MATHEMATICAL ANALAYSIS 2

Lecture 7

Prepared by
Dr. Sami INJROU

Line Integrals of Vector Fields

DEFINITION Let $\mathbf{F}$ be a vector field with continuous components defined along a smooth curve C parametrized by $\mathbf{r}(t)$, $a \le t \le b$. Then the **line integral of F along C** is

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \left(\mathbf{F} \cdot \frac{d\mathbf{r}}{ds}\right) ds = \int_C \mathbf{F} \cdot d\mathbf{r}. \qquad (1)$$

Evaluating the Line Integral of $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ Along $C\colon \mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$

1. Express the vector field $\mathbf{F}$ along the parametrized curve C as $\mathbf{F}(\mathbf{r}(t))$ by substituting the components $x = g(t)$, $y = h(t)$, $z = k(t)$ of $\mathbf{r}$ into the scalar components $M(x, y, z)$, $N(x, y, z)$, $P(x, y, z)$ of $\mathbf{F}$.
2. Find the derivative (velocity) vector $d\mathbf{r}/dt$.
3. Evaluate the line integral with respect to the parameter t, $a \le t \le b$, to obtain

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}\, dt. \qquad (2)$$

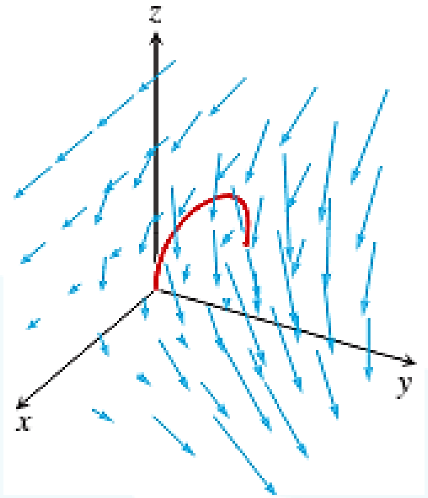
Line Integrals of Vector Fields

EXAMPLE 2 Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = z\mathbf{i} + xy\mathbf{j} - y^2\mathbf{k}$ along the curve C given by $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + \sqrt{t}\,\mathbf{k}$, $0 \le t \le 1$ and shown in Figure 16.18.

$$\mathbf{F}(\mathbf{r}(t)) = \sqrt{t}\,\mathbf{i} + t^3\mathbf{j} - t^2\mathbf{k} \qquad z = \sqrt{t},\ xy = t^3,\ -y^2 = -t^2$$

$$\frac{d\mathbf{r}}{dt} = 2t\mathbf{i} + \mathbf{j} + \frac{1}{2\sqrt{t}}\mathbf{k}.$$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t))\cdot\frac{d\mathbf{r}}{dt}\,dt = \int_0^1 \left(2t^{3/2} + t^3 - \frac{1}{2}t^{3/2}\right)dt = \frac{17}{20}.$$



The curve (in red) winds through the vector field in Example 2. The line integral is determined by the vectors that lie along the curve.

Line Integrals with Respect to dx, dy, or dz

$\mathbf{F} = M(x, y, z)\mathbf{i}$ having a component only in the x-direction

$\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$ for $a \le t \le b$, $x = g(t),\ dx = g'(t)\,dt$

$$\mathbf{F}\cdot d\mathbf{r} = \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt = M(x, y, z)\mathbf{i}\cdot\left(g'(t)\mathbf{i} + h'(t)\mathbf{j} + k'(t)\mathbf{k}\right)dt = M(x, y, z)\,g'(t)\,dt = M(x, y, z)\,dx.$$

$$\int_C M(x, y, z)\,dx = \int_C \mathbf{F}\cdot d\mathbf{r}, \quad \text{where} \quad \mathbf{F} = M(x, y, z)\mathbf{i}.$$

$$\int_C M(x, y, z)\,dx = \int_a^b M(g(t), h(t), k(t))\,g'(t)\,dt \tag{3}$$

$$\int_C N(x, y, z)\,dy = \int_a^b N(g(t), h(t), k(t))\,h'(t)\,dt \tag{4}$$

$$\int_C P(x, y, z)\,dz = \int_a^b P(g(t), h(t), k(t))\,k'(t)\,dt \tag{5}$$

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$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C M(x, y, z)\, dx + \int_C N(x, y, z)\, dy + \int_C P(x, y, z)\, dz = \int_C M\, dx + N\, dy + P\, dz.$$

Line Integrals with Respect to *dx*, *dy*, or *dz*

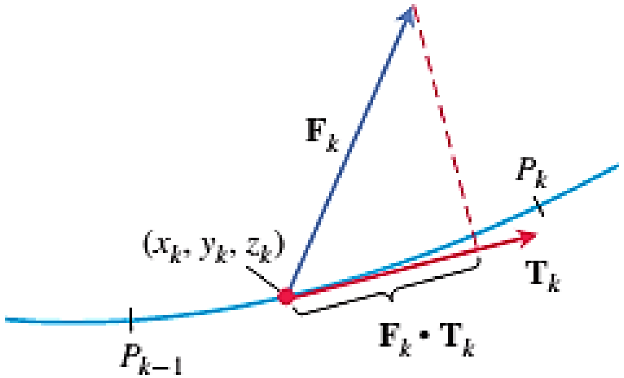
EXAMPLE 3 Evaluate the line integral $\int_C -y\, dx + z\, dy + 2x\, dz$, where C is the helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$, $0 \le t \le 2\pi$.

$x = \cos t,\ y = \sin t,\ z = t$, and $dx = -\sin t\, dt$, $dy = \cos t\, dt$, $dz = dt$.

$$\int_C -y\, dx + z\, dy + 2x\, dz = \int_0^{2\pi} [(-\sin t)(-\sin t) + t\cos t + 2\cos t]\, dt = \pi.$$

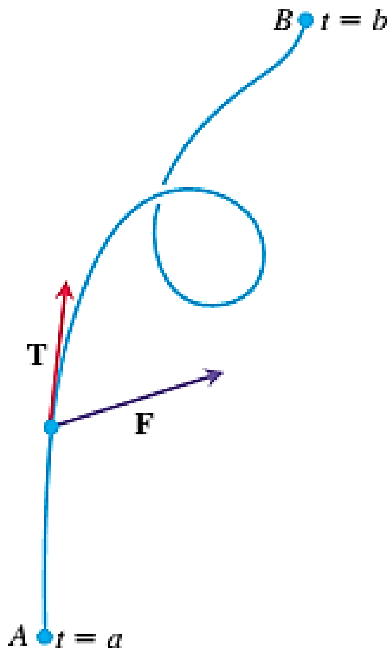
Work Done by a Force over a Curve in Space

$$W \approx \sum_{k=1}^{n} W_k \approx \sum_{k=1}^{n} \mathbf{F}(x_k, y_k, z_k) \cdot \mathbf{T}(x_k, y_k, z_k)\, \Delta s_k \xrightarrow[n \to \infty \text{ and } \Delta s_k \to 0,]{} \int_C \mathbf{F} \cdot \mathbf{T}\, ds.$$



DEFINITION Let C be a smooth curve parametrized by $\mathbf{r}(t)$, $a \le t \le b$, and let $\mathbf{F}$ be a continuous force field over a region containing C. Then the **work** done in moving an object from the point $A = \mathbf{r}(a)$ to the point $B = \mathbf{r}(b)$ along C is

$$W = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}\, dt. \qquad (6)$$



Work Done by a Force over a Curve in Space

TABLE 16.2 Different ways to write the work integral for $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ over the curve $C: \mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k},\ a \le t \le b$

$W = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$	The definition
$= \int_C \mathbf{F} \cdot d\mathbf{r}$	Vector differential form
$= \int_a^b \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt$	Parametric vector evaluation
$= \int_a^b \left(M g'(t) + N h'(t) + P k'(t)\right) dt$	Parametric scalar evaluation
$= \int_C M\, dx + N\, dy + P\, dz$	Scalar differential form

Work Done by a Force over a Curve in Space

EXAMPLE 4 Find the work done by the force field $\mathbf{F} = (y - x^2)\mathbf{i} + (z - y^2)\mathbf{j} + (x - z^2)\mathbf{k}$ in moving an object along the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, $0 \le t \le 1$, from $(0, 0, 0)$ to $(1, 1, 1)$ (Figure 16.21).

$$\begin{aligned} \mathbf{F} &= (y - x^2)\mathbf{i} + (z - y^2)\mathbf{j} + (x - z^2)\mathbf{k} \\ &= (t^2 - t^2)\mathbf{i} + (t^3 - t^4)\mathbf{j} + (t - t^6)\mathbf{k}. \end{aligned}$$

Substitute $x = t$, $y = t^2$, $z = t^3$.

$$\frac{d\mathbf{r}}{dt} = \frac{d}{dt}(t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}) = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}.$$

$$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = (t^3 - t^4)(2t) + (t - t^6)(3t^2) = 2t^4 - 2t^5 + 3t^3 - 3t^8.$$

$$\text{Work} = \int_a^b \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt = \int_0^1 (2t^4 - 2t^5 + 3t^3 - 3t^8)\, dt = \frac{29}{60}$$

Flow Integrals and Circulation for Velocity Fields

DEFINITION If $\mathbf{r}(t)$ parametrizes a smooth curve C in the domain of a continuous velocity field $\mathbf{F}$, the **flow** along the curve from $A = \mathbf{r}(a)$ to $B = \mathbf{r}(b)$ is

$$\text{Flow} = \int_C \mathbf{F}\cdot\mathbf{T}\,ds. \tag{7}$$

The integral is called a **flow integral**. If the curve starts and ends at the same point, so that $A = B$, the flow is called the **circulation** around the curve.

EXAMPLE 6 A fluid's velocity field is $\mathbf{F} = x\mathbf{i} + z\mathbf{j} + y\mathbf{k}$. Find the flow along the helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$, $0 \le t \le \pi/2$.

$\mathbf{F} = x\mathbf{i} + z\mathbf{j} + y\mathbf{k} = (\cos t)\mathbf{i} + t\mathbf{j} + (\sin t)\mathbf{k}$ Substitute $x = \cos t$, $z = t$, $y = \sin t$.

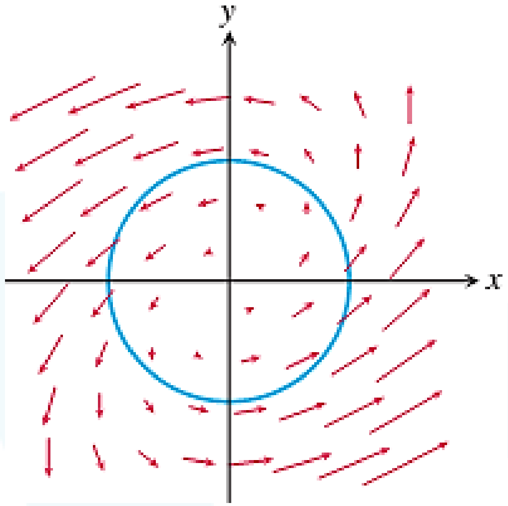
$$\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k}.$$

$$\mathbf{F}\cdot\frac{d\mathbf{r}}{dt} = -\sin t\cos t + t\cos t + \sin t.$$

$$\text{Flow} = \int_{t=a}^{t=b} \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt = \int_0^{\pi/2} (-\sin t\cos t + t\cos t + \sin t)\,dt = \frac{\pi}{2} - \frac{1}{2}$$

Flow Integrals and Circulation for Velocity Fields

EXAMPLE 7 Find the circulation of the field $\mathbf{F} = (x - y)\mathbf{i} + x\mathbf{j}$ around the circle $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \le t \le 2\pi$ (Figure 16.22).



$$\mathbf{F} = (x - y)\mathbf{i} + x\mathbf{j} = (\cos t - \sin t)\mathbf{i} + (\cos t)\mathbf{j},$$

$$\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}.$$

$$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -\sin t \cos t + \underbrace{\sin^2 t + \cos^2 t}_{1}$$

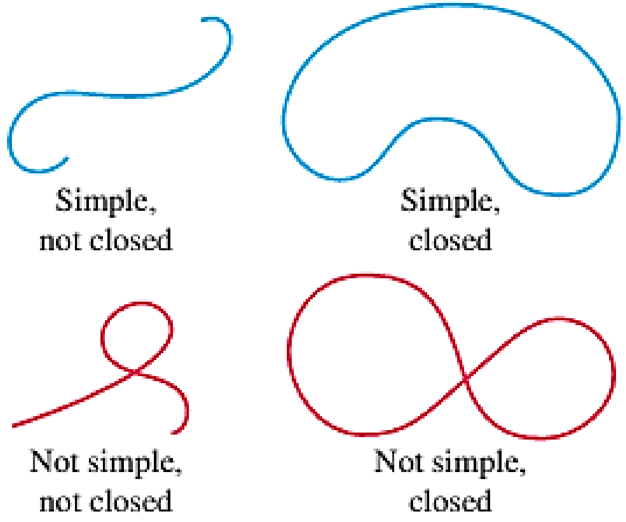
$$\text{Circulation} = \int_0^{2\pi} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\,dt = \int_0^{2\pi} (1 - \sin t \cos t)\,dt = 2\pi$$

As Figure 16.22 suggests, a fluid with this velocity field is circulating *counterclockwise* around the circle, so the circulation is positive. ■

Flux Across a Simple Closed Plane Curve

DEFINITION If C is a smooth simple closed curve in the domain of a continuous vector field $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ in the plane, and if $\mathbf{n}$ is the outward-pointing unit normal vector on C, the **flux** of $\mathbf{F}$ across C is

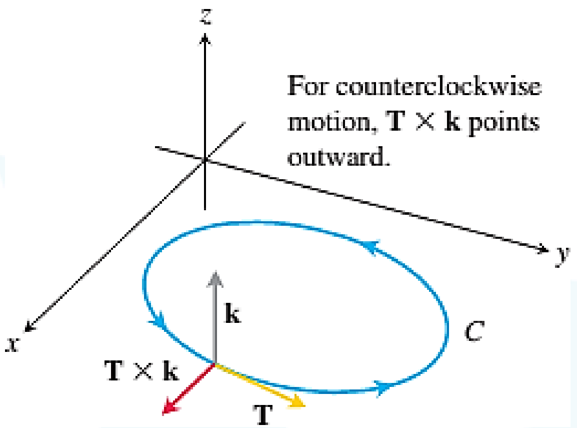
$$\text{Flux of } \mathbf{F} \text{ across C} = \int_C \mathbf{F} \cdot \mathbf{n}\, ds. \tag{8}$$



Flux Across a Simple Closed Plane Curve

$$\mathbf{n} = \mathbf{T} \times \mathbf{k} = \left(\frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j}\right) \times \mathbf{k} = \frac{dy}{ds}\mathbf{i} - \frac{dx}{ds}\mathbf{j}.$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{dx}{ds} & \frac{dy}{ds} & 0 \\ 0 & 0 & 1 \end{vmatrix} \Rightarrow \mathbf{n}|\mathbf{v}| = \frac{dy}{dt}\mathbf{i} - \frac{dx}{dt}\mathbf{j}$$



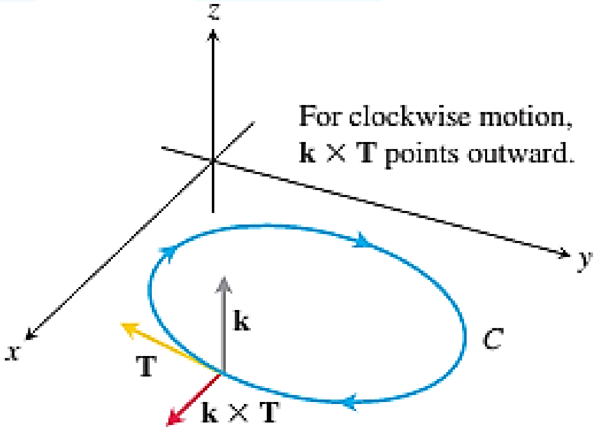
If $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$,

$$\mathbf{F} \cdot \mathbf{n} = M(x, y)\frac{dy}{ds} - N(x, y)\frac{dx}{ds}. \Rightarrow \int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_C \left(M\frac{dy}{ds} - N\frac{dx}{ds}\right) ds = \oint_C M\,dy - N\,dx.$$

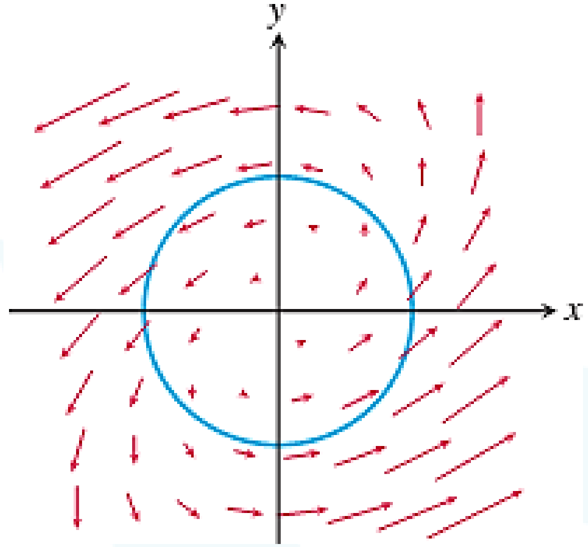
Calculating Flux Across a Smooth Closed Plane Curve

$$\text{Flux of } \mathbf{F} = M\mathbf{i} + N\mathbf{j} \text{ across } C = \oint_C M\,dy - N\,dx \tag{9}$$

The integral can be evaluated from any smooth parametrization $x = g(t)$, $y = h(t)$, $a \le t \le b$, that traces C counterclockwise exactly once.



Flux Across a Simple Closed Plane Curve



EXAMPLE 8 Find the flux of $\mathbf{F} = (x - y)\mathbf{i} + x\mathbf{j}$ across the circle $x^2 + y^2 = 1$ in the xy-plane. (The vector field and curve were shown previously in Figure 16.22.)

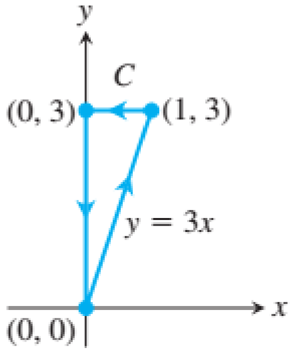
$$M = x - y = \cos t - \sin t, \qquad dy = d(\sin t) = \cos t\, dt,$$
$$N = x = \cos t, \qquad dx = d(\cos t) = -\sin t\, dt,$$

$$\text{Flux} = \oint_C M\,dy - N\,dx = \int_0^{2\pi} (\cos^2 t - \sin t \cos t + \cos t \sin t)\, dt = \pi$$

The flux of $\mathbf{F}$ across the circle is π. Since the answer is positive, the net flow across the curve is outward. A net inward flow would have given a negative flux. ■

Exercises

- $\int_C \sqrt{x+y}\,dx$, where C is given in the accompanying figure



$2\sqrt{3}-4$

- find the work done by $\mathbf{F}$ over the curve in the direction of increasing t.

$$\mathbf{F} = xy\mathbf{i} + y\mathbf{j} - yz\mathbf{k} \qquad \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 1$$

$\frac{1}{2}$

- Evaluate $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ for the vector field $\mathbf{F} = x^2\mathbf{i} - y\mathbf{j}$ along the curve $x = y^2$ from $(4, 2)$ to $(1, -1)$.

$-\frac{39}{2}$

- Find the flow of the velocity field $\mathbf{F} = (x+y)\mathbf{i} - (x^2+y^2)\mathbf{j}$ along each of the following paths from $(1, 0)$ to $(-1, 0)$ in the xy-plane.

a. The upper half of the circle $x^2 + y^2 = 1$

$-\frac{\pi}{2}$

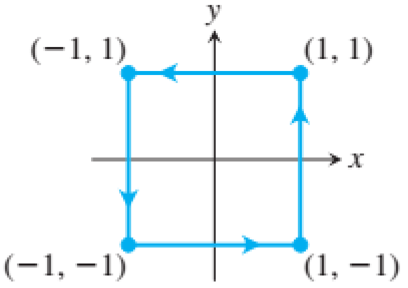
b. The line segment from $(1, 0)$ to $(-1, 0)$

0

Exercises

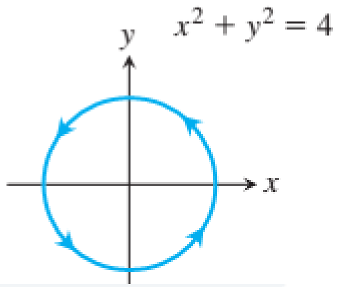
- Find the circulation of the field $\mathbf{F} = y\mathbf{i} + (x + 2y)\mathbf{j}$ around each of the following closed paths.

a.



0

b.



0

- $\mathbf{F}$ is the velocity field of a fluid flowing through a region in space. Find the flow along the given curve in the direction of increasing t.

$\mathbf{F} = -4xy\mathbf{i} + 8y\mathbf{j} + 2\mathbf{k} \qquad \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \mathbf{k}, \quad 0 \le t \le 2$ 48

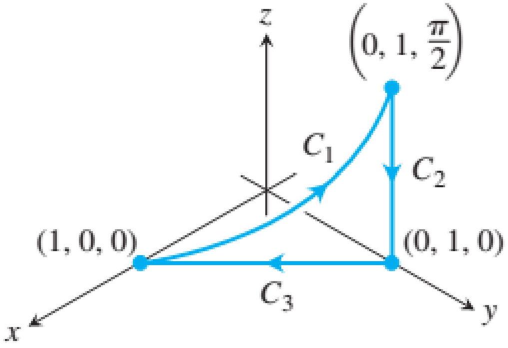
- Find the circulation of $\mathbf{F} = 2x\mathbf{i} + 2z\mathbf{j} + 2y\mathbf{k}$ around the closed path consisting of the following three curves

- C_1: $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le \pi/2$

  C_2: $\mathbf{r}(t) = \mathbf{j} + (\pi/2)(1 - t)\mathbf{k}, \quad 0 \le t \le 1$

  C_3: $\mathbf{r}(t) = t\mathbf{i} + (1 - t)\mathbf{j}, \quad 0 \le t \le 1$

  $\text{Circulation} = (-1+\pi) - \pi + 1 = 0$



Exercises

- Find the flux of the field $\mathbf{F} = (x + y)\mathbf{i} - (x^2 + y^2)\mathbf{j}$ outward across the triangle with vertices $(1, 0)$, $(0, 1)$, $(-1, 0)$.

$$\frac{1}{3}$$

- The flow of a gas with a density of $\delta = 0.001\,\mathrm{kg/m^2}$ over the closed curve $\mathbf{r}(t) = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j},\ 0 \leq t \leq 2\pi$, is given by the vector field $\mathbf{F} = \delta\mathbf{v}$, where $\mathbf{v} = x\mathbf{i} + y^2\mathbf{j}$ is a velocity field measured in meters per second. Find the flux of $\mathbf{F}$ across the curve $\mathbf{r}(t)$.

$(0.001)\pi$ kg/s ≈ 0.00314 kg/s

Path Independence, Conservative Fields, and Potential Functions

Path Independence

DEFINITIONS Let $\mathbf{F}$ be a vector field defined on an open region D in space, and suppose that for any two points A and B in D the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ along a path C from A to B in D is the same over all paths from A to B. Then the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is **path independent in D** and the field $\mathbf{F}$ is **conservative on D**.

DEFINITION If $\mathbf{F}$ is a vector field defined on D and $\mathbf{F} = \nabla f$ for some scalar function f on D, then f is called a **potential function for F**.

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_A^B \nabla f \cdot d\mathbf{r} = f(B) - f(A).$$

Path Independence, Conservative Fields, and Potential Functions

Line Integrals in Conservative Fields

THEOREM 1—Fundamental Theorem of Line Integrals
Let C be a smooth curve joining the point A to the point B in the plane or in space and parametrized by $\mathbf{r}(t)$. Let f be a differentiable function with a continuous gradient vector $\mathbf{F} = \nabla f$ on a domain D containing C. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A).$$

EXAMPLE 1 Suppose the force field $\mathbf{F} = \nabla f$ is the gradient of the function $f(x, y, z) = -\frac{1}{x^2 + y^2 + z^2}$.

Find the work done by $\mathbf{F}$ in moving an object along a smooth curve C joining $(1, 0, 0)$ to $(0, 0, 2)$ that does not pass through the origin.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(0, 0, 2) - f(1, 0, 0) = -\frac{1}{4} - (-1) = \frac{3}{4}.$$

Path Independence, Conservative Fields, and Potential Functions

Line Integrals in Conservative Fields

THEOREM 2—Conservative Fields are Gradient Fields
Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be a vector field whose components are continuous throughout an open connected region D in space. Then $\mathbf{F}$ is conservative if and only if $\mathbf{F}$ is a gradient field ∇f for a differentiable function f.

EXAMPLE 2 Find the work done by the conservative field

$$\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} = \nabla f, \quad \text{where} \quad f(x, y, z) = xyz,$$

in moving an object along any smooth curve C joining the point $A(-1, 3, 9)$ to $B(1, 6, -4)$.

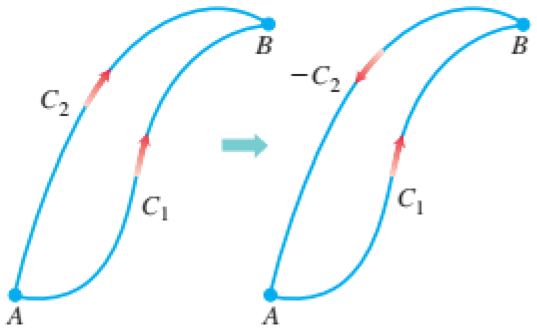
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_A^B \nabla f \cdot d\mathbf{r} = f(B) - f(A) = xyz\big|_{(1,6,-4)} - xyz\big|_{(-1,3,9)} = 3$$

Path Independence, Conservative Fields, and Potential Functions

Line Integrals in Conservative Fields

THEOREM 3—Loop Property of Conservative Fields
The following statements are equivalent.

1. $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ around every loop (that is, closed curve C) in D.
2. The field $\mathbf{F}$ is conservative on D.



Component Test for Conservative Fields
Let $\mathbf{F} = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$ be a field on an open simply connected domain whose component functions have continuous first partial derivatives. Then, $\mathbf{F}$ is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \qquad \text{and} \qquad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}. \tag{2}$$

Finding Potentials for Conservative Fields

$$\nabla f = \mathbf{F} \;\Rightarrow\; \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k} \;\Rightarrow\; \frac{\partial f}{\partial x} = M, \quad \frac{\partial f}{\partial y} = N, \quad \frac{\partial f}{\partial z} = P$$

Path Independence, Conservative Fields, and Potential Functions

Line Integrals in Conservative Fields

EXAMPLE 3 Show that $\mathbf{F} = (e^x \cos y + yz)\mathbf{i} + (xz - e^x \sin y)\mathbf{j} + (xy + z)\mathbf{k}$ is conservative over its natural domain and find a potential function for it.

$$M = e^x \cos y + yz, \qquad N = xz - e^x \sin y, \qquad P = xy + z$$

$$\frac{\partial P}{\partial y} = x = \frac{\partial N}{\partial z}, \qquad \frac{\partial M}{\partial z} = y = \frac{\partial P}{\partial x}, \qquad \frac{\partial N}{\partial x} = -e^x \sin y + z = \frac{\partial M}{\partial y}. \quad \Rightarrow \quad \mathbf{F} \text{ is conservative}$$

$$\frac{\partial f}{\partial x} = e^x \cos y + yz, \qquad \frac{\partial f}{\partial y} = xz - e^x \sin y, \qquad \frac{\partial f}{\partial z} = xy + z$$

$$f(x, y, z) = e^x \cos y + xyz + g(y, z).$$

$$-e^x \sin y + xz + \frac{\partial g}{\partial y} = xz - e^x \sin y \quad \Rightarrow \quad \partial g/\partial y = 0$$

$$\Rightarrow \quad f(x, y, z) = e^x \cos y + xyz + h(z).$$

$$\frac{\partial f}{\partial z} = xy + z$$

$$\Rightarrow \quad xy + \frac{dh}{dz} = xy + z, \quad \Rightarrow \quad h(z) = \frac{z^2}{2} + C$$

$$f(x, y, z) = e^x \cos y + xyz + \frac{z^2}{2} + C$$

Path Independence, Conservative Fields, and Potential Functions

Line Integrals in Conservative Fields

EXAMPLE 5 Show that the vector field $\mathbf{F} = \dfrac{-y}{x^2 + y^2}\mathbf{i} + \dfrac{x}{x^2 + y^2}\mathbf{j} + 0\mathbf{k}$

satisfies the equations in the Component Test, but is not conservative over its natural domain. Explain why this is possible.

$$M = -y/(x^2 + y^2),\ N = x/(x^2 + y^2), \text{ and } P = 0.$$

$$\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \qquad \frac{\partial P}{\partial x} = 0 = \frac{\partial M}{\partial z}, \qquad \text{and} \qquad \frac{\partial M}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial N}{\partial x}.$$

But

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j},\ 0 \le t \le 2\pi.$$

$$\mathbf{F} = \frac{-y}{x^2 + y^2}\mathbf{i} + \frac{x}{x^2 + y^2}\mathbf{j} = \frac{-\sin t}{\sin^2 t + \cos^2 t}\mathbf{i} + \frac{\cos t}{\sin^2 t + \cos^2 t}\mathbf{j} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}.$$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt = \int_0^{2\pi} \left(\sin^2 t + \cos^2 t\right) dt = 2\pi.$$

Since the line integral of $\mathbf{F}$ around the loop C is not zero, the field $\mathbf{F}$ is not conservative

Path Independence, Conservative Fields, and Potential Functions

DEFINITIONS Any expression $M(x, y, z)\,dx + N(x, y, z)\,dy + P(x, y, z)\,dz$ is a **differential form**. A differential form is **exact** on a domain D in space if

$$M\,dx + N\,dy + P\,dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = df$$

for some scalar function f throughout D.

Component Test for Exactness of $M\,dx + N\,dy + P\,dz$

The differential form $M\,dx + N\,dy + P\,dz$ is exact on an open simply connected domain if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \qquad \text{and} \qquad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

This is equivalent to saying that the field $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is conservative.

Path Independence, Conservative Fields, and Potential Functions

Exact Differential Forms

EXAMPLE 6 Show that $y\,dx + x\,dy + 4\,dz$ is exact and evaluate the integral $\int_{(1,1,1)}^{(2,3,-1)} y\,dx + x\,dy + 4\,dz$ over any path from $(1, 1, 1)$ to $(2, 3, -1)$.

$M = y, N = x, P = 4 \longrightarrow \frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = 1 = \frac{\partial M}{\partial y}. \longrightarrow y\,dx + x\,dy + 4\,dz$ is exact.

$y\,dx + x\,dy + 4\,dz = df \longrightarrow \frac{\partial f}{\partial x} = y, \quad \frac{\partial f}{\partial y} = x, \quad \frac{\partial f}{\partial z} = 4. \longrightarrow f(x, y, z) = xy + 4z + C.$

The value of the line integral is independent of the path taken from $(1, 1, 1)$ to $(2, 3, -1)$.

$$f(2, 3, -1) - f(1, 1, 1) = 2 + C - (5 + C) = -3.$$

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Exercises

- Which fields are conservative, and which are not?

  $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ $\quad\quad$ $\mathbf{F} = y\mathbf{i} + (x + z)\mathbf{j} - y\mathbf{k}$

- find a potential function f for the field $\mathbf{F}$. $\mathbf{F} = (y \sin z)\mathbf{i} + (x \sin z)\mathbf{j} + (xy \cos z)\mathbf{k}$ $\quad$ $xy \sin z + C$

- show that the differential forms in the integrals are exact. Then evaluate the integrals.

$$\int_{(0,0,0)}^{(2,3,-6)} 2x\,dx + 2y\,dy + 2z\,dz$$

$f(x, y, z) = x^2 + y^2 + z^2 + C$ $\quad$ 49

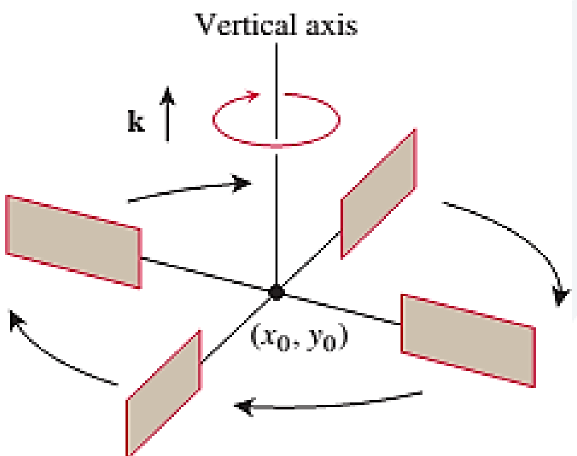
Green's Theorem in the Plane

Spin Around an Axis: The k-Component of Curl

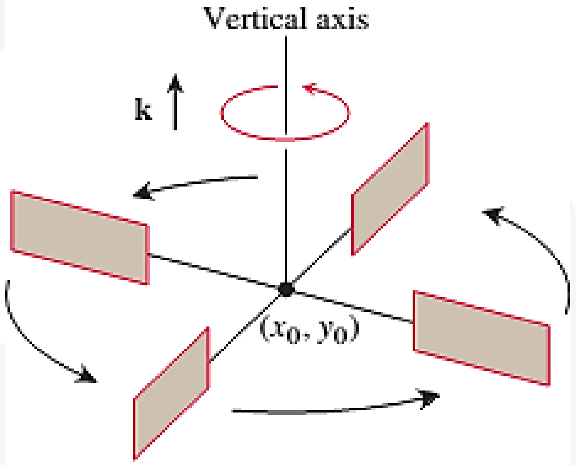
DEFINITION The **circulation density** of a vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ at the point (x, y) is the scalar expression

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}. \tag{1}$$

This expression is also called **the k-component of the curl**, denoted by $(\text{curl } \mathbf{F}) \cdot \mathbf{k}$.



Curl $\mathbf{F}(x_0, y_0) \cdot \mathbf{k} < 0$
Clockwise circulation



Curl $\mathbf{F}(x_0, y_0) \cdot \mathbf{k} > 0$
Counterclockwise circulation

Green's Theorem in the Plane

Divergence

DEFINITION The **divergence (flux density)** of a vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ at the point (x, y) is

$$\text{div}\,\mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}. \tag{2}$$

EXAMPLE 2 Find the divergence, and interpret what it means, for each vector field in Example 1 representing the velocity of a gas flowing in the xy-plane.

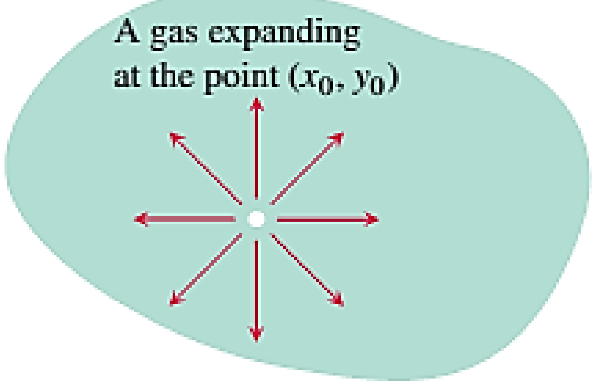
(a) *Uniform expansion or compression:* $\mathbf{F}(x, y) = cx\mathbf{i} + cy\mathbf{j}$ $\quad$ c a constant

(b) *Uniform rotation:* $\mathbf{F}(x, y) = -cy\mathbf{i} + cx\mathbf{j}$

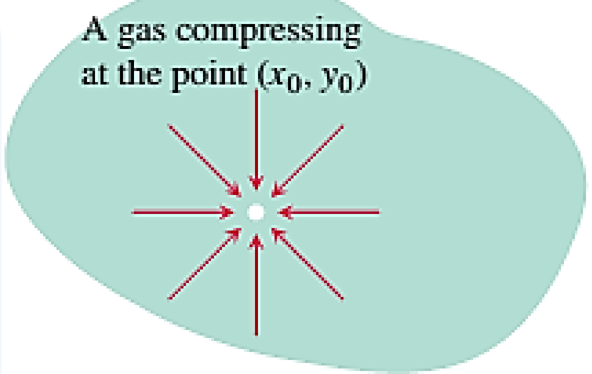
(c) *Shearing flow:* $\mathbf{F}(x, y) = y\mathbf{i}$

(d) *Whirlpool effect:* $\mathbf{F}(x, y) = \dfrac{-y}{x^2 + y^2}\mathbf{i} + \dfrac{x}{x^2 + y^2}\mathbf{j}$

Source: div $\mathbf{F}(x_0, y_0) > 0$



Sink: div $\mathbf{F}(x_0, y_0) < 0$



Green's Theorem in the Plane

Divergence

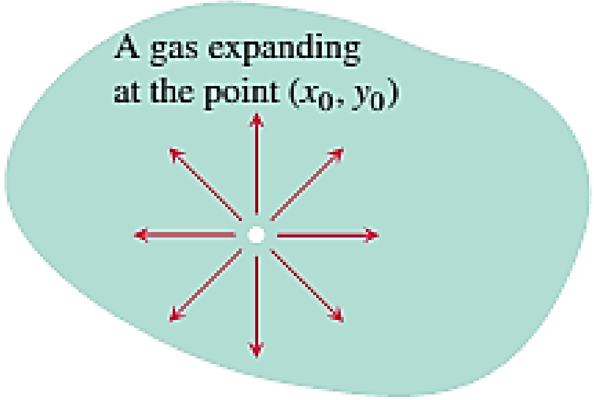
(a) div $\mathbf{F} = \frac{\partial}{\partial x}(cx) + \frac{\partial}{\partial y}(cy) = 2c$: If $c > 0$, the gas is undergoing uniform expansion; if $c < 0$, it is undergoing uniform compression.

(b) div $\mathbf{F} = \frac{\partial}{\partial x}(-cy) + \frac{\partial}{\partial y}(cx) = 0$: The gas is neither expanding nor compressing.

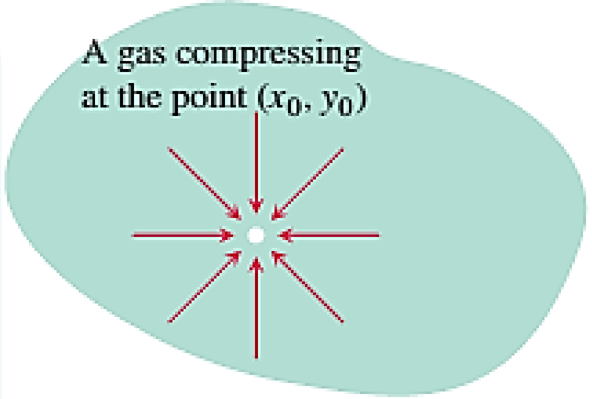
(c) div $\mathbf{F} = \frac{\partial}{\partial x}(y) = 0$: The gas is neither expanding nor compressing.

(d) div $\mathbf{F} = \frac{\partial}{\partial x}\left(\frac{-y}{x^2 + y^2}\right) + \frac{\partial}{\partial y}\left(\frac{x}{x^2 + y^2}\right) = \frac{2xy}{(x^2 + y^2)^2} - \frac{2xy}{(x^2 + y^2)^2} = 0$: Again, the divergence is zero at all points in the domain of the velocity field. ■

Source: div $\mathbf{F}(x_0, y_0) > 0$



Sink: div $\mathbf{F}(x_0, y_0) < 0$



Green's Theorem in the Plane

Two Forms for Green's Theorem

THEOREM 4—Green's Theorem (Circulation-Curl or Tangential Form)
Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field with M and N having continuous first partial derivatives in an open region containing R. Then the counterclockwise circulation of $\mathbf{F}$ around C equals the double integral of $(\text{curl } \mathbf{F}) \cdot \mathbf{k}$ over R.

$$\underbrace{\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \oint_C M\, dx + N\, dy}_{\text{Counterclockwise circulation}} = \underbrace{\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy}_{\text{Curl integral}} \qquad (3)$$

Green's Theorem in the Plane

Two Forms for Green's Theorem

THEOREM 5—Green's Theorem (Flux-Divergence or Normal Form)
Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field with M and N having continuous first partial derivatives in an open region containing R. Then the outward flux of $\mathbf{F}$ across C equals the double integral of div $\mathbf{F}$ over the region R enclosed by C.

$$\underbrace{\oint_C \mathbf{F}\cdot\mathbf{n}\,ds}_{\text{Outward flux}} = \oint_C M\,dy - N\,dx = \underbrace{\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy}_{\text{Divergence integral}} \qquad (4)$$

Green’s Theorem in the Plane

Two Forms for Green’s Theorem

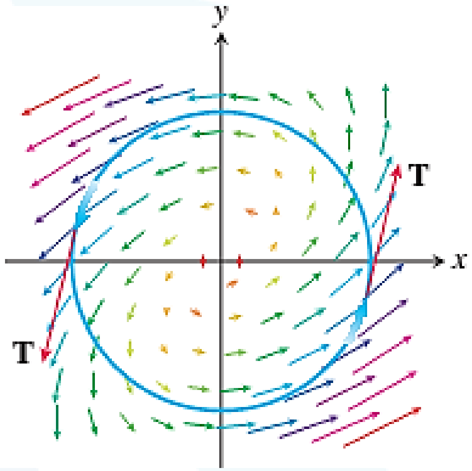
EXAMPLE 3 Verify both forms of Green’s Theorem for the vector field $\mathbf{F}(x, y) = (x - y)\mathbf{i} + x\mathbf{j}$

and the region R bounded by the unit circle $C: \quad \mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}, \quad 0 \le t \le 2\pi.$

$$M = x - y = \cos t - \sin t, \qquad dx = d(\cos t) = -\sin t\, dt,$$

$$N = x = \cos t, \qquad dy = d(\sin t) = \cos t\, dt.$$

$$\oint_C \mathbf{F}\cdot\mathbf{T}\, ds = \oint_C M\, dx + N\, dy = \int_{t=0}^{t=2\pi} (\cos t - \sin t)(-\sin t)\, dt + (\cos t)(\cos t)\, dt = 2\pi$$



$$\frac{\partial M}{\partial x} = 1, \qquad \frac{\partial M}{\partial y} = -1, \qquad \frac{\partial N}{\partial x} = 1, \qquad \frac{\partial N}{\partial y} = 0.$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \iint_R (1 - (-1))\, dx\, dy = 2\iint_R dx\, dy = 2(\text{area inside the unit circle}) = 2\pi$$

Green's Theorem in the Plane

Two Forms for Green's Theorem

$$\oint_C M\,dy - N\,dx = \int_{t=0}^{t=2\pi} (\cos t - \sin t)(\cos t\,dt) - (\cos t)(-\sin t\,dt) = \int_0^{2\pi} \cos^2 t\,dt = \pi$$

$$\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy = \iint_R (1 + 0)\,dx\,dy = \iint_R dx\,dy = \pi.$$

EXAMPLE 4 Evaluate the line integral $\oint_C xy\,dy - y^2\,dx,$

where C is the square cut from the first quadrant by the lines $x = 1$ and $y = 1$

1. *With the Tangential Form* Equation (3): Taking $M = -y^2$ and $N = xy$ gives the result:

$$\oint_C -y^2\,dx + xy\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (y - (-2y))\,dx\,dy = \int_0^1\int_0^1 3y\,dx\,dy = \frac{3}{2}.$$

2. *With the Normal Form* Equation (4): Taking $M = xy$, $N = y^2$, gives the same result:

$$\oint_C xy\,dy - y^2\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy = \iint_R (y + 2y)\,dx\,dy = \frac{3}{2}$$

Green's Theorem in the Plane

Two Forms for Green's Theorem

EXAMPLE 5 Calculate the outward flux of the vector field $\mathbf{F}(x, y) = 2e^{xy}\mathbf{i} + y^3\mathbf{j}$ across the square bounded by the lines $x = \pm 1$ and $y = \pm 1$.

$$\text{Flux} = \oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy$$

$$= \int_{-1}^{1}\int_{-1}^{1} \left(2ye^{xy} + 3y^2\right) dx\,dy = 4$$

Exercises

- use Green's Theorem to find the counterclockwise circulation and outward flux for the field $\mathbf{F}$ and curve C.

  $\mathbf{F} = (x - y)\mathbf{i} + (y - x)\mathbf{j}$ C: The square bounded by $x = 0, x = 1, y = 0, y = 1$ 2; 0

- find the work done by $\mathbf{F}$ in moving a particle once counterclockwise around the given curve.

  $\mathbf{F} = 2xy^3\mathbf{i} + 4x^2y^2\mathbf{j}$ C: The boundary of the "triangular" region in the first quadrant enclosed by the x-axis, the line $x = 1$, and the curve $y = x^3$ $\frac{2}{33}$

- Apply Green's Theorem to evaluate the integral $\oint_C (y^2\,dx + x^2\,dy)$ 0

  C: The triangle bounded by $x = 0, x + y = 1, y = 0$