Exercise 8: LaPlace Transform

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Laplace Transform

The Laplace transform of a continuous-time signal $x(t)$ is defined as

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

where s, the independent variable of the transform, is a complex variable.

where $s = \sigma + j\omega$, the independent variable of the transform.

σ: damping factor, ω: frequency variable.

Unilateral (or one-sided): $X(s) = \int_{0^-}^{\infty} x(t) e^{-st} dt;$

Bilateral (or two sided): $X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt;$

Characteristics of the Region of Convergence

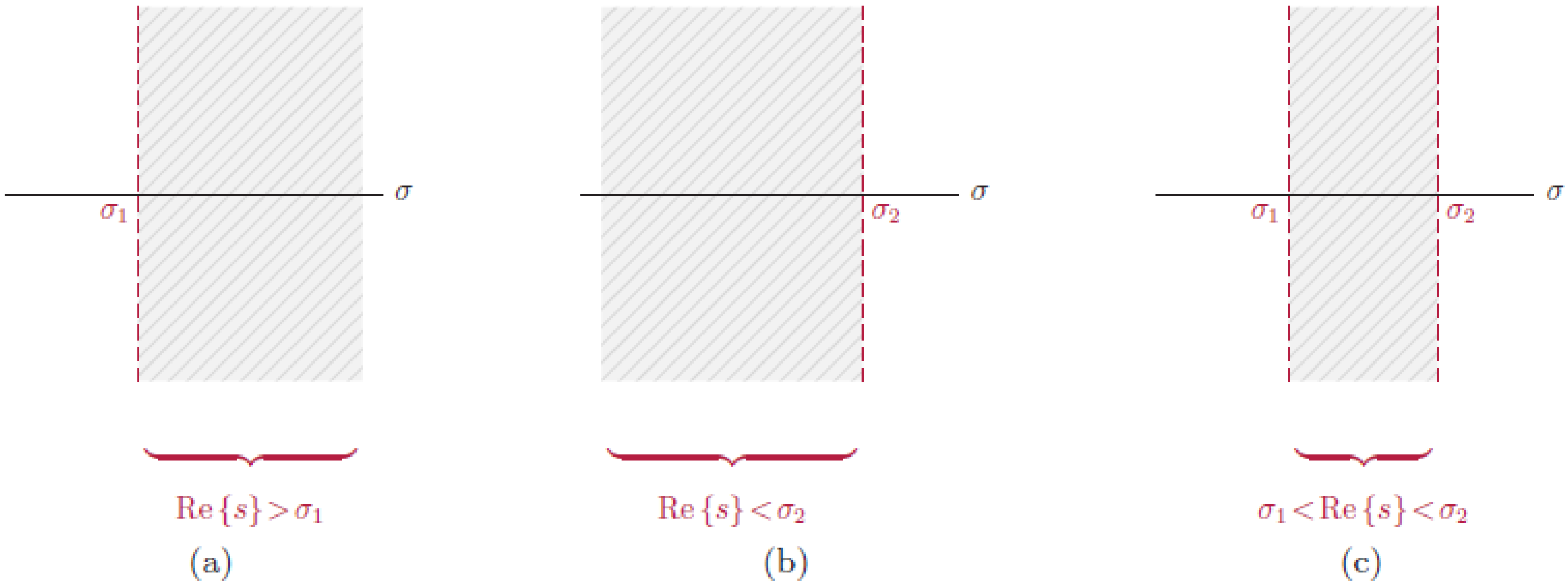


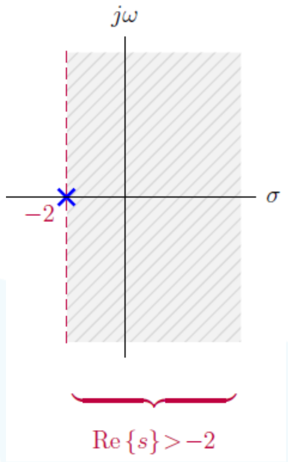
Figure 7.19 – Shape of the region of convergence.

causal signal | anticausal signal | noncausal signal

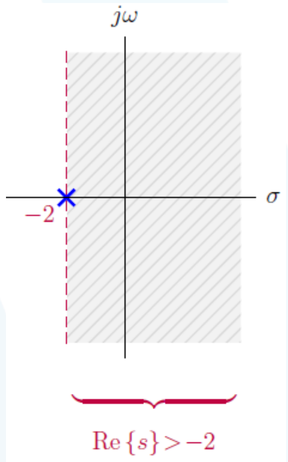
1. Using the Laplace transform definition determine the transform of each signal listed below. For each transform construct a pole-zero diagram and specify the ROC.
   a. $x(t) = e^{-2t}u(t)$
   b. $x(t) = e^{-2t}u(t-1)$
   c. $x(t) = e^{2t}u(-t)$

$$e.\ x(t) = \begin{cases} 1, & 0 < t < 1 \\ -1, & t > 1 \\ 0, & t < 0 \end{cases}$$

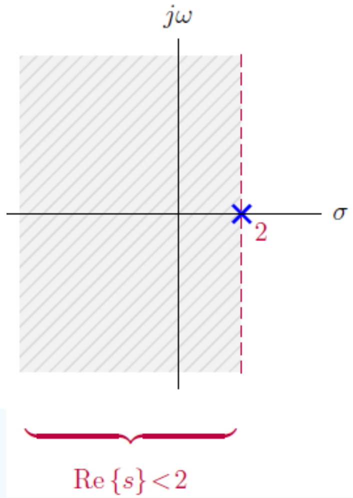
$$a.\ X(s) = \int_{-\infty}^{\infty} e^{-2t} u(t) e^{-st} dt = \int_{0}^{\infty} e^{-2t} e^{-st} dt = \left. \frac{e^{-(s+2)t}}{-(s+2)} \right|_{0}^{\infty} = \frac{1}{s+2}$$



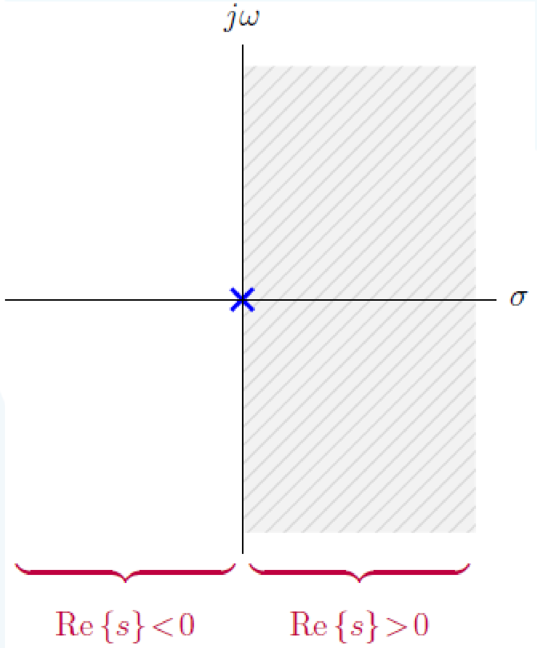
$$b. X(s) = \int_{-\infty}^{\infty} e^{-2t} u(t-1) e^{-st}\, dt = \int_{1}^{\infty} e^{-2t} e^{-st}\, dt = \left. \frac{e^{-(s+2)t}}{-(s+2)} \right|_{1}^{\infty} = \frac{e^{-2} e^{-s}}{s+2}$$



$$c.\ X(s)=\int_{-\infty}^{\infty} e^{2t}u(-t)e^{-st}dt=\int_{-\infty}^{0} e^{2t}e^{-st}dt=\left.\frac{e^{(2-s)t}}{(2-s)}\right|_{-\infty}^{0}=\frac{1}{2-s}$$



$$e.\ X(s)=\int_{0}^{1}(1)e^{-st}dt+\int_{1}^{\infty}(-1)e^{-st}dt=\left.\frac{e^{-st}}{-s}\right|_{0}^{1}+\left.\frac{e^{-st}}{s}\right|_{1}^{\infty}=\frac{1}{s}[1-2e^{-s}]$$



2. Construct a pole-zero diagram and specify the ROC for each of the transforms given below. Also, determine the Fourier transform $X(\omega)$ if it exists.

$$a.\ X(s) = \frac{s-2}{s^2+3s+2}, \qquad x(t) \text{ is causal}$$

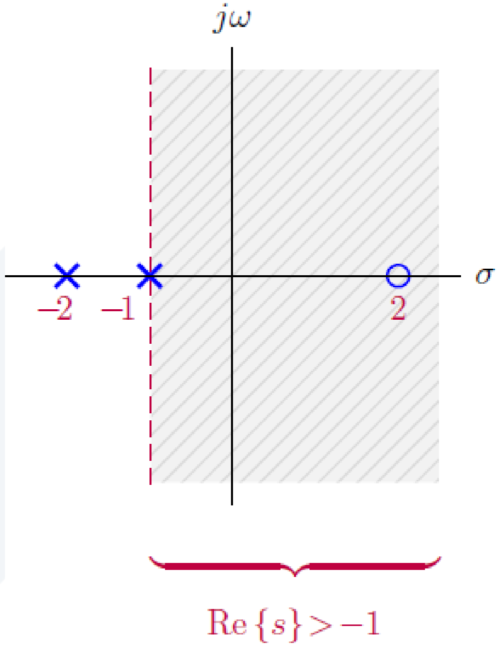
$$b.\ X(s) = \frac{s}{s^2-1}, \qquad x(t) \text{ is causal}$$

$$c.\ X(s) = \frac{s+1}{s^2-4s+3}, \qquad x(t) \text{ is anti-causal}$$

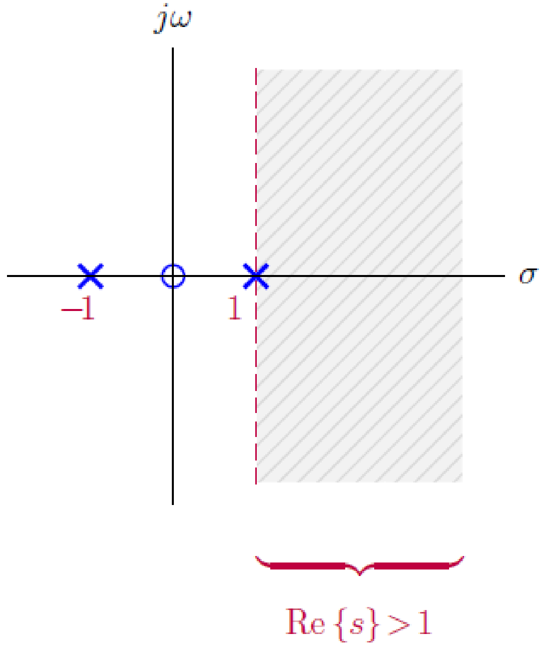
$$d.\ X(s) = \frac{s^2-s}{s^2-s-6}, \qquad x(t) \text{ is anti-causal}$$

a. The transform has a zero at $s = 2$ and poles at $s = -1, -2$. Since $x(t)$ is causal, the ROC is $\operatorname{Re}\{s\} > -1$. The Fourier transform $X(\omega)$ is

$$X(\omega) = X(s)\big|_{s=j\omega} = \frac{j\omega - 2}{(j\omega)^2 + j3\omega + 2} = \frac{j\omega - 2}{(2-\omega^2) + j3\omega}$$

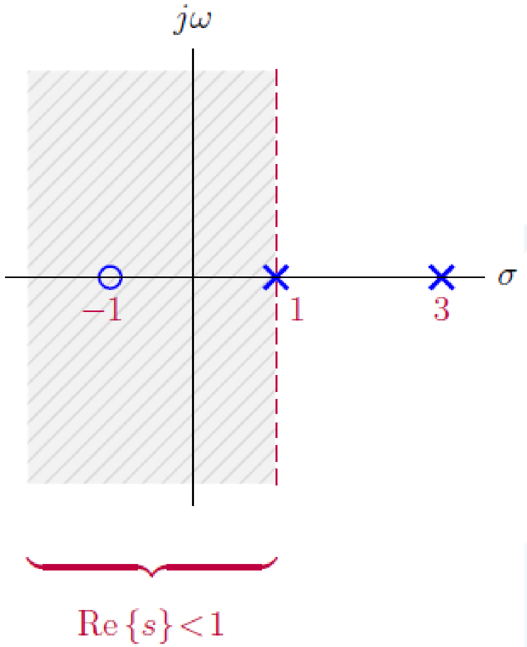


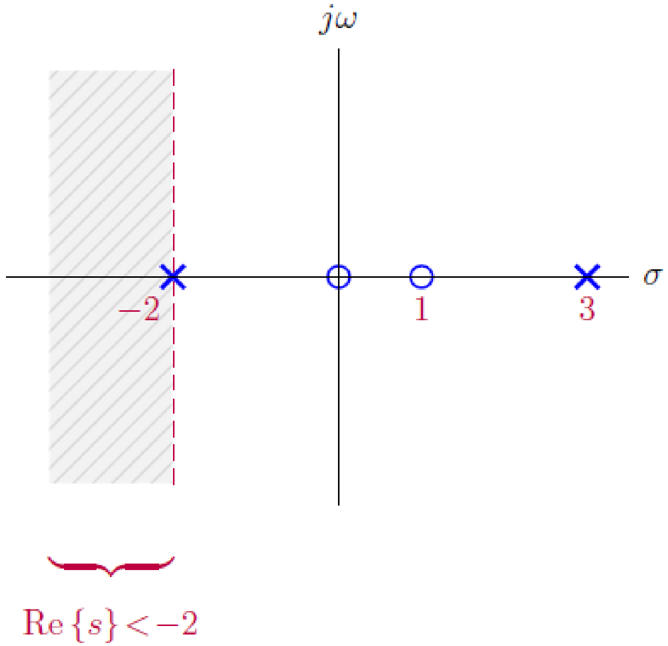
b. The transform has a zero at $s = 0$ and poles at $s = \pm 1, -2$. Since $x(t)$ is causal, the ROC is $\mathrm{Re}\,\{s\} > 1$. The ROC does not include the jω-axis of the s-plane. Therefore the Fourier transform does not exist.



c. The transform has a zero at $s = -1$ and poles at $s = 1, 3$. Since $x(t)$ is anti-causal, the ROC is $\mathrm{Re}\,\{s\} < 1$. The Fourier transform $X(\omega)$ is

$$X(\omega) = X(s)\big|_{s=j\omega} = \frac{j\omega + 1}{(j\omega)^2 - j4\omega + 3} = \frac{j\omega + 1}{(3 - \omega^2) - j4\omega}$$





d. The transform has a zero at $s = 0$, 1 and poles at $s = -2$, 3. Since $x(t)$ is anti-causal, the ROC is Re $\{s\} < -2$. The ROC does not include the $j\omega$-axis of the s-plane. Therefore the Fourier transform does not exist.

Properties of Laplace Transform

Property	$x(t)$	$X(s)$	ROC
Linearity	$ax_1(t)+bx_2(t)$	$aX_1(s)+bX_2(s)$	$\supset (R_1 \cap R_2)$
Delay by T	$x(t-T)$	$X(s)e^{-sT}$	R
Multiply by t	$tx(t)$	$-dX(s)/ds$	R
Multiply by $e^{-\alpha t}$	$x(t)e^{-\alpha t}$	$X(s+\alpha)$	Shift R by $-\alpha$
Scaling in t	$x(at)$	$\frac{1}{\lvert a\rvert}X(\frac{s}{a})$	aR
Differentiate in t	$dx(t)/dt$	$sX(s)$	$\supset R$
Integrate in t	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{X(s)}{s}$	$\supset (R \cap (\mathrm{Re}(s)>0))$
Convolve in t	$x_1 * x_2(t)$	$X_1(s)\,X_2(s)$	$\supset (R_1 \cap R_2)$

3. Using the property of the Laplace transform, determine $X(s)$ for each of the signals listed below. Also indicate the ROC in each case.

a. $x(t) = \delta(t) + 2e^{-t}\, u(t)$

b. $x(t) = (1 - e^{-t})\; u(t)$

c. $x(t) = e^{-2t} \cos(3t)\; u(t)$

d. $x(t) = e^{-2(t-1)}\, u(t - 1)$

e. $x(t) = e^{-2t}\, u(t - 1)$

a. $x(t) = \delta(t) + 2e^{-t}\, u(t)$

$$a.\, L\{\delta(t)\} = 1, \quad \text{all } s \qquad L\{e^{-t}u(t)\} = \frac{1}{s+1}, \quad \text{Re}\{s\} > -1$$

$$\boxed{L\{u(t)\} = \frac{1}{s}}$$

Using linearity $X(s) = 1 + \dfrac{2}{s+1} = \dfrac{s+3}{s+1}, \quad \text{Re}\{s\} > -1$

b. $x(t) = (1 - e^{-t})\, u(t)$

$$b.\, L\{u(t)\} = \frac{1}{s}, \quad \text{Re}\{s\} > 0 \qquad L\{e^{-t}u(t)\} = \frac{1}{s+1}, \quad \text{Re}\{s\} > -1$$

Using linearity $X(s) = \dfrac{1}{s} - \dfrac{1}{s+1} = \dfrac{1}{s(s+1)}, \quad \text{Re}\{s\} > 0$

c. $x(t) = e^{-2t}\cos(3t)\, u(t)$

$$c.\, x(t) = \frac{1}{2}e^{t(-2+j3)}u(t) + \frac{1}{2}e^{t(-2-j3)}u(t)$$

$$L\{e^{-t(2-j3)}u(t)\} = \frac{1}{s+2-j3}, \quad \text{Re}\{s\} > -2$$

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$$L\{e^{-t(2+j3)}u(t)\} = \frac{1}{s+2+j3}, \quad \mathrm{Re}\{s\} > -2$$

Using linearity $X(s) = \dfrac{\frac{1}{2}}{s+2-j3} + \dfrac{1/2}{s+2+j3} = \dfrac{s+2}{s^2+4s+13}, \quad \mathrm{Re}\{s\} > -2$

d. $x(t) = e^{-2(t-1)}u(t-1)$

$$d.\ L\{e^{-2t}u(t)\} = \frac{1}{s+2}, \quad \mathrm{Re}\{s\} > -2$$

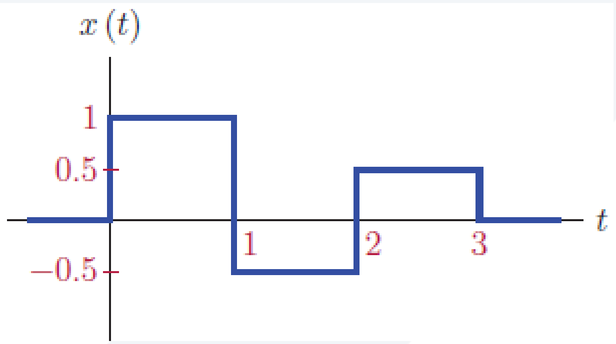
Using time shifting property $X(s) = L\{e^{-2(t-1)}u(t-1)\} = \dfrac{e^{-s}}{s+2}, \quad \mathrm{Re}\{s\} > -2$

e. $x(t) = e^{-2t}u(t-1)$

$$e.\ L\{e^{-2t}u(t-1)\} = e^{-2}L\{e^{-2(t-1)}u(t-1)\} = \frac{e^{-s-2}}{s+2}, \quad \mathrm{Re}\{s\} > -2$$

Using scaling property

4. Using the linearity and time shifting properties, determine the Laplace transform of the signal shown. Specify the ROC in each case.



$$x(t) = u(t) - 1.5u(t-1) + u(t-2) - 0.5u(t-3)$$

$$X(s) = \frac{1}{s}\left(1 - 1.5e^{-s} + e^{-2s} - 0.5e^{-3s}\right), \quad \mathrm{Re}\{s\} > -\infty$$

5. **Find a transfer function for each CTLTI system described below by means of a differential equation.**

$$a.\ \frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} - x(t)$$

$$b.\ \frac{d^2y(t)}{dt^2} + 4y(t) = \frac{d^2x(t)}{dt^2} + \frac{dx(t)}{dt} + 3x(t)$$

$$a.\ H(s) = \frac{Y(s)}{X(s)} = \frac{s-1}{s^2+4s+3}$$

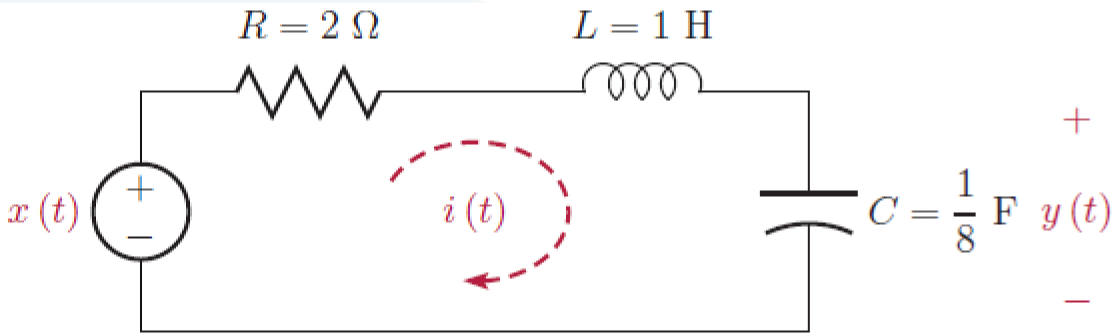
$$b.\ H(s) = \frac{Y(s)}{X(s)} = \frac{s^2+s+3}{s^2+4}$$

6. **Find a differential equation for each CTLTI system described below by means of a transfer function.**

$$a.\ H(s) = \frac{s}{s+4} \qquad a.\ \frac{dy(t)}{dt} + 4y(t) = \frac{dx(t)}{dt}$$

$$b.\ H(s) = \frac{s+1}{s^2+5s+6} \qquad b.\ \frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = \frac{dx(t)}{dt} + x(t)$$

7. For the RLC circuit shown,



a. Find a differential equation between the input $x(t)$ and the output $y(t)$.

b. Obtain the transfer function $H(s)$ from the DE found in part (a).

a. $$x(t) = y(t) + L\frac{di(t)}{dt} + Ri(t)$$

$$i(t) = C\frac{dy(t)}{dt} \Rightarrow \frac{di(t)}{dt} = C\frac{d^2y(t)}{dt^2} \quad \Rightarrow \quad LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t) = x(t)$$

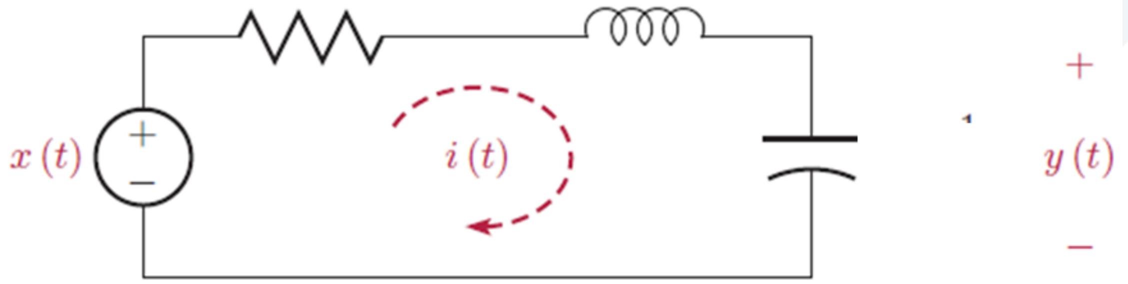
$$\frac{d^2y(t)}{dt^2} + \frac{R}{L}\frac{dy(t)}{dt} + \frac{1}{LC}y(t) = \frac{1}{LC}x(t)$$

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 8y(t) = 8x(t)$$

b. $$(s^2 + 2s + 8)\,Y(s) = 8X(s) \Rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{8}{s^2 + 2s + 8}$$

8. For the RLC circuit shown, the initial values are *relaxed*

R = 5 Ω, L = 1 H and C = 1/6 F.



a. Find a differential equation between the input $x(t)$ and the output $y(t)$.

b. Obtain the transfer function $H(s)$ from the DE found in part (a).

c. Obtain the impulse response $h(t)$ by using Inverse Laplace Transform

a. $$\boldsymbol{x(t) = y(t) + L\frac{di(t)}{dt} + Ri(t)}$$

$$\boldsymbol{i(t) = C\frac{dy(t)}{dt} \Rightarrow \frac{di(t)}{dt} = C\frac{d^2y(t)}{dt^2}} \quad \Rightarrow \quad \boldsymbol{LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t) = x(t)}$$

$$\boldsymbol{\frac{d^2y(t)}{dt^2} + \frac{R}{L}\frac{dy(t)}{dt} + \frac{1}{LC}y(t) = \frac{1}{LC}x(t)}$$

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = 6x(t)$$

b. $$(s^2 + 5s + 6)Y(s) = 6X(s) \Rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{6}{s^2 + 5s + 6}$$

c.

$$H(s) = \frac{Y(s)}{X(s)} = \frac{6}{s^2 + 5s + 6} = \frac{6}{(s+2)(s+3)}$$

The poles are : $s = -3$; $s = -2$; : $ROC : RE\{s\} > -2$

The Poles Are Real and Distinct

$$H(s) = \frac{A}{s+2} + \frac{B}{s+3}$$

$$A = (s+2)H(s)|_{s=-2} \gg A = \frac{6}{s+3}|_{s=-2} = 6$$

$$B = (s+3)H(s)|_{s=-3} \gg B = \frac{6}{s+2}|_{s=-3} = -6$$

The transfer function $\Rightarrow$ $H(s) = \frac{6}{s+2} - \frac{6}{s+3}$ $\Rightarrow$

The impulse response

$h(t) = 6e^{-2t}u(t) - 6e^{-3t}u(t)$

d. Find the output if the input is $x(t) = u(t)$

$$x(t) = u(t)$$

$$X(s) = \frac{1}{s}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{6}{s^2 + 5s + 6}$$

$$Y(s) = H(s).X(s) = \frac{6}{s^2 + 5s + 6}.\frac{1}{s}$$

The Poles Are Real and Distinct

The poles are : $s = -3$; $s = -2$; $s = 0$;

$ROC : RE\{s\} > 0$

$$Y(s) = \frac{6}{s^2 + 5s + 6}.\frac{1}{s} = \frac{A}{s+2} + \frac{B}{s+3} + \frac{C}{s}$$

$$A = (s+2)Y(s)|_{s=-2} \gg A = \frac{6}{s(s+3)}|_{s=-2} = -3$$

$$B = (s+3)Y(s)|_{s=-3} \gg B = \frac{6}{s(s+2)}|_{s=-3} = 2$$

$$C = sY(s)|_{s=0} \gg C = \frac{6}{(s+3)(s+2)}|_{s=0} = \frac{6}{5}$$

$$Y(s) = \frac{-3}{s+2} + \frac{2}{s+3} + \frac{6/5}{s}$$

$$y(t) = -3e^{-2t}u(t) + 2e^{-3t}u(t) + \frac{6}{5}u(t)$$

e. Find the impulse response

$$x(t) = \delta(t)$$

$$\boxed{y(t) = Trans\,(u(t)) \rightarrow \frac{dy(t)}{dt} = Trans\left(\frac{du(t)}{dt}\right) = Trans(\delta(t)) = h(t)}$$

$$h(t) = \frac{dy(t)}{dt} = \frac{d\left(-3e^{-2t}u(t) + 2e^{-3t}u(t) + \frac{6}{5}u(t)\right)}{dt}$$

$$h(t) = \left(\frac{3}{2}e^{-2t} - \frac{2}{3}e^{-3t}\right)u(t)$$

Extra examples

Inverse Laplace transform

9. For a causal signal x(t) the Laplace transform is : $X(s) = \dfrac{1}{s^2 + 3s + 2}$

Find the inverse Laplace transform

The poles are : $s = -1$; $s = -2$; : $ROC : RE\{s\} > -1$

The Poles Are Real and Distinct

$$X(s) = \frac{A}{s+2} + \frac{B}{s+1}$$

$$A = (s+2)X(s)|_{s=-2} \gg A = \frac{1}{s+1}|_{s=-2} = -1$$

$$B = (s+1)X(s)|_{s=-1} \gg B = \frac{1}{s+2}|_{s=-1} = 1$$

$$\Rightarrow X(s) = -\frac{1}{s+2} + \frac{1}{s+1} \Rightarrow x(t) = -e^{-2t}u(t) + e^{-t}u(t)$$

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Inverse Laplace transform

10. For a causal signal x(t) the Laplace transform is : $X(s) = \dfrac{s+1}{s(s+2)}$

Find the inverse Laplace transform

$The\ poles\ are:\ s=-2\ ;\ s=\ 0\ ;\ :\quad ROC:\ RE\{s\}>0$

THE POLES ARE REAL AND DISTINCT

$$X(s) = \frac{k_1}{s} + \frac{k2}{s+2}$$

$$k_1 = s\,X(s)\Big|_{s=0} = \frac{s+1}{s+2}\Big|_{s=0} = \frac{1}{2}$$

$$k_2 = (s+2)\,X(s)\Big|_{s=-2} = \frac{s+1}{s}\Big|_{s=-2} = \frac{1}{2}$$

$$\Rightarrow\quad X(s) = \frac{1/2}{s} + \frac{1/2}{s+2} \quad\Rightarrow\quad x(t) = \frac{1}{2}u(t) + \frac{1}{2}e^{-2t}u(t)$$

Inverse Laplace transform

The Laplace transform of a signal $x(t)$ is

$$X(s) = \frac{s+1}{s(s^2+9)}$$

with the ROC specified as

$$\mathrm{Re}\{s\} > 0$$

Determine $x(t)$.

Please note :
The poles are complex

$$X(s) = \frac{s+1}{s\,(s+j3)\,(s-j3)}$$

$$X(s) = \frac{k_1}{s} + \frac{k_2}{s+j3} + \frac{k_3}{s-j3}$$

$$k_1 = s\,X(s)\Big|_{s=0} = \frac{s+1}{(s+j3)\,(s-j3)}\Big|_{s=0} = \frac{1}{9}$$

$$k_2 = (s+j3)\,X(s)\Big|_{s=-j3} = \frac{s+1}{s\,(s-j3)}\Big|_{s=-j3} = -\frac{1}{18} + j\frac{1}{6}$$

$$k_3 = k_2^* = -\frac{1}{18} - j\frac{1}{6}$$

$$\begin{aligned} x(t) =& k_1\,u(t) + k_2\,e^{-j3t}\,u(t) + k_3\,e^{j3t}\,u(t) \\ =& \frac{1}{9}\,u(t) + \left(-\frac{1}{18} + j\frac{1}{6}\right)\,e^{-j3t}\,u(t) + \left(-\frac{1}{18} - j\frac{1}{6}\right)\,e^{j3t}\,u(t) \end{aligned}$$

Inverse Laplace transform

A causal signal $x(t)$ has the Laplace transform $X(s) = \dfrac{s(s+1)}{(s+1)^3(s+2)}$

- Multiple-order poles

$$X(s) = \frac{s(s+1)}{(s+1)^3(s+2)} = \frac{-3}{s+1} + \frac{3}{(s+1)^2} - \frac{2}{(s+1)^3} + \frac{3}{s+2}$$

$$L\{e^{-t}u(t)\} = \frac{1}{s+1}, \qquad L\{te^{-t}u(t)\} = -\frac{d}{ds}\left[\frac{1}{s+1}\right] = \frac{1}{(s+1)^2}$$

$$L\{t^2e^{-t}u(t)\} = -\frac{d}{ds}\left[\frac{1}{(s+1)^2}\right] = \frac{2}{(s+1)^3}$$

$$x(t) = -3e^{-t}u(t) + 3te^{-t}u(t) - t^2e^{-3t}u(t) + 3e^{-2t}u(t)$$