جامعة
المنارة
MANARA UNIVERSITY

MATHEMATICAL ANALAYSIS 2

Lecture 8

Prepared by

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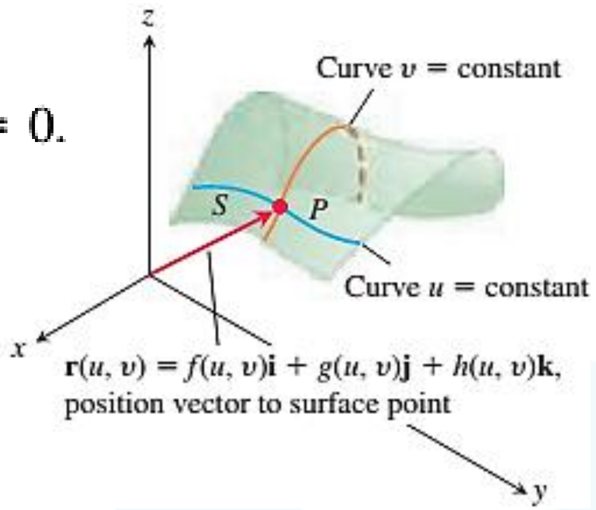
Surfaces and Area

Parametrizations of Surfaces

Explicit form: $z = f(x, y)$ Implicit form: $F(x, y, z) = 0.$

$$\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}$$

$$x = f(u, v), \quad y = g(u, v), \quad z = h(u, v). \quad a \le u \le b \quad c \le v \le d$$

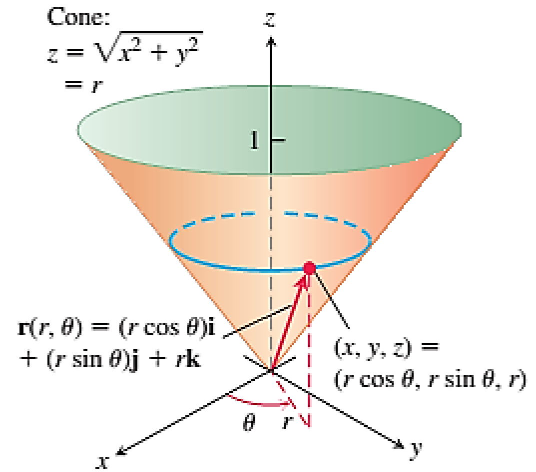


EXAMPLE 1 Find a parametrization of the cone

$$z = \sqrt{x^2 + y^2}, \qquad 0 \le z \le 1.$$

$x = r\cos\theta$, $y = r\sin\theta$, and $z = \sqrt{x^2 + y^2} = r$ $\quad 0 \le r \le 1$ and $0 \le \theta \le 2\pi.$

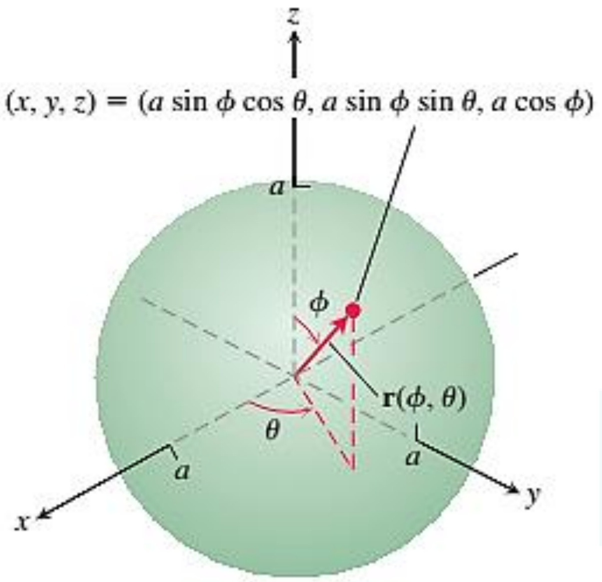
$$\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}, \qquad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi.$$



Surfaces and Area

Parametrizations of Surfaces

EXAMPLE 2 Find a parametrization of the sphere $x^2 + y^2 + z^2 = a^2$.



$$x = a \sin\phi \cos\theta, \quad y = a \sin\phi \sin\theta, \quad \text{and} \quad z = a\cos\phi, \; 0 \le \phi \le \pi, \; 0 \le \theta \le 2\pi.$$

$$\mathbf{r}(\phi, \theta) = (a \sin\phi \cos\theta)\mathbf{i} + (a \sin\phi \sin\theta)\mathbf{j} + (a \cos\phi)\mathbf{k}, \quad 0 \le \phi \le \pi, \; 0 \le \theta \le 2\pi.$$

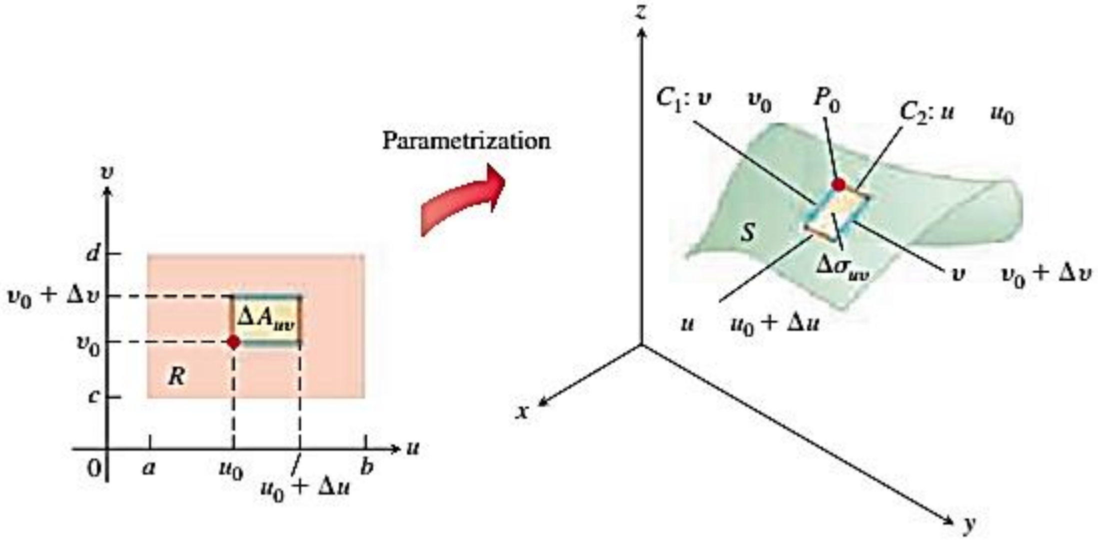
Surface Area

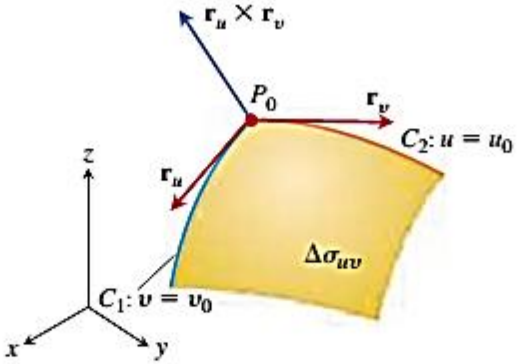
DEFINITION A parametrized surface $\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}$ is **smooth** if $\mathbf{r}_u$ and $\mathbf{r}_v$ are continuous and $\mathbf{r}_u \times \mathbf{r}_v$ is never zero on the interior of the parameter domain.

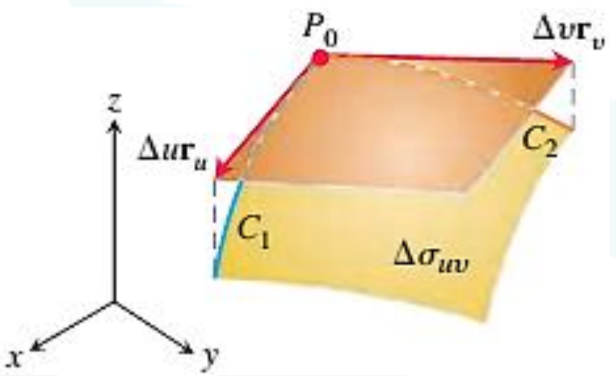
$$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u} = \frac{\partial f}{\partial u}\mathbf{i} + \frac{\partial g}{\partial u}\mathbf{j} + \frac{\partial h}{\partial u}\mathbf{k} \qquad \mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v} = \frac{\partial f}{\partial v}\mathbf{i} + \frac{\partial g}{\partial v}\mathbf{j} + \frac{\partial h}{\partial v}\mathbf{k}.$$

Surfaces and Area

Surface Area







$$|\Delta u\,\mathbf{r}_u \times \Delta v\,\mathbf{r}_v| = |\mathbf{r}_u \times \mathbf{r}_v|\,\Delta u\,\Delta v.$$

$$\sum_n |\mathbf{r}_u \times \mathbf{r}_v|\,\Delta u\,\Delta v.$$

Surfaces and Area
Surface Area

DEFINITION The **area** of the smooth surface

$$\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}, \qquad a \le u \le b, \quad c \le v \le d$$

is

$$A = \iint_R |\mathbf{r}_u \times \mathbf{r}_v|\, dA = \int_c^d \int_a^b |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv. \tag{4}$$

Surface Area Differential for a Parametrized Surface

$$d\sigma = |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv \qquad \iint_S d\sigma \tag{5}$$

Surface area differential, also called surface area element

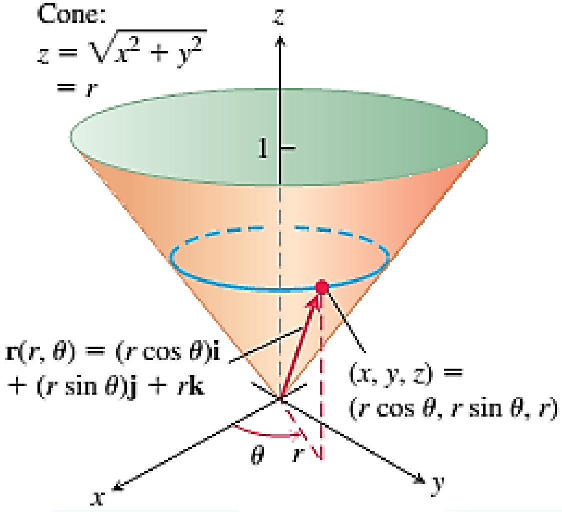
Differential formula for surface area

Surfaces and Area

Surface Area

EXAMPLE 4 Find the surface area of the cone

$$\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}, \qquad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi.$$



$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = -(r\cos\theta)\mathbf{i} - (r\sin\theta)\mathbf{j} + \underbrace{(r\cos^2\theta + r\sin^2\theta)}_{r}\mathbf{k}.$$

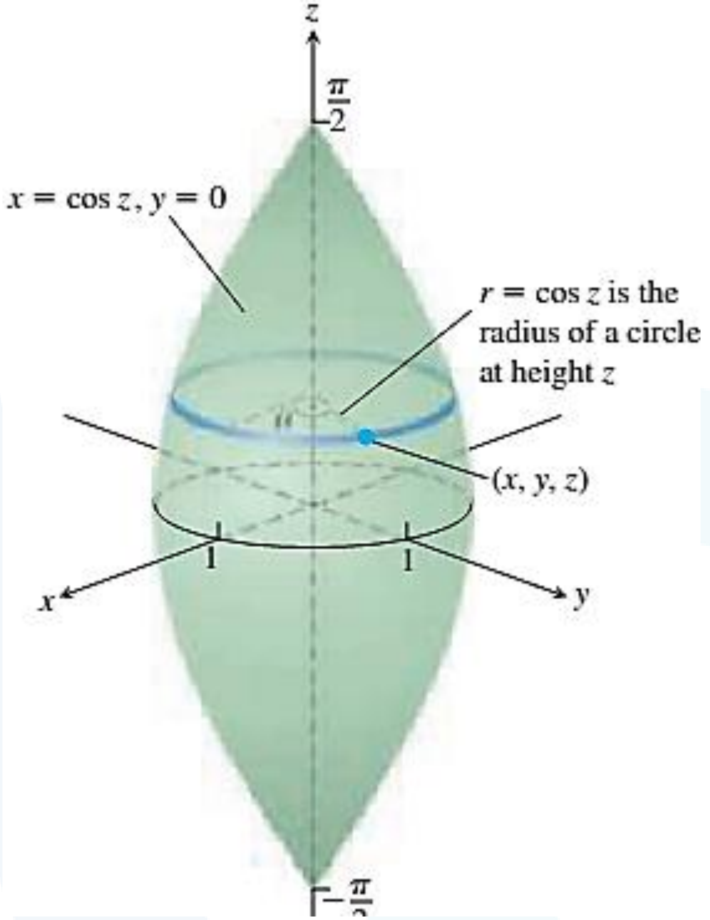
$$|\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} = \sqrt{2r^2} = \sqrt{2}r.$$

$$A = \int_0^{2\pi}\int_0^1 |\mathbf{r}_r \times \mathbf{r}_\theta|\, dr\, d\theta = \int_0^{2\pi}\int_0^1 \sqrt{2}\, r\, dr\, d\theta = \pi\sqrt{2} \text{ square units.}$$

Surfaces and Area

Surface Area

EXAMPLE 6 Let S be the "football" surface formed by rotating the curve $x = \cos z$, $y = 0, -\pi/2 \le z \le \pi/2$ around the z-axis (see Figure 16.46). Find a parametrization for S and compute its surface area.



Solution

$r = \cos z$ $\quad u = z$ and $v = \theta$. $\quad 0 \le \theta \le 2\pi$

$x = r\cos\theta = \cos u \cos v,\ \ y = r\sin\theta = \cos u \sin v,\ z = u$

$$\mathbf{r}(u, v) = \cos u \cos v\mathbf{i} + \cos u \sin v\mathbf{j} + u\mathbf{k},\quad -\frac{\pi}{2} \le u \le \frac{\pi}{2},\quad 0 \le v \le 2\pi.$$

$\mathbf{r}_u = -\sin u \cos v\,\mathbf{i} - \sin u \sin v\,\mathbf{j} + \mathbf{k} \qquad \mathbf{r}_v = -\cos u \sin v\,\mathbf{i} + \cos u \cos v\,\mathbf{j}.$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin u \cos v & -\sin u \sin v & 1 \\ -\cos u \sin v & \cos u \cos v & 0 \end{vmatrix}$$

$$= -\cos u \cos v\,\mathbf{i} - \cos u \sin v\,\mathbf{j} - (\sin u \cos u \cos^2 v + \cos u \sin u \sin^2 v)\mathbf{k}$$

$$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{\cos^2 u\,(\cos^2 v + \sin^2 v) + \sin^2 u \cos^2 u} = \cos u\sqrt{1 + \sin^2 u}.$$ $\quad \cos u \ge 0$ for $-\frac{\pi}{2} \le u \le \frac{\pi}{2}$

Surfaces and Area

Surface Area

$$A = \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \cos u \sqrt{1 + \sin^2 u}\, du\, dv.$$

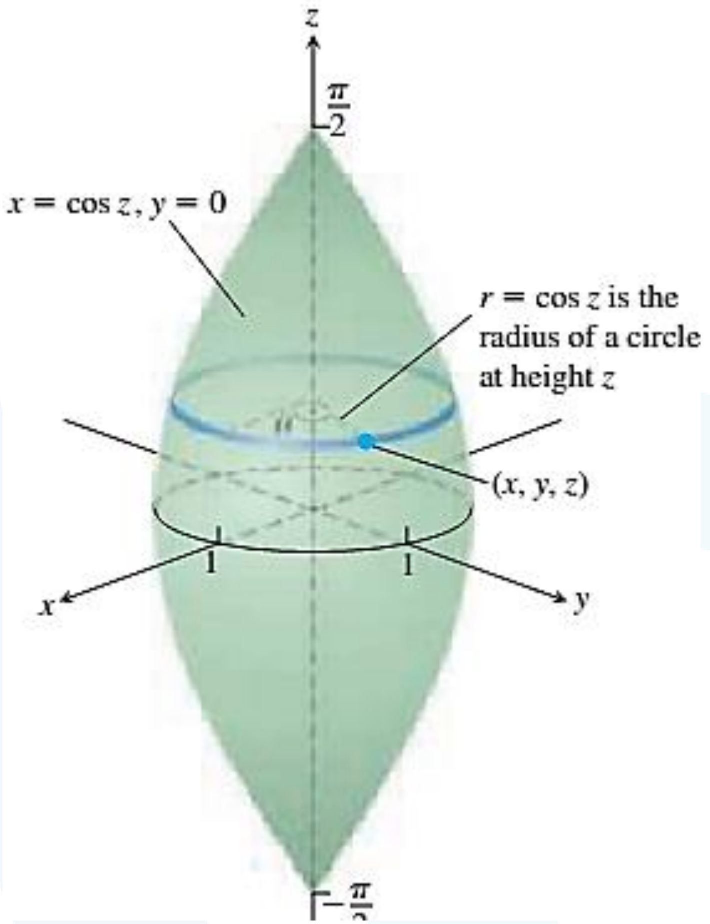
$$w = \sin u \quad \text{and} \quad dw = \cos u\, du, \quad -1 \le w \le 1$$

Since the surface S is symmetric across the xy-plane,

$$A = 2\int_0^{2\pi} \int_0^{1} \sqrt{1 + w^2}\, dw\, dv$$

$$= 2\int_0^{2\pi} \left[\frac{w}{2}\sqrt{1 + w^2} + \frac{1}{2}\ln\left(w + \sqrt{1 + w^2}\right) \right]_0^1 dv$$

$$= 2\pi\left[\sqrt{2} + \ln\left(1 + \sqrt{2}\right) \right]$$



Surfaces and Area

Implicit Surfaces

$u = x$ and $v = y$. Then $z = h(u, v)$

$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + h(u, v)\mathbf{k}$

$$\mathbf{r}_u = \mathbf{i} + \frac{\partial h}{\partial u}\mathbf{k} \quad \text{and} \quad \mathbf{r}_v = \mathbf{j} + \frac{\partial h}{\partial v}\mathbf{k}.$$

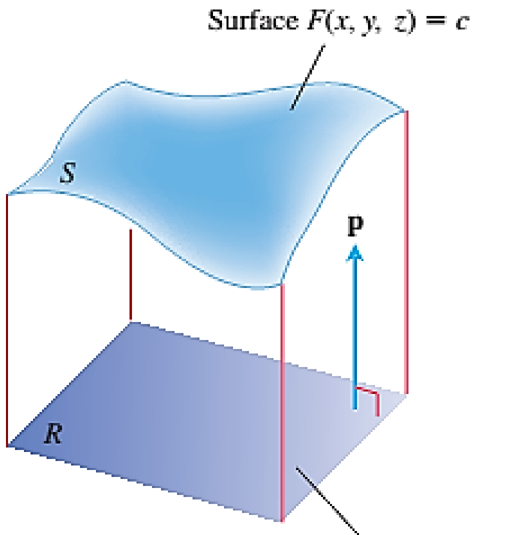
$$\frac{\partial h}{\partial u} = -\frac{F_x}{F_z} \quad \text{and} \quad \frac{\partial h}{\partial v} = -\frac{F_y}{F_z}. \qquad F_z \neq 0$$

$$\mathbf{r}_u = \mathbf{i} - \frac{F_x}{F_z}\mathbf{k} \quad \text{and} \quad \mathbf{r}_v = \mathbf{j} - \frac{F_y}{F_z}\mathbf{k}.$$

$$\mathbf{r}_u \times \mathbf{r}_v = \frac{F_x}{F_z}\mathbf{i} + \frac{F_y}{F_z}\mathbf{j} + \mathbf{k} = \frac{1}{F_z}(F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}) = \frac{\nabla F}{\nabla F \cdot \mathbf{k}} = \frac{\nabla F}{\nabla F \cdot \mathbf{p}}.$$

$\mathbf{p} = \mathbf{k}$

$$d\sigma = |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv = \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|}dx\,dy. \qquad u = x \text{ and } v = y$$



Formula for the Surface Area of an Implicit Surface

The area of the surface $F(x, y, z) = c$ over a closed and bounded plane region R is

$$\text{Surface area} = \iint_R \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|}\, dA, \tag{7}$$

where $\mathbf{p} = \mathbf{i}, \mathbf{j}$, or $\mathbf{k}$ is normal to R and $\nabla F \cdot \mathbf{p} \neq 0$.

Surfaces and Area

Implicit Surfaces

EXAMPLE 7 Find the area of the surface cut from the bottom of the paraboloid $x^2 + y^2 - z = 0$ by the plane $z = 4$.

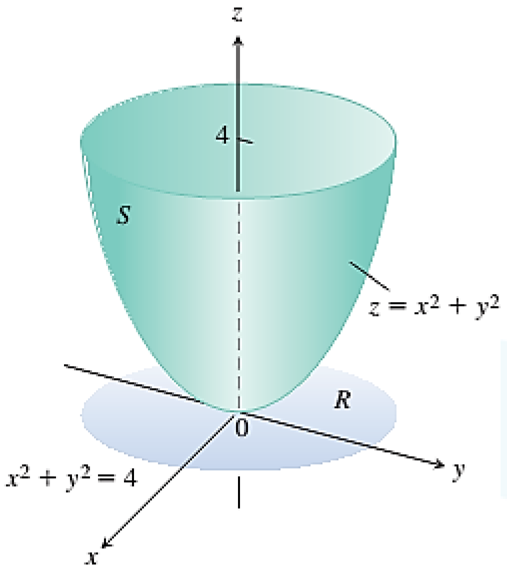
$$F(x, y, z) = x^2 + y^2 - z$$

$$\nabla F = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \quad \Rightarrow \quad |\nabla F| = \sqrt{(2x)^2 + (2y)^2 + (-1)^2} = \sqrt{4x^2 + 4y^2 + 1}$$

$$|\nabla F \cdot \mathbf{p}| = |\nabla F \cdot \mathbf{k}| = |-1| = 1.$$

$$\text{Surface area} = \iint_R \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|}\, dA = \iint_{x^2+y^2 \le 4} \sqrt{4x^2 + 4y^2 + 1}\, dx\, dy$$

$$= \int_0^{2\pi}\int_0^2 \sqrt{4r^2 + 1}\, r\, dr\, d\theta = \frac{\pi}{6}\left(17\sqrt{17} - 1\right)$$



Surfaces and Area

Explicit Surfaces

We parametrize the surface by taking $x = u$, $y = v$, and $z = f(x, y)$ over R. This gives the parametrization

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + f(u, v)\mathbf{k}. \Longrightarrow \quad \mathbf{r}_u = \mathbf{i} + f_u\mathbf{k},\ \mathbf{r}_v = \mathbf{j} + f_v\mathbf{k}$$

$$\mathbf{r}_u \times \mathbf{r}_v = -f_u\mathbf{i} - f_v\mathbf{j} + \mathbf{k}. \quad \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_u \\ 0 & 1 & f_v \end{vmatrix} \Longrightarrow \quad |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv = \sqrt{f_u^2 + f_v^2 + 1}\, du\, dv.$$

$$d\sigma = \sqrt{f_x^2 + f_y^2 + 1}\, dx\, dy.$$

Formula for the Surface Area of a Graph $z = f(x, y)$

For a graph $z = f(x, y)$ over a region R in the xy-plane, the surface area formula is

$$A = \iint_R \sqrt{f_x^2 + f_y^2 + 1}\, dx\, dy. \tag{8}$$

Exercises

- find a parametrization of the surface.

The paraboloid $z = 9 - x^2 - y^2, z \geq 0$

$$\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \left(9 - r^2\right)\mathbf{k} \qquad 0 \leq \theta \leq 2\pi. \qquad 0 \leq r \leq 3$$

- use a parametrization to express the area of the surface as a double integral. Then evaluate the integral

The portion of the plane $y + 2z = 2$ inside the cylinder $x^2 + y^2 = 1$

$$\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \left(\frac{2 - r\sin\theta}{2}\right)\mathbf{k}, \quad 0 \leq r \leq 1 \text{ and } 0 \leq \theta \leq 2\pi$$

$$\frac{\pi\sqrt{5}}{2}$$

- find an equation for the plane tangent to the surface at P_0.

Cone The cone $\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}, r \geq 0, 0 \leq \theta \leq 2\pi$ at the point $P_0(\sqrt{2}, \sqrt{2}, 2)$ corresponding to $(r, \theta) = (2, \pi/4)$

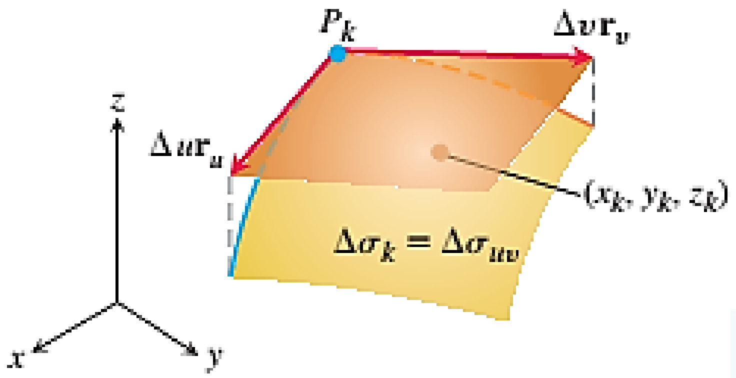
$$\sqrt{2}x + \sqrt{2}y - 2z = 0,$$

- Find the area of the surface cut from the paraboloid $x^2 + y^2 - z = 0$ by the plane $z = 2$. $\frac{13}{3}\pi$

Surface Integrals

$$\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}, \qquad (u, v) \in R.$$

$$\Delta\sigma_{uv} \approx |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv.$$



$$\sum_{k=1}^{n} G(x_k, y_k, z_k)\,\Delta\sigma_k.$$

$$\iint_S G(x, y, z)\, d\sigma = \lim_{n\to\infty} \sum_{k=1}^{n} G(x_k, y_k, z_k)\,\Delta\sigma_k. \qquad (1)$$

Surface Integrals

Formulas for a Surface Integral of a Scalar Function

1. For a smooth surface S defined **parametrically** as $\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}$, $(u, v) \in R$, and a continuous function $G(x, y, z)$ defined on S, the surface integral of G over S is given by the double integral over R,

$$\iint_S G(x, y, z)\, d\sigma = \iint_R G(f(u, v), g(u, v), h(u, v)) \left| \mathbf{r}_u \times \mathbf{r}_v \right| du\, dv. \qquad (2)$$

2. For a surface S given **implicitly** by $F(x, y, z) = c$, where F is a continuously differentiable function, with S lying above its closed and bounded shadow region R in the coordinate plane beneath it, the surface integral of the continuous function G over S is given by the double integral over R,

$$\iint_S G(x, y, z)\, d\sigma = \iint_R G(x, y, z) \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|} dA, \qquad (3)$$

where $\mathbf{p}$ is a unit vector normal to R and $\nabla F \cdot \mathbf{p} \neq 0$.

3. For a surface S given **explicitly** as the graph of $z = f(x, y)$, where f is a continuously differentiable function over a region R in the xy-plane, the surface integral of the continuous function G over S is given by the double integral over R,

$$\iint_S G(x, y, z)\, d\sigma = \iint_R G(x, y, f(x, y)) \sqrt{f_x^{\,2} + f_y^{\,2} + 1}\, dx\, dy. \qquad (4)$$

Surface Integrals

EXAMPLE 1 Integrate $G(x, y, z) = x^2$ over the cone $z = \sqrt{x^2 + y^2}$, $0 \le z \le 1$.

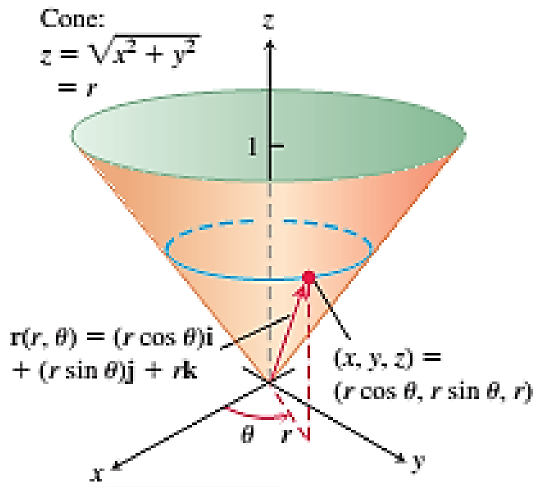
$$\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}, \qquad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi.$$

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = -(r\cos\theta)\mathbf{i} - (r\sin\theta)\mathbf{j} + \underbrace{(r\cos^2\theta + r\sin^2\theta)}_{r}\mathbf{k}$$

$$|\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} = \sqrt{2r^2} = \sqrt{2}r$$

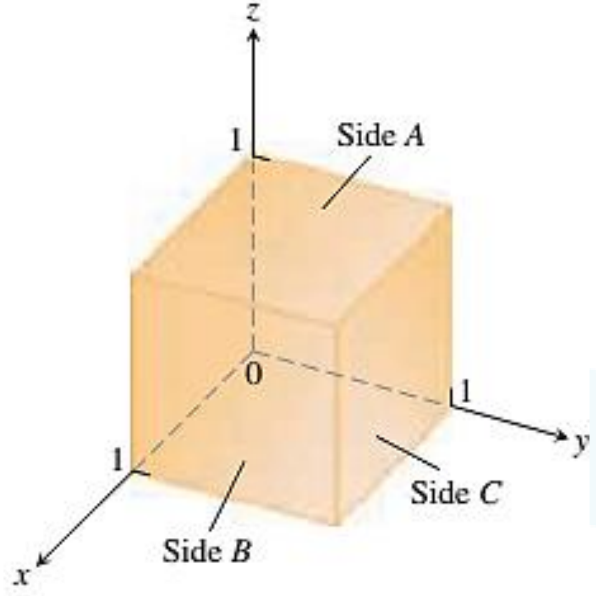
$$\iint_S x^2\, d\sigma = \int_0^{2\pi}\int_0^1 \left(r^2\cos^2\theta\right)\left(\sqrt{2}r\right) dr\, d\theta = \sqrt{2}\int_0^{2\pi}\int_0^1 r^3\cos^2\theta\, dr\, d\theta$$

$$= \frac{\sqrt{2}}{4}\int_0^{2\pi} \cos^2\theta\, d\theta = \frac{\sqrt{2}}{4}\left[\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{\pi\sqrt{2}}{4}.$$



Surface Integrals

$$\iint_S G\,d\sigma = \iint_{S_1} G\,d\sigma + \iint_{S_2} G\,d\sigma + \cdots + \iint_{S_n} G\,d\sigma.$$



EXAMPLE 2 Integrate $G(x, y, z) = xyz$ over the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$ (Figure 16.50).

$$\iint_{\substack{\text{Cube}\\ \text{surface}}} xyz\,d\sigma = \iint_{\text{Side } A} xyz\,d\sigma + \iint_{\text{Side } B} xyz\,d\sigma + \iint_{\text{Side } C} xyz\,d\sigma.$$

Side A is the surface $f(x, y, z) = z = 1$ over the square region R_{xy}: $0 \le x \le 1$, $0 \le y \le 1$, in the xy-plane. For this surface and region,

$$\mathbf{p} = \mathbf{k}, \qquad \nabla f = \mathbf{k}, \qquad |\nabla f| = 1, \qquad |\nabla f \cdot \mathbf{p}| = |\mathbf{k} \cdot \mathbf{k}| = 1$$

$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}\,dA = \frac{1}{1}\,dx\,dy = dx\,dy$$

$$xyz = xy(1) = xy$$

$$\iint_{\text{Side } A} xyz\,d\sigma = \iint_{R_{xy}} xy\,dx\,dy = \int_0^1\int_0^1 xy\,dx\,dy = \int_0^1 \frac{y}{2}\,dy = \frac{1}{4}.$$

Symmetry

$$\iint_{\substack{\text{Cube}\\ \text{surface}}} xyz\,d\sigma = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}.$$

Surface Integrals

EXAMPLE 4 Evaluate $\iint_S \sqrt{x(1+2z)}\, d\sigma$ on the portion of the cylinder $z = y^2/2$ over the triangular region R: $x \ge 0$, $y \ge 0$, $x + y \le 1$ in the xy-plane (Figure 16.51).

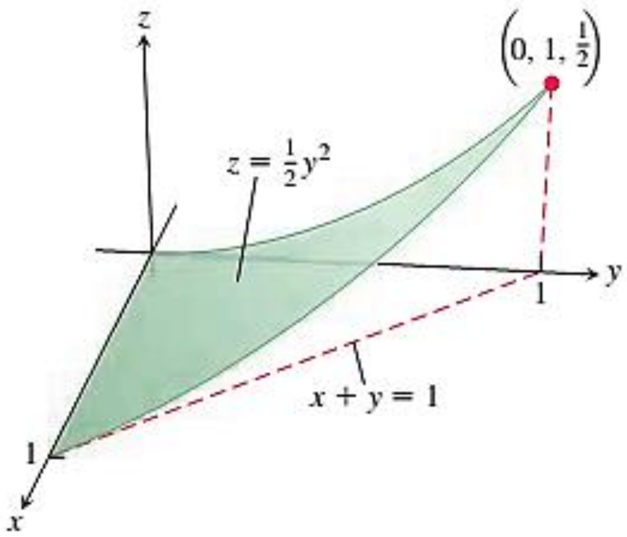
$$G(x, y, z) = \sqrt{x(1+2z)} = \sqrt{x}\sqrt{1+y^2}.$$

With $z = f(x, y) = y^2/2$, we use Equation (4) to evaluate the surface integral:

$$d\sigma = \sqrt{f_x^{\,2} + f_y^{\,2} + 1}\, dx\, dy = \sqrt{0 + y^2 + 1}\, dx\, dy$$

$$\iint_S G(x, y, z)\, d\sigma = \iint_R \left(\sqrt{x}\sqrt{1+y^2}\right)\sqrt{1+y^2}\, dx\, dy$$

$$= \int_0^1 \int_0^{1-x} \sqrt{x}\,(1+y^2)\, dy\, dx = \frac{284}{945} \approx 0.30.$$



Surface Integrals

Surface Integrals of Vector Fields

DEFINITION Let $\mathbf{F}$ be a vector field in three-dimensional space with continuous components defined over a smooth surface S having a chosen field of normal unit vectors $\mathbf{n}$ orienting S. Then the **surface integral of F over S** is

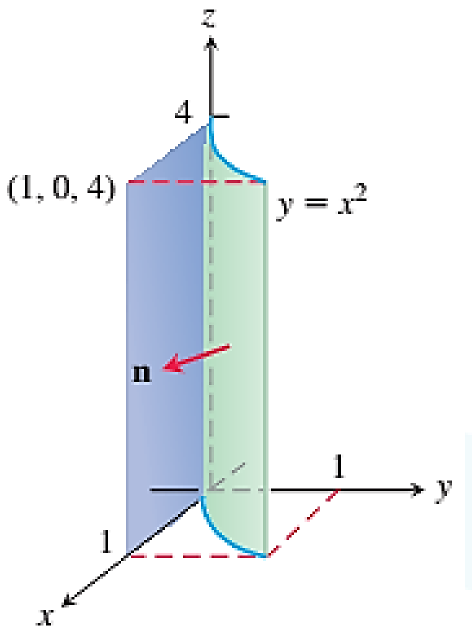
$$\iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma. \tag{5}$$

This integral is also called the **flux** of the vector field $\mathbf{F}$ across S.

Surface Integrals

Computing a Surface Integral for a Parametrized Surface

EXAMPLE 5 Find the flux of $\mathbf{F} = yz\mathbf{i} + x\mathbf{j} - z^2\mathbf{k}$ through the parabolic cylinder $y = x^2$, $0 \le x \le 1$, $0 \le z \le 4$, in the direction $\mathbf{n}$ indicated in Figure 16.54.



On the surface we have $x = x$, $y = x^2$, and $z = z$,

$\mathbf{r}(x, z) = x\mathbf{i} + x^2\mathbf{j} + z\mathbf{k}$, $0 \le x \le 1$, $0 \le z \le 4$.

$$\mathbf{r}_x \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2x\mathbf{i} - \mathbf{j}. \qquad \mathbf{n} = \frac{\mathbf{r}_x \times \mathbf{r}_z}{|\mathbf{r}_x \times \mathbf{r}_z|} = \frac{2x\mathbf{i} - \mathbf{j}}{\sqrt{4x^2 + 1}}.$$

On the surface, $y = x^2$, so the vector field there is $\mathbf{F} = yz\mathbf{i} + x\mathbf{j} - z^2\mathbf{k} = x^2z\mathbf{i} + x\mathbf{j} - z^2\mathbf{k}.$

$$\mathbf{F} \cdot \mathbf{n} = \frac{1}{\sqrt{4x^2 + 1}}((x^2z)(2x) + (x)(-1) + (-z^2)(0)) = \frac{2x^3z - x}{\sqrt{4x^2 + 1}}.$$

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \int_0^4 \int_0^1 \frac{2x^3z - x}{\sqrt{4x^2 + 1}} |\mathbf{r}_x \times \mathbf{r}_z|\, dx\, dz \qquad d\sigma = |\mathbf{r}_x \times \mathbf{r}_z|\, dx\, dz$$

$$= \int_0^4 \int_0^1 \frac{2x^3z - x}{\sqrt{4x^2 + 1}} \sqrt{4x^2 + 1}\, dx\, dz = 2.$$

Surface Integrals

Computing a Surface Integral for a Level Surface

$$g(x, y, z) = c,$$

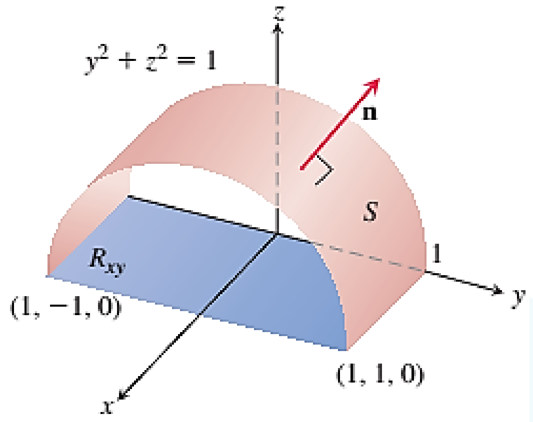
$$\mathbf{n} = \pm \frac{\nabla g}{|\nabla g|},$$

$$\text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iint_R \left(\mathbf{F} \cdot \frac{\pm \nabla g}{|\nabla g|} \right) \frac{|\nabla g|}{|\nabla g \cdot \mathbf{p}|} dA$$

$$= \iint_R \mathbf{F} \cdot \frac{\pm \nabla g}{|\nabla g \cdot \mathbf{p}|} dA.$$

Surface Integrals

EXAMPLE 6 Find the flux of $\mathbf{F} = yz\mathbf{j} + z^2\mathbf{k}$ outward through the surface S cut from the cylinder $y^2 + z^2 = 1$, $z \geq 0$, by the planes $x = 0$ and $x = 1$.



$$g(x, y, z) = y^2 + z^2$$

$$\mathbf{n} = +\frac{\nabla g}{|\nabla g|} = \frac{2y\mathbf{j} + 2z\mathbf{k}}{\sqrt{4y^2 + 4z^2}} = \frac{2y\mathbf{j} + 2z\mathbf{k}}{2\sqrt{1}} = y\mathbf{j} + z\mathbf{k}.$$

With $\mathbf{p} = \mathbf{k}$,

$$d\sigma = \frac{|\nabla g|}{|\nabla g \cdot \mathbf{k}|}dA = \frac{2}{|2z|}dA = \frac{1}{z}dA. \qquad z \geq 0$$

$$\mathbf{F} \cdot \mathbf{n} = (yz\mathbf{j} + z^2\mathbf{k}) \cdot (y\mathbf{j} + z\mathbf{k}) = z$$

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_{R_{xy}} (z)\left(\frac{1}{z}dA\right) = \iint_{R_{xy}} dA = \text{area}(R_{xy}) = 2$$

Surface Integrals

Moments and Masses of Thin Shells

Mass: $M = \iint_S \delta \, d\sigma$ $\quad \delta = \delta(x, y, z) =$ density at (x, y, z) is mass per unit area

First moments about the coordinate planes:

$$M_{yz} = \iint_S x\,\delta\,d\sigma, \qquad M_{xz} = \iint_S y\,\delta\,d\sigma, \qquad M_{xy} = \iint_S z\,\delta\,d\sigma$$

Coordinates of center of mass:

$$\bar{x} = M_{yz}/M, \qquad \bar{y} = M_{xz}/M, \qquad \bar{z} = M_{xy}/M$$

Moments of inertia about coordinate axes:

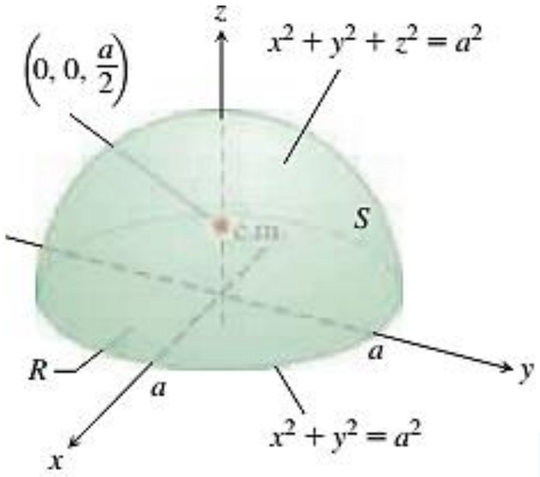
$$I_x = \iint_S (y^2 + z^2)\,\delta\,d\sigma, \quad I_y = \iint_S (x^2 + z^2)\,\delta\,d\sigma, \quad I_z = \iint_S (x^2 + y^2)\,\delta\,d\sigma,$$

$$I_L = \iint_S r^2 \delta \, d\sigma$$ $\quad r(x, y, z) =$ distance from point (x, y, z) to line L

Surface Integrals

Moments and Masses of Thin Shells

EXAMPLE 7 Find the center of mass of a thin hemispherical shell of radius a and constant density δ.



$$f(x, y, z) = x^2 + y^2 + z^2 = a^2, \qquad z \ge 0$$

The symmetry of the surface about the z-axis tells us that $\bar{x} = \bar{y} = 0$.

$$M = \iint_S \delta \, d\sigma = \delta \iint_S d\sigma = (\delta)(\text{area of } S) = 2\pi a^2 \delta.$$

$$\mathbf{p} = \mathbf{k} \qquad |\nabla f| = |2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}| = 2\sqrt{x^2 + y^2 + z^2} = 2a \qquad |\nabla f \cdot \mathbf{p}| = |\nabla f \cdot \mathbf{k}| = |2z| = 2z$$

$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \frac{a}{z} dA$$

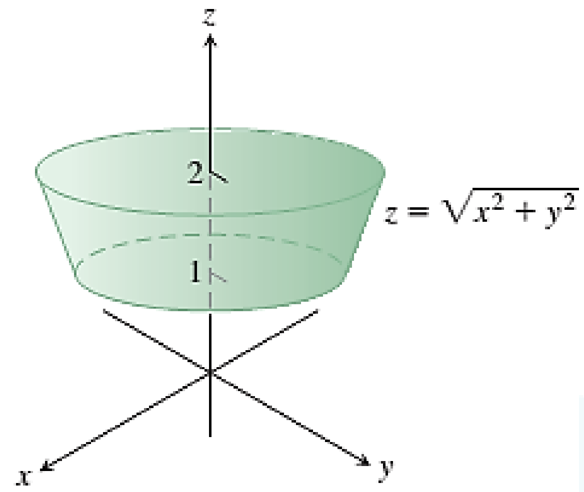
$$M_{xy} = \iint_S z\delta \, d\sigma = \delta \iint_R z \frac{a}{z} dA = \delta a \iint_R dA = \delta a(\pi a^2) = \delta \pi a^3 \qquad \bar{z} = \frac{M_{xy}}{M} = \frac{\pi a^3 \delta}{2\pi a^2 \delta} = \frac{a}{2}.$$

The shell's center of mass is the point $(0, 0, a/2)$.

Surface Integrals

Moments and Masses of Thin Shells

EXAMPLE 8 Find the center of mass of a thin shell of density $\delta = 1/z^2$ cut from the cone $z = \sqrt{x^2 + y^2}$ by the planes $z = 1$ and $z = 2$ (Figure 16.57).



The symmetry of the surface about the z-axis tells us that $\bar{x} = \bar{y} = 0$.

$$\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}, \qquad 1 \le r \le 2, \quad 0 \le \theta \le 2\pi,$$

$$|\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{2}r.$$

$$M = \iint_S \delta\, d\sigma = \int_0^{2\pi}\int_1^2 \frac{1}{r^2}\sqrt{2}r\, dr\, d\theta = 2\pi\sqrt{2}\ln 2,$$

$$M_{xy} = \iint_S \delta z\, d\sigma = \int_0^{2\pi}\int_1^2 \frac{1}{r^2} r\sqrt{2}r\, dr\, d\theta = \frac{1}{\ln 2}.$$

The shell's center of mass is the point $(0, 0, 1/\ln 2)$.

Exercises

- integrate the given function over the given surface.

  $G(x, y, z) = x$, over the parabolic cylinder $y = x^2, 0 \le x \le 2, 0 \le z \le 3$ $\frac{17\sqrt{17}-1}{4}$

- Integrate $G(x, y, z) = x\sqrt{y^2 + 4}$ over the surface cut from the parabolic cylinder $y^2 + 4z = 16$ by the planes $x = 0$, $x = 1$ and $z = 0$. $\frac{56}{3}$

- , use a parametrization to find the flux $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$ across the surface in the specified direction.

  $\mathbf{F} = z^2\mathbf{i} + x\mathbf{j} - 3z\mathbf{k}$ outward (normal away from the x-axis) through the surface cut from the parabolic cylinder $z = 4 - y^2$ by the planes $x = 0$, $x = 1$, and $z = 0$ -32

- Find the center of mass and the moment of inertia about the z-axis of a thin shell of constant density δ cut from the cone $x^2 + y^2 - z^2 = 0$ by the planes $z = 1$ and $z = 2$.

$$(\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{14}{9}\right) \qquad \frac{\sqrt{10}}{2}$$