Relations and Their Properties

What is a relation

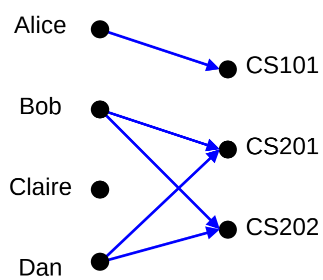
- Let A and B be sets. A binary relation R is a subset of $A \times B$
- Example
  - Let A be the students in a the CS major
    - A = {Alice, Bob, Claire, Dan}
  - Let B be the courses the department offers
    - B = {CS101, CS201, CS202}
  - We specify relation $R = A \times B$ as the set that lists all students $a \in A$ enrolled in class $b \in B$
  - R = { (Alice, CS101), (Bob, CS201), (Bob, CS202), (Dan, CS201), (Dan, CS202) }

More relation examples

- Another relation example:
  - Let *A* be the cities in the US
  - Let *B* be the states in the US
  - We define *R* to mean *a* is a city in state *b*
  - Thus, the following are in our relation:
    - (C'ville, VA)
    - (Philadelphia, PA)
    - (Portland, MA)
    - (Portland, OR)
    - etc…
- Most relations we will see deal with ordered pairs of integers

Representing relations

We can represent relations graphically:



We can represent relations in a table:

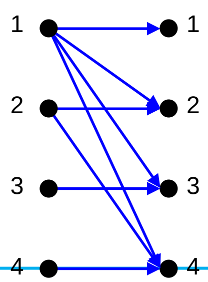
	CS101	CS201	CS202
Alice	X		
Bob		X	X
Claire			
Dan		X	X

Relations on a set

- A relation on the set *A* is a relation from *A* to *A*
  - In other words, the domain and co-domain are the same set
  - We will generally be studying relations of this type

Relations on a set

- Let *A* be the set { 1, 2, 3, 4 }
- Which ordered pairs are in the relation *R* = { (*a*,*b*) | *a* divides *b* }
- R = { (1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4) }



R	1	2	3	4
1	X	X	X	X
2		X		X
3			X	
4				X

More examples

- Consider some relations on the set **Z**
- Are the following ordered pairs in the relation?

	(1,1)	(1,2)	(2,1)	(1,-1)	(2,2)
• $R_1 = \{ (a,b) \mid a \leq b \}$	X	X			X
• $R_2 = \{ (a,b) \mid a > b \}$			X	X	
• $R_3 = \{ (a,b) \mid a = \lvert b \rvert \}$	X			X	X
• $R_4 = \{ (a,b) \mid a = b \}$	X				X
• $R_5 = \{ (a,b) \mid a = b+1 \}$			X		
• $R_6 = \{ (a,b) \mid a+b \leq 3 \}$	X	X	X	X	

Relation properties

- Six properties of relations we will study:
  - انعكاسية Reflexive
  - لاانعكاسيةIrreflexive
  - متماثلةSymmetric
  - غير متماثلةAsymmetric
  - ضد متماثلةAntisymmetric
  - متعدية Transitive

Reflexivity

- A relation is reflexive if every element is related to itself
  - Or, $(a,a) \in R$
- Examples of reflexive relations:
  - =, ≤, ≥
- Examples of relations that are not reflexive:
  - <, >

Irreflexivity

- A relation is irreflexive if every element is *not* related to itself
  - Or, $(a,a) \notin R$
  - Irreflexivity is the opposite of reflexivity
- Examples of irreflexive relations:
  - <, >
- Examples of relations that are not irreflexive:
  - =, ≤, ≥

Reflexivity vs. Irreflexivity

- A relation can be neither reflexive nor irreflexive
  - Some elements are related to themselves, others are not
- We will see an example of this later on

Symmetry

- A relation is symmetric if, for every $(a,b)\in R$, then $(b,a)\in R$
- Examples of symmetric relations:
  - =, isTwinOf()
- Examples of relations that are not symmetric:
  - $<, >, \leq, \geq$

Asymmetry

- A relation is asymmetric if, for every $(a,b) \in R$, then $(b,a) \notin R$
  - Asymmetry is the opposite of symmetry
- Examples of asymmetric relations:
  - <, >
- Examples of relations that are not asymmetric:
  - =, isTwinOf(), ≤, ≥

Antisymmetry

- A relation is antisymmetric if, for every $(a,b) \in R$, then $(b,a) \in R$ is true only when $a=b$
  - Antisymmetry is *not* the opposite of symmetry
- Examples of antisymmetric relations:
  - =, ≤, ≥
- Examples of relations that are not antisymmetric:
  - <, >, isTwinOf()

Notes on *symmetric relations

- A relation can be neither symmetric or asymmetric
  - R = { (a,b) | a=|b| }
  - This is not symmetric
    - -4 is not related to itself
  - This is not asymmetric
    - 4 is related to itself
  - Note that it is antisymmetric

Transitivity

- A relation is transitive if, for every $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$
- If $a < b$ and $b < c$, then $a < c$
  - Thus, < is transitive
- If $a = b$ and $b = c$, then $a = c$
  - Thus, = is transitive

Transitivity examples

- Consider isAncestorOf()
  - Let Alice be Bob's parent, and Bob be Claire's parent
  - Thus, Alice is an ancestor of Bob, and Bob is an ancestor of Claire
  - Thus, Alice is an ancestor of Claire
  - Thus, isAncestorOf() is a transitive relation
- Consider isParentOf()
  - Let Alice be Bob's parent, and Bob be Claire's parent
  - Thus, Alice is a parent of Bob, and Bob is a parent of Claire
  - However, Alice is *not* a parent of Claire
  - Thus, isParentOf() is *not* a transitive relation

Relations of relations summary

	=	<	>	≤	≥
Reflexive	X			X	X
Irreflexive		X	X		
Symmetric	X				
Asymmetric		X	X		
Antisymmetric	X			X	X
Transitive	X	X	X	X	X

Combining relations

- There are two ways to combine relations R_1 and R_2
  - Via Boolean operators
  - Via relation "composition"

Combining relations via Boolean operators

- Consider two relations $R_{\geq}$ and $R_{\leq}$
- We can combine them as follows:
  - $R_{\geq} \cup R_{\leq}$ = all numbers ≥ OR ≤
    - That's all the numbers
  - $R_{\geq} \cap R_{\leq}$ = all numbers ≥ AND ≤
    - That's all numbers equal to
  - $R_{\geq} \oplus R_{\leq}$ = all numbers ≥ or ≤, but not both
    - That's all numbers not equal to
  - $R_{\geq} - R_{\leq}$ = all numbers ≥ that are not also ≤
    - That's all numbers strictly greater than
  - $R_{\leq} - R_{\geq}$ = all numbers ≤ that are not also ≥
    - That's all numbers strictly less than
- Note that it's possible the result is the empty set

Combining relations via relational composition

- Let R be a relation from A to B, and S be a relation from B to C
  - Let $a \in A$, $b \in B$, and $c \in C$
  - Let $(a,b) \in R$, and $(b,c) \in S$
  - Then the composite of R and S consists of the ordered pairs (a,c)
    - We denote the relation by $S \circ R$
    - Note that S comes first when writing the composition!

Combining relations via relational composition

- Let M be the relation "is mother of"
- Let F be the relation "is father of"
- What is $M \circ F$?
  - If $(a,b) \in F$, then a is the father of b
  - If $(b,c) \in M$, then b is the mother of c
  - Thus, $M \circ F$ denotes the relation "maternal grandfather"
- What is $F \circ M$?
  - If $(a,b) \in M$, then a is the mother of b
  - If $(b,c) \in F$, then b is the father of c
  - Thus, $F \circ M$ denotes the relation "paternal grandmother"
- What is $M \circ M$?
  - If $(a,b) \in M$, then a is the mother of b
  - If $(b,c) \in M$, then b is the mother of c
  - Thus, $M \circ M$ denotes the relation "maternal grandmother"
- Note that M and F are not transitive relations!!!

Combining relations via relational composition

- Given relation R
  - $R \circ R$ can be denoted by R^2
  - $R^2 \circ R = (R \circ R) \circ R = R^3$
  - Example: M^3 is your mother's mother's mother

Representing Relations

In this slide set…

- Matrix review
- Two ways to represent relations
  - Via matrices
  - Via directed graphs

Matrix review

- We will only be dealing with zero-one matrices
  - Each element in the matrix is either a 0 or a 1

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

- These matrices will be used for Boolean operations
  - 1 is true, 0 is false

Matrix transposition

- Given a matrix **M**, the transposition of **M**, denoted **M**^{t}, is the matrix obtained by switching the columns and rows of **M**

$$\mathbf{M} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$\mathbf{M}^t = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

$$\mathbf{M}^t = \begin{bmatrix} 1 & 5 & 9 & 13 \\ 2 & 6 & 10 & 14 \\ 3 & 7 & 11 & 15 \\ 4 & 8 & 12 & 16 \end{bmatrix}$$

- In a “square” matrix, the main diagonal stays unchanged

Matrix join

- A *join* of two matrices performs a Boolean OR on each relative entry of the matrices
  – Matrices must be the same size
  – Denoted by the or symbol: $\vee$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Matrix meet

- A *meet* of two matrices performs a Boolean AND on each relative entry of the matrices
  - Matrices must be the same size
  - Denoted by the or symbol: $\wedge$

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\1&0&1&0\\1&0&1&0\end{bmatrix}\wedge\begin{bmatrix}0&1&1&0\\0&1&1&0\\0&0&1&1\\1&1&0&0\end{bmatrix}=\begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\1&0&0&0\end{bmatrix}$$

Matrix Boolean product

- A *Boolean product* of two matrices is similar to matrix multiplication

$$c_{1,1} = a_{1,1} * b_{1,1} + a_{1,2} * b_{2,1} + a_{1,3} * b_{3,1} + a_{1,4} * b_{4,1}$$

- Instead of the sum of the products, it's the conjunction (and) of the disjunctions (ors)

$$c_{1,1} = a_{1,1} \wedge b_{1,1} \vee a_{1,2} \wedge b_{2,1} \vee a_{1,3} \wedge b_{3,1} \vee a_{1,4} \wedge b_{4,1}$$

- Denoted by the or symbol:

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\1&0&1&0\\1&0&1&0\end{bmatrix}\odot\begin{bmatrix}0&1&1&0\\0&1&1&0\\0&0&1&1\\1&1&0&0\end{bmatrix}=\begin{bmatrix}0&1&1&0\\0&1&1&0\\0&1&1&1\\0&1&1&1\end{bmatrix}$$

Relations using matrices

- List the elements of sets *A* and *B* in a particular order
  – Order doesn't matter, but we'll generally use ascending order
- Create a matrix

$$\mathbf{M}_R = [m_{ij}]$$

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

Relations using matrices

- Consider the relation of who is enrolled in which class
  – Let A = { Alice, Bob, Claire, Dan }
  – Let B = { CS101, CS201, CS202 }
  – R = { (*a,b*) | person *a* is enrolled in course *b* }

	CS101	CS201	CS202
Alice	X		
Bob		X	X
Claire			
Dan		X	X

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Relations using matrices

- What is it good for?
  - It is how computers view relations
    - A 2-dimensional array
  - Very easy to view relationship properties
- We will generally consider relations on a single set
  - In other words, the domain and co-domain are the same set
  - And the matrix is square

Reflexivity

- Consider a reflexive relation: ≤
  - One which every element is related to itself
  - Let A = { 1, 2, 3, 4, 5 }

$$M_{\leq} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

If the center (main) diagonal is all 1's, a relation is reflexive

Irreflexivity

- Consider a reflexive relation: <
  - One which every element is *not* related to itself
  - Let A = { 1, 2, 3, 4, 5 }

$$\mathbf{M}_{\leq} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

If the center (main) diagonal is all 0's, a relation is irreflexive

Symmetry

- Consider an symmetric relation *R*
  - One which if *a* is related to *b* then *b* is related to *a* for all (*a*,*b*)
  - Let A = { 1, 2, 3, 4, 5 }

$$\mathbf{M}_{\leq} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

- If, for *every* value, it is the equal to the value in its transposed position, then the relation is symmetric

Asymmetry

- Consider an asymmetric relation: $<$
  - One which if *a* is related to *b* then *b* is *not* related to *a* for all (*a,b*)
  - Let A = { 1, 2, 3, 4, 5 }

$$\mathbf{M}_{\le} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- If, for every value and the value in its transposed position, if they are not both 1, then the relation is asymmetric
- An asymmetric relation must also be irreflexive
- Thus, the main diagonal must be all 0's

Antisymmetry

- Consider an antisymmetric relation: ≤
  - One which if *a* is related to *b* then *b* is *not* related to *a* unless *a*=*b* for all (*a,b*)
  - Let A = { 1, 2, 3, 4, 5 }

$$\mathbf{M}_{\le} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- If, for every value and the value in its transposed position, if they are not both 1, then the relation is antisymmetric
- The center diagonal can have both 1's and 0's

Transitivity

- Consider an transitive relation: ≤
  - One which if *a* is related to *b* and *b* is related to *c* then *a* is related to *c* for all (*a,b*), (*b,c*) and (*a,c*)
  - Let A = { 1, 2, 3, 4, 5 }

$$\mathbf{M}_{\leq} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- If, for every spot (*a,b*) and (*b,c*) that each have a 1, there is a 1 at (*a,c*), then the relation is transitive
- Matrices don't show this property easily

Combining relations: via Boolean operators

- Let: $\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $\mathbf{M}_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$
- Join: $\mathbf{M}_{R \cup S} = \mathbf{M}_R \vee \mathbf{M}_S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$
- Meet: $\mathbf{M}_{R \cap S} = \mathbf{M}_R \wedge \mathbf{M}_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Combining relations: via relation composition

- Let:

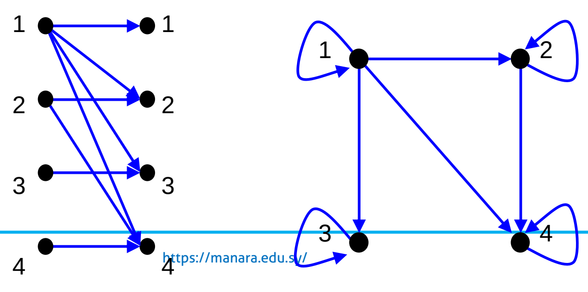
$$\mathbf{M}_R = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} d & e & f \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array} \qquad \mathbf{M}_S = \begin{array}{c} \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{ccc} g & h & i \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

- But why is this the case?

$$\mathbf{M}_{S \circ R} = \mathbf{M}_R \odot \mathbf{M}_S = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} g & h & i \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

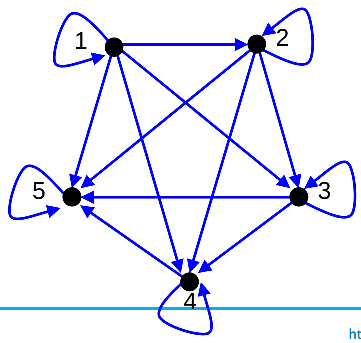
Representing relations using directed graphs

- A directed graph consists of:
  - A set *V* of vertices (or nodes)
  - A set *E* of edges (or arcs)
  - If (*a*, *b*) is in the relation, then there is an arrow from *a* to *b*
- Will generally use relations on a single set
- Consider our relation *R* = { (*a*,*b*) | *a* divides *b* }
- Old way:



Reflexivity

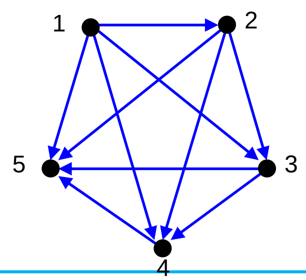
- Consider a reflexive relation: ≤
  - One which every element is related to itself
  - Let A = { 1, 2, 3, 4, 5 }



If every node has a loop, a relation is reflexive

Irreflexivity

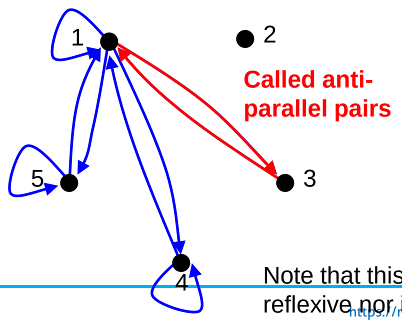
- Consider a reflexive relation: <
  - One which every element is *not* related to itself
  - Let A = { 1, 2, 3, 4, 5 }



If every node does *not* have a loop, a relation is irreflexive

Symmetry

- Consider an symmetric relation *R*
  - One which if *a* is related to *b* then *b* is related to *a* for all (*a,b*)
  - Let A = { 1, 2, 3, 4, 5 }

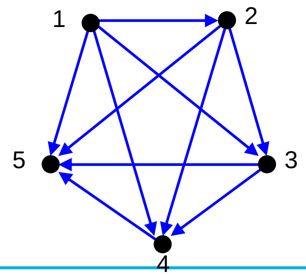


- If, for every edge, there is an edge in the other direction, then the relation is symmetric
- Loops are allowed, and do not need edges in the "other" direction

Note that this relation is neither reflexive nor irreflexive!

Asymmetry

- Consider an asymmetric relation: <
  - One which if *a* is related to *b* then *b* is *not* related to *a* for all (*a,b*)
  - Let A = { 1, 2, 3, 4, 5 }

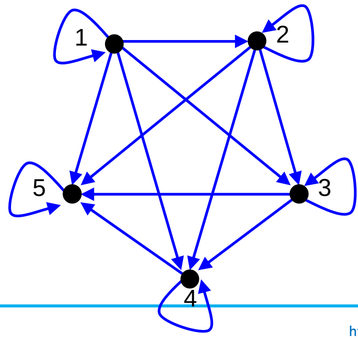


- A digraph is asymmetric if:

1. If, for every edge, there is *not* an edge in the other direction, then the relation is asymmetric
2. Loops are *not* allowed in an asymmetric digraph (recall it must be irreflexive)

Antisymmetry

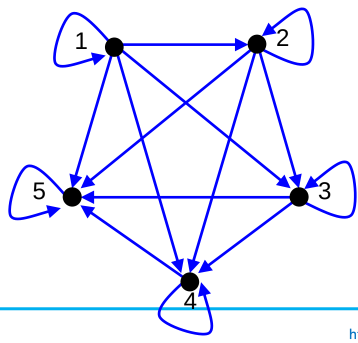
- Consider an antisymmetric relation: ≤
  – One which if *a* is related to *b* then *b* is *not* related to *a* unless *a*=*b* for all (*a,b*)
  – Let A = { 1, 2, 3, 4, 5 }



- If, for every edge, there is *not* an edge in the other direction, then the relation is antisymmetric
- Loops are allowed in the digraph

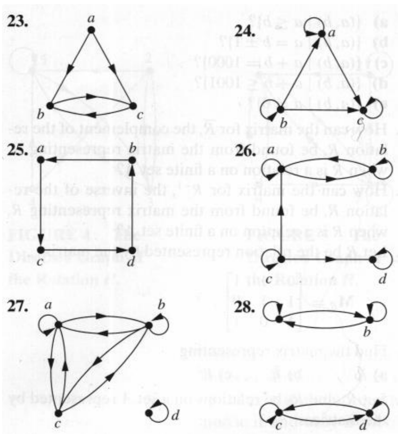
Transitivity

- Consider an transitive relation: ≤
  – One which if *a* is related to *b* and *b* is related to *c* then *a* is related to *c* for all (*a,b*), (*b,c*) and (*a,c*)
  – Let A = { 1, 2, 3, 4, 5 }



- A digraph is transitive if, for there is a edge from *a* to *c* when there is a edge from *a* to *b* and from *b* to *c*

Sample questions



Which of the graphs are reflexive, irreflexive, symmetric, asymmetric, antisymmetric, or transitive

	23	24	25	26	27	28
Reflexive		Y		Y		Y
Irreflexive	Y		Y			
Symmetric					Y	Y
Asymmetric			Y			
Anti-symmetric		Y	Y			
Transitive						Y

Equivalence Relations

Introduction

- Certain combinations of relation properties are very useful
  - We won't have a chance to see many applications in this course
- In this set we will study equivalence relations
  - A relation that is reflexive, symmetric and transitive
- Next slide set we will study partial orderings
  - A relation that is reflexive, antisymmetric, and transitive
- The difference is whether the relation is symmetric or antisymmetric

Equivalence relations

- A relation on a set *A* is called an *equivalence relation* if it is reflexive, symmetric, and transitive
- Consider relation $R = \{ (a,b) \mid len(a) = len(b) \}$
  - Where *len*(*a*) means the length of string *a*
  - It is reflexive: $len(a) = len(a)$
  - It is symmetric: if $len(a) = len(b)$, then $len(b) = len(a)$
  - It is transitive: if $len(a) = len(b)$ and $len(b) = len(c)$, then $len(a) = len(c)$
  - Thus, *R* is a equivalence relation

Equivalence relation example

- Consider the relation $R = \{ (a,b) \mid m \mid a\text{-}b \}$
  - Called "congruence modulo m"
- Is it reflexive: $(a,a) \in R$ means that $m \mid a\text{-}a$
  - $a\text{-}a = 0$, which is divisible by m
- Is it symmetric: if $(a,b) \in R$ then $(b,a) \in R$
  - (a,b) means that $m \mid a\text{-}b$
  - Or that $km = a\text{-}b$. Negating that, we get $b\text{-}a = -km$
  - Thus, $m \mid b\text{-}a$, so $(b,a) \in R$
- Is it transitive: if $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$
  - (a,b) means that $m \mid a\text{-}b$, or that $km = a\text{-}b$
  - (b,c) means that $m \mid b\text{-}c$, or that $lm = b\text{-}c$
  - (a,c) means that $m \mid a\text{-}c$, or that $nm = a\text{-}c$
  - Adding these two, we get $km+lm = (a\text{-}b) + (b\text{-}c)$
  - Or $(k+l)m = a\text{-}c$
  - Thus, m divides $a\text{-}c$, where $n = k+l$
- Thus, congruence modulo m is an equivalence relation

Sample questions

- Which of these relations on {0, 1, 2, 3} are equivalence relations? Determine the properties of an equivalence relation that the others lack

a) { (0,0), (1,1), (2,2), (3,3) }
- Has all the properties, thus, is an equivalence relation

b) { (0,0), (0,2), (2,0), (2,2), (2,3), (3,2), (3,3) }
- Not reflexive: (1,1) is missing
- Not transitive: (0,2) and (2,3) are in the relation, but not (0,3)

c) { (0,0), (1,1), (1,2), (2,1), (2,2), (3,3) }
- Has all the properties, thus, is an equivalence relation

d) { (0,0), (1,1), (1,3), (2,2), (2,3), (3,1), (3,2) (3,3) }
- Not transitive: (1,3) and (3,2) are in the relation, but not (1,2)

e) { (0,0), (0,1) (0,2), (1,0), (1,1), (1,2), (2,0), (2,2), (3,3) }
- Not symmetric: (1,2) is present, but not (2,1)
- Not transitive: (2,0) and (0,1) are in the relation, but not (2,1)