^2} = 2z^{-3} - 0 = 2z^{-3} = \frac{2}{z^3}$$

$$50. \quad p = \frac{q^2 + 3}{(q-1)^3 + (q+1)^3} = \frac{q^2 + 3}{(q^3 - 3q^2 + 3q - 1) + (q^3 + 3q^2 + 3q + 1)} = \frac{q^2 + 3}{2q^3 + 6q} = \frac{q^2 + 3}{2q(q^2 + 3)} = \frac{1}{2q} = \frac{1}{2}q^{-1} \Rightarrow \frac{dp}{dq} = -\frac{1}{2}q^{-2} = -\frac{1}{2q^2}$$

$$\Rightarrow \frac{d^2p}{dq^2} = q^{-3} = \frac{1}{q^3}$$

Exercises

53. Suppose u and v are functions of x that are differentiable at $x = 0$ and that

$$u(0) = 5, \quad u'(0) = -3, \quad v(0) = -1, \quad v'(0) = 2.$$

Find the values of the following derivatives at $x = 0$.

- a. $\frac{d}{dx}(uv)$ b. $\frac{d}{dx}\left(\frac{u}{v}\right)$ c. $\frac{d}{dx}\left(\frac{v}{u}\right)$ d. $\frac{d}{dx}(7v - 2u)$

54. Suppose u and v are differentiable functions of x and that

$$u(1) = 2, \quad u'(1) = 0, \quad v(1) = 5, \quad v'(1) = -1.$$

Find the values of the following derivatives at $x = 1$.

- a. $\frac{d}{dx}(uv)$ b. $\frac{d}{dx}\left(\frac{u}{v}\right)$ c. $\frac{d}{dx}\left(\frac{v}{u}\right)$ d. $\frac{d}{dx}(7v - 2u)$

53. $u(0) = 5, u'(0) = -3, v(0) = -1, v'(0) = 2$

(a) $\frac{d}{dx}(uv) = uv' + vu' \Rightarrow \frac{d}{dx}(uv)\Big|_{x=0} = u(0)v'(0) + v(0)u'(0) = 5 \cdot 2 + (-1)(-3) = 13$

(b) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2} \Rightarrow \frac{d}{dx}\left(\frac{u}{v}\right)\Big|_{x=0} = \frac{v(0)u'(0) - u(0)v'(0)}{(v(0))^2} = \frac{(-1)(-3) - (5)(2)}{(-1)^2} = -7$

(c) $\frac{d}{dx}\left(\frac{v}{u}\right) = \frac{uv' - vu'}{u^2} \Rightarrow \frac{d}{dx}\left(\frac{v}{u}\right)\Big|_{x=0} = \frac{u(0)v'(0) - v(0)u'(0)}{(u(0))^2} = \frac{(5)(2) - (-1)(-3)}{(5)^2} = \frac{7}{25}$

(d) $\frac{d}{dx}(7v - 2u) = 7v' - 2u' \Rightarrow \frac{d}{dx}(7v - 2u)|_{x=0} = 7v'(0) - 2u'(0) = 7 \cdot 2 - 2(-3) = 20$

54. $u(1) = 2, u'(1) = 0, v(1) = 5, v'(1) = -1$

(a) $\frac{d}{dx}(uv)|_{x=1} = u(1)v'(1) + v(1)u'(1) = 2 \cdot (-1) + 5 \cdot 0 = -2$

(b) $\frac{d}{dx}\left(\frac{u}{v}\right)\Big|_{x=1} = \frac{v(1)u'(1) - u(1)v'(1)}{(v(1))^2} = \frac{5 \cdot 0 - 2 \cdot (-1)}{(5)^2} = \frac{2}{25}$

(c) $\frac{d}{dx}\left(\frac{v}{u}\right)\Big|_{x=1} = \frac{u(1)v'(1) - v(1)u'(1)}{(u(1))^2} = \frac{2(-1) - 5 \cdot 0}{(2)^2} = -\frac{1}{2}$

(d) $\frac{d}{dx}(7v - 2u)|_{x=1} = 7v'(1) - 2u'(1) = 7 \cdot (-1) - 2 \cdot 0 = -7$

53. $u(0) = 5, u'(0) = -3, v(0) = -1, v'(0) = 2$

(a) $\frac{d}{dx}(uv) = uv' + vu' \Rightarrow \frac{d}{dx}(uv)\Big|_{x=0} = u(0)v'(0) + v(0)u'(0) = 5 \cdot 2 + (-1)(-3) = 13$

(b) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2} \Rightarrow \frac{d}{dx}\left(\frac{u}{v}\right)\Big|_{x=0} = \frac{v(0)u'(0) - u(0)v'(0)}{(v(0))^2} = \frac{(-1)(-3) - (5)(2)}{(-1)^2} = -7$

(c) $\frac{d}{dx}\left(\frac{v}{u}\right) = \frac{uv' - vu'}{u^2} \Rightarrow \frac{d}{dx}\left(\frac{v}{u}\right)\Big|_{x=0} = \frac{u(0)v'(0) - v(0)u'(0)}{(u(0))^2} = \frac{(5)(2) - (-1)(-3)}{(5)^2} = \frac{7}{25}$

(d) $\frac{d}{dx}(7v - 2u) = 7v' - 2u' \Rightarrow \frac{d}{dx}(7v - 2u)|_{x=0} = 7v'(0) - 2u'(0) = 7 \cdot 2 - 2(-3) = 20$

54. $u(1) = 2, u'(1) = 0, v(1) = 5, v'(1) = -1$

(a) $\frac{d}{dx}(uv)|_{x=1} = u(1)v'(1) + v(1)u'(1) = 2 \cdot (-1) + 5 \cdot 0 = -2$

(b) $\frac{d}{dx}\left(\frac{u}{v}\right)\Big|_{x=1} = \frac{v(1)u'(1) - u(1)v'(1)}{(v(1))^2} = \frac{5 \cdot 0 - 2 \cdot (-1)}{(5)^2} = \frac{2}{25}$

(c) $\frac{d}{dx}\left(\frac{v}{u}\right)\Big|_{x=1} = \frac{u(1)v'(1) - v(1)u'(1)}{(u(1))^2} = \frac{2 \cdot (-1) - 5 \cdot 0}{(2)^2} = -\frac{1}{2}$

(d) $\frac{d}{dx}(7v - 2u)|_{x=1} = 7v'(1) - 2u'(1) = 7 \cdot (-1) - 2 \cdot 0 = -7$

69. Find the value of a that makes the following function differentiable for all x -values.

$$g(x) = \begin{cases} ax, & \text{if } x < 0 \\ x^2 - 3x, & \text{if } x \geq 0 \end{cases}$$

70. Find the values of a and b that make the following function differentiable for all x -values.

$$f(x) = \begin{cases} ax + b, & x > -1 \\ bx^2 - 3, & x \leq -1 \end{cases}$$

69. $g'(x) = \begin{cases} 2x-3 & x>0 \\ a & x<0 \end{cases}$, since g is differentiable at $x=0 \Rightarrow \lim_{x \rightarrow 0^+} (2x-3) = -3$ and $\lim_{x \rightarrow 0^-} a = a \Rightarrow a = -3$

70. $f'(x) = \begin{cases} a & x>-1 \\ 2bx & x<-1 \end{cases}$, since f is differentiable at $x=-1 \Rightarrow \lim_{x \rightarrow -1^+} a = a$ and $\lim_{x \rightarrow -1^-} (2bx) = -2b \Rightarrow a = -2b$, and
since f is continuous at $x=-1 \Rightarrow \lim_{x \rightarrow -1^+} (ax+b) = -a+b$ and $\lim_{x \rightarrow -1^-} (bx^2-3) = b-3 \Rightarrow -a+b = b-3$
 $\Rightarrow a = 3 \Rightarrow 3 = -2b \Rightarrow b = -\frac{3}{2}$.

Derivatives

In Exercises 1–18, find dy/dx .

$$1. \ y = -10x + 3 \cos x$$

$$2. \ y = \frac{3}{x} + 5 \sin x$$

$$3. \ y = x^2 \cos x$$

$$4. \ y = \sqrt{x} \sec x + 3$$

$$5. \ y = \csc x - 4\sqrt{x} + \frac{7}{e^x}$$

$$6. \ y = x^2 \cot x - \frac{1}{x^2}$$

$$7. \ f(x) = \sin x \tan x$$

$$8. \ g(x) = \frac{\cos x}{\sin^2 x}$$

$$9. \ y = xe^{-x} \sec x$$

$$10. \ y = (\sin x + \cos x) \sec x$$

$$11. \ y = \frac{\cot x}{1 + \cot x}$$

$$12. \ y = \frac{\cos x}{1 + \sin x}$$

$$13. \ y = \frac{4}{\cos x} + \frac{1}{\tan x}$$

$$14. \ y = \frac{\cos x}{x} + \frac{x}{\cos x}$$

$$15. \ y = (\sec x + \tan x)(\sec x - \tan x)$$

$$16. \ y = x^2 \cos x - 2x \sin x - 2 \cos x$$

$$17. \ f(x) = x^3 \sin x \cos x$$

$$18. \ g(x) = (2 - x) \tan^2 x$$

In Exercises 19–22, find ds/dt .

$$19. \ s = \tan t - e^{-t}$$

$$20. \ s = t^2 - \sec t + 5e^t$$

$$21. \ s = \frac{1 + \csc t}{1 - \csc t}$$

$$22. \ s = \frac{\sin t}{1 - \cos t}$$

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In Exercises 1–18, find dy/dx .

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$$22. \ s = \frac{\sin t}{1 - \cos t}$$

$$1. \quad y = -10x + 3 \cos x \Rightarrow \frac{dy}{dx} = -10 + 3 \frac{d}{dx}(\cos x) = -10 - 3 \sin x$$

$$2. \quad y = \frac{3}{x} + 5 \sin x \Rightarrow \frac{dy}{dx} = \frac{-3}{x^2} + 5 \frac{d}{dx}(\sin x) = \frac{-3}{x^2} + 5 \cos x$$

$$3. \quad y = x^2 \cos x \Rightarrow \frac{dy}{dx} = x^2(-\sin x) + 2x \cos x = -x^2 \sin x + 2x \cos x$$

$$4. \quad y = \sqrt{x} \sec x + 3 \Rightarrow \frac{dy}{dx} = \sqrt{x} \sec x \tan x + \frac{\sec x}{2\sqrt{x}} + 0 = \sqrt{x} \sec x \tan x + \frac{\sec x}{2\sqrt{x}}$$

$$5. \quad y = \csc x - 4\sqrt{x} + \frac{7}{e^x} \Rightarrow \frac{dy}{dx} = -\csc x \cot x - \frac{4}{2\sqrt{x}} - \frac{7}{e^x}$$

$$6. \quad y = x^2 \cot x - \frac{1}{x^2} \Rightarrow \frac{dy}{dx} = x^2 \frac{d}{dx}(\cot x) + \cot x \cdot \frac{d}{dx}(x^2) + \frac{2}{x^3} = -x^2 \csc^2 x + (\cot x)(2x) + \frac{2}{x^3}$$

$$= -x^2 \csc^2 x + 2x \cot x + \frac{2}{x^3}$$

$$7. \quad f(x) = \sin x \tan x \Rightarrow f'(x) = \sin x \sec^2 x + \cos x \tan x = \sin x \sec^2 x + \cos x \frac{\sin x}{\cos x} = \sin x (\sec^2 x + 1)$$

$$8. \quad g(x) = \frac{\cos x}{\sin^2 x} = \frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} = \csc x \cot x \Rightarrow g'(x) = \csc x (-\csc^2 x) + (-\csc x \cot x) \cot x = -\csc^3 x - \csc x \cot^2 x$$

$$= -\csc x (\csc^2 x + \cot^2 x)$$



$$9. \quad y = xe^{-x} \sec x \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x)e^{-x} \sec x + x \frac{d}{dx}(e^{-x})\sec x + xe^{-x} \frac{d}{dx}(\sec x) = e^{-x} \sec x - xe^{-x} \sec x + xe^{-x} \sec x \tan x = e^{-x} \sec x(1 - x + x \tan x)$$

$$\begin{aligned}10. \quad y &= (\sin x + \cos x) \sec x \Rightarrow \frac{dy}{dx} = (\sin x + \cos x) \frac{d}{dx}(\sec x) + \sec x \frac{d}{dx}(\sin x + \cos x) \\&= (\sin x + \cos x)(\sec x \tan x) + (\sec x)(\cos x - \sin x) = \frac{(\sin x + \cos x)\sin x}{\cos^2 x} + \frac{\cos x - \sin x}{\cos x} \\&= \frac{\sin^2 x + \cos x \sin x + \cos^2 x - \cos x \sin x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x\end{aligned}$$

(Note also that $y = \sin x \sec x + \cos x \sec x = \tan x + 1 \Rightarrow \frac{dy}{dx} = \sec^2 x$.)

$$\begin{aligned}11. \quad y &= \frac{\cot x}{1+\cot x} \Rightarrow \frac{dy}{dx} = \frac{(1+\cot x)\frac{d}{dx}(\cot x) - (\cot x)\frac{d}{dx}(1+\cot x)}{(1+\cot x)^2} = \frac{(1+\cot x)(-\csc^2 x) - (\cot x)(-\csc^2 x)}{(1+\cot x)^2} \\&= \frac{-\csc^2 x - \csc^2 x \cot x + \csc^2 x \cot x}{(1+\cot x)^2} = \frac{-\csc^2 x}{(1+\cot x)^2}\end{aligned}$$

$$\begin{aligned}12. \quad y &= \frac{\cos x}{1+\sin x} \Rightarrow \frac{dy}{dx} = \frac{(1+\sin x)\frac{d}{dx}(\cos x) - (\cos x)\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2} = \frac{(1+\sin x)(-\sin x) - (\cos x)(\cos x)}{(1+\sin x)^2} = \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2} \\&= \frac{-\sin x - 1}{(1+\sin x)^2} = \frac{-(1+\sin x)}{(1+\sin x)^2} = \frac{-1}{1+\sin x}\end{aligned}$$

$$13. \quad y = \frac{4}{\cos x} + \frac{1}{\tan x} = 4 \sec x + \cot x \Rightarrow \frac{dy}{dx} = 4 \sec x \tan x - \csc^2 x$$

$$14. \quad y = \frac{\cos x}{x} + \frac{x}{\cos x} \Rightarrow \frac{dy}{dx} = \frac{x(-\sin x) - (\cos x)(1)}{x^2} + \frac{(\cos x)(1) - x(-\sin x)}{\cos^2 x} = \frac{-x \sin x - \cos x}{x^2} + \frac{\cos x + x \sin x}{\cos^2 x}$$

$$15. \quad y = (\sec x + \tan x)(\sec x - \tan x) \Rightarrow \frac{dy}{dx} = (\sec x + \tan x) \frac{d}{dx}(\sec x - \tan x) + (\sec x - \tan x) \frac{d}{dx}(\sec x + \tan x)$$

$$= (\sec x + \tan x)(\sec x \tan x - \sec^2 x) + (\sec x - \tan x)(\sec x \tan x + \sec^2 x)$$

$$= (\sec^2 x \tan x + \sec x \tan^2 x - \sec^3 x - \sec^2 x \tan x) + (\sec^2 x \tan x - \sec x \tan^2 x + \sec^3 x - \tan x \sec^2 x) = 0.$$

(Note also that $y = \sec^2 x - \tan^2 x = (\tan^2 x + 1) - \tan^2 x = 1 \Rightarrow \frac{dy}{dx} = 0.$)

$$16. \quad y = x^2 \cos x - 2x \sin x - 2 \cos x \Rightarrow \frac{dy}{dx} = (x^2(-\sin x) + (\cos x)(2x)) - (2x \cos x + (\sin x)(2)) - 2(-\sin x)$$

$$= -x^2 \sin x + 2x \cos x - 2x \cos x - 2 \sin x + 2 \sin x = -x^2 \sin x$$

$$17. \quad f(x) = x^3 \sin x \cos x \Rightarrow f'(x) = x^3 \sin x(-\sin x) + x^3 \cos x(\cos x) + 3x^2 \sin x \cos x$$

$$= -x^3 \sin^2 x + x^3 \cos^2 x + 3x^2 \sin x \cos x$$

$$18. \quad g(x) = (2-x) \tan^2 x \Rightarrow g'(x) = (2-x)(2 \tan x \sec^2 x) + (-1) \tan^2 x = 2(2-x) \tan x \sec^2 x - \tan^2 x \\ = 2(2-x) \tan x (\sec^2 x - \tan x)$$

$$19. \quad s = \tan t - e^{-t} \Rightarrow \frac{ds}{dt} = \sec^2 t + e^{-t}$$

$$20. \quad s = t^2 - \sec t + 5e^t \Rightarrow \frac{ds}{dt} = 2t - \sec t \tan t + 5e^t$$

$$21. \quad s = \frac{1+\csc t}{1-\csc t} \Rightarrow \frac{ds}{dt} = \frac{(1-\csc t)(-\csc t \cot t) - (1+\csc t)(\csc t \cot t)}{(1-\csc t)^2} = \frac{-\csc t \cot t + \csc^2 t \cot t - \csc t \cot t - \csc^2 t \cot t}{(1-\csc t)^2} = \frac{-2 \csc t \cot t}{(1-\csc t)^2}$$

$$22. \quad s = \frac{\sin t}{1-\cos t} \Rightarrow \frac{ds}{dt} = \frac{(1-\cos t)(\cos t) - (\sin t)(-\sin t)}{(1-\cos t)^2} = \frac{\cos t - \cos^2 t - \sin^2 t}{(1-\cos t)^2} = \frac{\cos t - 1}{(1-\cos t)^2} = -\frac{1}{1-\cos t} = \frac{1}{\cos t - 1}$$

33. Find y'' if

a. $y = \csc x.$

b. $y = \sec x.$

34. Find $y^{(4)} = d^4 y/dx^4$ if

a. $y = -2 \sin x.$

b. $y = 9 \cos x.$

33. (a) $y = \csc x \Rightarrow y' = -\csc x \cot x \Rightarrow y'' = -((\csc x)(-\csc^2 x) + (\cot x)(-\csc x \cot x)) = \csc^3 x + \csc x \cot^2 x$
 $= (\csc x)(\csc^2 x + \cot^2 x) = (\csc x)(\csc^2 x + \csc^2 x - 1) = 2 \csc^3 x - \csc x$

(b) $y = \sec x \Rightarrow y' = \sec x \tan x \Rightarrow y'' = (\sec x)(\sec^2 x) + (\tan x)(\sec x \tan x) = \sec^3 x + \sec x \tan^2 x$
 $= (\sec x)(\sec^2 x + \tan^2 x) = (\sec x)(\sec^2 x + \sec^2 x - 1) = 2 \sec^3 x - \sec x$

34. (a) $y = -2 \sin x \Rightarrow y' = -2 \cos x \Rightarrow y'' = -2(-\sin x) = 2 \sin x \Rightarrow y''' = 2 \cos x \Rightarrow y^{(4)} = -2 \sin x$

(b) $y = 9 \cos x \Rightarrow y' = -9 \sin x \Rightarrow y'' = -9 \cos x \Rightarrow y''' = -9(-\sin x) = 9 \sin x \Rightarrow y^{(4)} = 9 \cos x$



Thank you for your attention