MATHEMATICAL ANALAYSIS 2

Lecture 10

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Uniform Convergence

Definition

Let $\{u_n(x)\}, n = 1, 2, 3, \ldots$ be a sequence of functions defined in $[a, b]$. The sequence is said to converge to $F(x)$, or to have the limit $F(x)$ in $[a, b]$, if for each $\epsilon > 0$ and each x in $[a, b]$ we can find $N > 0$ such that $|u_n(x) - F(x)| < \epsilon$ for all $n > N$. In such case we write $\lim_{n\to\infty} u_n(x) = F(x)$. The number N may depend on x as well as ϵ. Pointwise convergence

If it depends *only* on ϵ and not on x, the sequence is said to converge to $F(x)$ *uniformly* in $[a, b]$ or to be *uniformly convergent* in $[a, b]$. Uniform convergence

Example

$u_n = x^n/n$ uniformly converges to $F(x) = 0$ on $0 \leqq x \leqq 1$,

For any $\epsilon > 0$ and any x in the interval, there is N such that for all $n > N |u_n - F(x)| < \epsilon$, i.e., $|x^n/n| < \epsilon$. Since the limit does not depend on x, the sequence is uniformly convergent.

Uniform Convergence

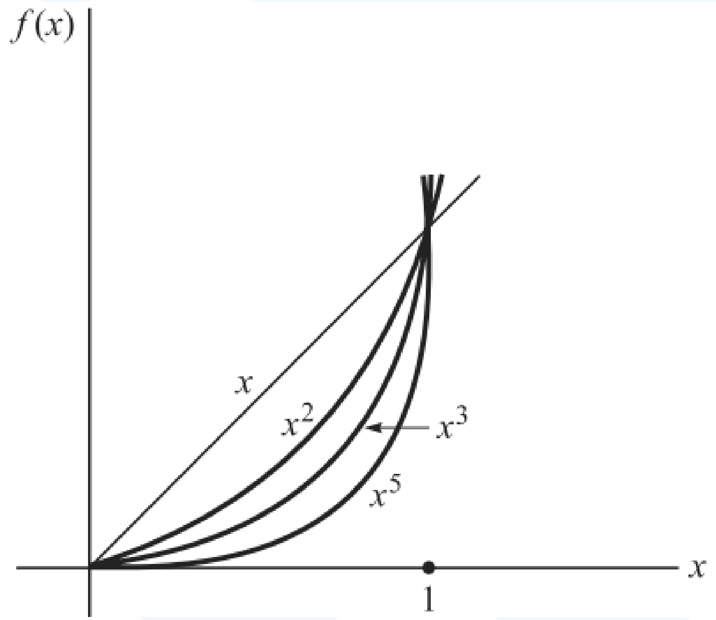
Example

If $u_n = x^n$ and $0 \leqq x \leqq 1$, the sequence is not uniformly convergent because (think of the function $F(x) = 0,\ 0 \leqq x < 1,\ F(1) = 1$)

$$|x^n - 0| < \epsilon \text{ when } x^n < \epsilon, \quad \Rightarrow \quad n \ln x < \ln \epsilon.$$

On the interval $0 \leqq x < 1$, and for $0 < \epsilon < 1$, both members of the inequality are negative, therefore, $n > \dfrac{\ln \epsilon}{\ln x}$. Since $\dfrac{\ln \epsilon}{\ln x} = \dfrac{\ln 1 - \ln \epsilon}{\ln 1 - nn\, x} = \dfrac{\ln(/\epsilon)}{\ln(1/x)}$, it follows that we must choose N such that

$$n > N > \frac{\ln 1/\epsilon}{\ln 1/x}$$



Improper Integrals Depending on a Parameter

Let $\phi(\alpha) = \int_a^\infty f(x,\alpha)\,dx$

This integral is analogous to an infinite series of functions.

Definition

The integral $\phi(\alpha) = \int_a^\infty f(x,\alpha)\,dx$ is said to be *uniformly convergent* in $[\alpha_1, \alpha_2]$

if for each $\epsilon > 0$ we can find a number N depending on ϵ but not on α, such that

$$\left|\phi(\alpha) - \int_a^u f(x,\alpha)\,dx\right| < \epsilon \qquad \text{for all } u > N \text{ and all } \alpha \text{ in } [\alpha_1, \alpha_2]$$

This can be restated by noting that $\left|\phi(\alpha) - \int_a^u f(x,\alpha)\,dx\right| = \left|\int_u^\infty f(x,\alpha)\,dx\right|$, which is analogous

in an infinite series to the absolute value of the remainder after N terms.

Improper Integrals Depending on a Parameter

Example

The integral $\displaystyle\int_1^{+\infty} \frac{dx}{x^2+y^2}$

converges uniformly on the entire set $\mathbb{R}$ of values of the parameter $y \in \mathbb{R}$, since for every $y \in \mathbb{R}$

$$\int_b^{+\infty} \frac{dx}{x^2+y^2} \leq \int_b^{+\infty} \frac{dx}{x^2} = \frac{1}{b} < \varepsilon,$$

provided $b > 1/\varepsilon$.

Improper Integrals Depending on a Parameter

Example

The integral $\int_0^{+\infty} e^{-xy}\,dx$, obviously converges only when $y > 0$.

$$F(y) = \int_0^{\infty} e^{-xy}\,dx = \lim_{b\to\infty}\int_0^{b} e^{-xy}\,dx = \lim_{b\to\infty}\left(-\frac{1}{y}e^{-xy}\Big|_0^b\right) = \lim_{b\to\infty}\left(-\frac{1}{y}e^{-yb} + \frac{1}{y}\right) = \frac{1}{y}\ , \ \forall\ y > 0$$

Moreover it converges uniformly on every set $\{y \in \mathbb{R} \mid y \geq y_0 > 0\}$.

$$0 \leq \int_b^{+\infty} e^{-xy}\,dx = \frac{1}{y}e^{-by} \leq \frac{1}{y_0}e^{-by_0} \to 0 \quad \text{as } b \to +\infty.$$

At the same time, the convergence is not uniform on the entire set $\mathbb{R}_+ = \{y \in \mathbb{R} \mid y > 0\}$.

$$\int_b^{+\infty} e^{-xy}\,dx = \frac{1}{y}e^{-by} \to +\infty, \quad \text{as } y \to +0, \quad \text{for every fixed value of } b \in [0, +\infty[.$$

Special Tests for Uniform Convergence of Improper Integrals Depending on a Parameter

The Weierstrass test

If we can find a function $M(x) \geqq 0$ such that

(a) $|f(x,\alpha)| \leqq M(x) \qquad \alpha_1 \leqq \alpha \leqq \alpha_2, x > a$

(b) $\int_a^\infty M(x)\,dx$ converges,

then $\int_a^\infty f(x,\alpha)\,dx$ is uniformly and absolutely convergent in $\alpha_1 \leqq \alpha \leqq \alpha_2$.

Special Tests for Uniform Convergence of Improper Integrals Depending on a Parameter

Example $\int_0^\infty \frac{\cos \alpha x}{x^2+1}\,dx$

Since $\left|\frac{\cos \alpha x}{x^2+1}\right| \leqq \frac{1}{x^2+1}$ and $\int_0^\infty \frac{dx}{x^2+1}$ converges, it follows that $\int_0^\infty \frac{\cos \alpha x}{x^2+1}\,dx$ is uniformly and absolutely convergent for all real values of α.

Example

$$F(\alpha) = \int_0^\infty e^{-x\alpha} \sin x\, dx$$

We have $\left|e^{-x\alpha} \sin x\right| \le e^{-x\alpha}$ $\Longrightarrow$ $\int_0^\infty \left|e^{-x\alpha} \sin x\right| dx \le \int_0^\infty e^{-x\alpha} dx$

Since $\int_0^\infty e^{-x\alpha} dx$ converges uniformly for $\alpha \ge \alpha_0 > 0$ Therefore, by Wierestrass test $F(\alpha) = \int_0^\infty e^{-x\alpha} \sin x\, dx$

converges uniformly for $\alpha \ge \alpha_0 > 0$

Special Tests for Uniform Convergence of Improper Integrals Depending on a Parameter

Dirichlet test

Let f and φ are defined for $a < x < \infty$ and $\alpha \in Y$. Assume that f is continuous with respect to x, and φ have continuous first order partial derivative $\dfrac{\partial \varphi}{\partial x}$, if :

a) $\varphi(x,\alpha)$ is a positive decreasing function with respect to x, and $\varphi(x,\alpha) \xrightarrow[x \to \infty]{} 0 \; ; \; \forall \alpha \in Y$

b) $\left| \int_a^u f(x,\alpha)dx \right| < P$ for every $u > a$ and $\alpha \in Y$.

Then $\int_a^\infty f(x,\alpha)\varphi(x,\alpha)dx$ converges uniformly on Y.

Special Tests for Uniform Convergence of Improper Integrals Depending on a Parameter

Example $F(\alpha)=\int_0^{\infty} e^{-x^2\alpha}\sin x\,dx$

Let $f(x,\alpha)=\sin x,\ \varphi(x,\alpha)=e^{-x^2\alpha}$

$f(x,\alpha)=\sin x$ is continuous with respect to x. $\varphi(x,\alpha)=e^{-x^2\alpha}$ is positive continuous function with respect to x

$\frac{\partial\varphi}{\partial x}=-2\alpha x e^{-\alpha x^2}\le 0\ ,\ \forall x\ge 0\ ,\ \forall\alpha>0 \Longrightarrow \varphi(x,\alpha)=e^{-x^2\alpha}$ is decreasing

Moreover, $\varphi(x,\alpha)\xrightarrow[x\to\infty]{}0\ ;\ \forall\alpha>0$

$\left|\int_0^b \sin x\,dx\right|=\left|-\cos x\,\Big|_0^b\right|=|1-\cos b|<2\ ;\ \forall b>0\ ,\ \forall x\ge 0\ ,\ \forall\alpha>0 \Longrightarrow$

By Dirichlet test, $F(\alpha)=\int_0^{\infty} e^{-x^2\alpha}\sin x\,dx$ converges uniformly for every $\alpha>0$

Special Tests for Uniform Convergence of Improper Integrals Depending on a Parameter

Abel test

If

1) Let $\varphi(x,\alpha)$ is monotonic function with respect to x for $a \le x < \infty$, and

$$\exists\, M > 0 : \left|\varphi(x,\alpha)\right| \le M \quad ; \ a \le x < \infty \ , \ \alpha \in Y$$

2) The integral $\int_a^\infty f(x,\alpha)dx$ converges uniformly on Y

Then the integral $\int_a^\infty f(x,\alpha)\varphi(x,\alpha)dx$ converges uniformly on Y.

Special Tests for Uniform Convergence of Improper Integrals Depending on a Parameter

Example $F(\alpha)=\int_0^\infty e^{-x^2\alpha}\frac{\sin x}{x}dx$

$\varphi(x,\alpha)=e^{-\alpha x^2}$ is monotonic function with respect to x, where $0\le x<\infty$ and $\alpha\ge 0$

Moreover, $|\varphi(x,\alpha)|=\left|e^{-\alpha x^2}\right|\le 1$ $\Longrightarrow$ $\varphi(x,\alpha)=e^{-\alpha x^2}$ is bounded for $0\le x<\infty$ and $\alpha\ge 0$

By Dirichlet test, $\int_0^\infty \frac{\sin x}{x}dx$ converges uniformly for every $\alpha\ge 0$

By Abel test, $F(\alpha)=\int_0^\infty e^{-x^2\alpha}\frac{\sin x}{x}dx$ converges uniformly on $\{\alpha\ ;\ \alpha\ge 0\}$

Special Tests for Uniform Convergence of Improper Integrals Depending on a Parameter

Remark: The uniform convergence of an integral whose only singularity is at the lower limit of integration can be defined and studied similarly.

Remark: If the integral has a singularity at both limits of integration, it can be represented as

$$\int_{-\infty}^{\infty} f(x,\alpha)dx = \int_{-\infty}^{c} f(x,\alpha)dx + \int_{c}^{\infty} f(x,\alpha)dx$$

if both of the integrals on the right-hand side of the equality converge uniformly

Limiting Passage Under the Sign of an Improper Integral Depending on a Parameter

Theorem

If $f(x,\alpha)$ is continuous for $x \geqq a$ and $\alpha_1 \leqq \alpha \leqq \alpha_2$, and if $\int_a^\infty f(x,\alpha)\,dx$ is uniformly convergent for $\alpha_1 \leqq \alpha \leqq \alpha_2$, then $\phi(\alpha) = \int_a^\infty f(x,\alpha)\,dx$ is continous in $\alpha_1 \leqq \alpha \leqq \alpha_2$. In particular, if α_0 is any point of $\alpha_1 \leqq \alpha \leqq \alpha_2$, we can write

$$\lim_{\alpha\to\alpha_0} \phi(\alpha) = \lim_{\alpha\to\alpha_0} \int_a^\infty f(x,\alpha)\,dx = \int_a^\infty \lim_{\alpha\to\alpha_0} f(x,\alpha)\,dx$$

If α_0 is one of the end points, we use right or left hand limits.

Differentiation and Integration of an Improper Integral with Respect to a Parameter

Theorem

If $f(x, \alpha)$ is continuous and has a continuous partial derivative with respect to α for $x \geqq a$ and $\alpha_1 \leqq \alpha \leqq \alpha_2$, and if $\int_a^\infty \frac{\partial f}{\partial \alpha}\, dx$ converges uniformly in $\alpha_1 \leqq \alpha \leqq \alpha_2$, then if a does not depend on α,

$$\frac{d\phi}{d\alpha} = \int_a^\infty \frac{\partial f}{\partial \alpha}\, dx$$

Theorem

Under the conditions of Theorem we can integrate $\phi(\alpha)$ with respect to α from α_1 to α_2 to

$$\int_{\alpha_1}^{\alpha_2} \phi(\alpha)\, d\alpha = \int_{\alpha_1}^{\alpha_2} \left\{ \int_a^\infty f(x, \alpha)\, dx \right\} d\alpha = \int_a^\infty \left\{ \int_{\alpha_1}^{\alpha_2} f(x, \alpha)\, d\alpha \right\} dx$$

which corresponds to a change of the order of integration.

Example Compute the Dirichlet integral $\int_0^{+\infty} \frac{\sin x}{x}\,dx$

We use the integral $F(y) = \int_0^{+\infty} \frac{\sin x}{x} e^{-xy}\,dx$ which converges uniformly on the interval $0 \le y < +\infty$.

$$\lim_{y \to +0} \int_0^{+\infty} \frac{\sin x}{x} e^{-xy}\,dx = \int_0^{+\infty} \frac{\sin x}{x}\,dx.$$

since the integral $F'(y) = -\int_0^{+\infty} \sin x e^{-xy}\,dx$, converges uniformly on every set of the form $\{y \in \mathbb{R} \mid y \ge y_0 > 0\}$.

The integral $F'(y) = -\int_0^{+\infty} \sin x e^{-xy}\,dx$, is easily computed from the primitive of the integrand,

$F'(y) = -\frac{1}{1+y^2}$ for $y > 0$, $\Rightarrow$ $F(y) = -\arctan y + c$ for $y > 0$.

We have $F(y) \to 0$ as $y \to +\infty$, $\Rightarrow$ $c = \pi/2$. $\Rightarrow$ $F(0) = \pi/2$. $\Rightarrow$ $\int_0^{+\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}$.

Example Compute the Euler–Poisson integral

$$I=\int_0^\infty e^{-x^2}dx$$

We use the integral $I(\alpha)=\int_0^\infty \alpha e^{-x^2\alpha^2}dx$

Let $u=x\alpha$ in $I(\alpha)=\int_0^\infty \alpha e^{-x^2\alpha^2}dx \Rightarrow I(\alpha)=\int_0^\infty \alpha e^{-x^2\alpha^2}dx=\int_0^\infty e^{-u^2}du=I$

Let $x=\alpha$ in $I=\int_0^\infty e^{-x^2}dx$ and taking the product

$$I^2=\left(\int_0^\infty e^{-u^2}du\right)\left(\int_0^\infty e^{-\alpha^2}d\alpha\right)=\left(\int_0^\infty \alpha e^{-x^2\alpha^2}dx\right)\left(\int_0^\infty e^{-\alpha^2}d\alpha\right)=\int_0^\infty\left(\int_0^\infty \alpha e^{-\alpha^2(1+x^2)}dx\right)d\alpha$$

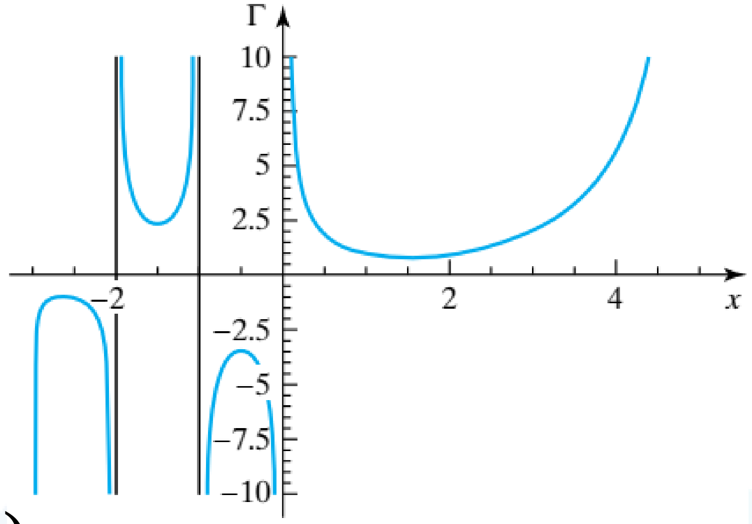
Reversing the order of integration

$$I^2=\int_0^\infty\left(\int_0^\infty \alpha e^{-\alpha^2(1+x^2)}d\alpha\right)dx=\int_0^\infty \left.\frac{-e^{-\alpha^2(1+x^2)}}{2(1+x^2)}\right|_0^\infty dx=\frac{1}{2}\int_0^\infty\frac{1}{(1+x^2)}dx=\frac{1}{2}\arctan x\,|_0^\infty=\frac{\pi}{4}\Rightarrow I=\frac{\sqrt{\pi}}{2}$$

The Eulerian Integrals

The Gamma Function

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt \quad \text{for } x > 0.$$



Properties of The Gamma Function

1. $\Gamma(x+1) = x\Gamma(x)$ Integration by part $\Gamma(x) = \dfrac{\Gamma(x+1)}{x}$

2. $\Gamma(1) = 1$ $\Gamma(1) = \int_0^\infty t^0 e^{-t} dt = 1$

3. $\Gamma(n+1) = n!$ n is a positive integer

$\xrightarrow{n=0}$ $0! = 1$

The Eulerian Integrals

The Gamma Function

4 $$\Gamma(x)\Gamma(1-x)=\frac{\pi}{\sin \pi x} \qquad x\in(0,1)$$

$$\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)=\frac{\pi}{\sin \pi/2}=\pi \quad \Longrightarrow \quad \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$$

5 Let $x=-y \quad \Longrightarrow \quad \Gamma(-y)=\dfrac{-1}{y}\Gamma(1-y)$

The Eulerian Integrals

The Gamma Function

Example Evaluate $\Gamma\left(\frac{-1}{2}\right), \Gamma\left(\frac{-3}{2}\right)$

$$\Gamma\left(\frac{-1}{2}\right)=\frac{\Gamma(1/2)}{-1/2}=-2\sqrt{\pi} \quad , \quad \Gamma\left(\frac{-3}{2}\right)=\frac{\Gamma(-1/2)}{-3/2}=\frac{4}{3}\sqrt{\pi}$$

Example Use Gamma function to compute Euler-Poisson integral $I=\int_0^\infty e^{-x^2}dx$

$$x^2=t \Rightarrow x=0 \Leftrightarrow t=0 \quad , \quad x\to\infty \Leftrightarrow t\to\infty \qquad x=t^{\frac{1}{2}} \qquad dx=\frac{1}{2}t^{-\frac{1}{2}}dt$$

$$\int_0^\infty e^{-x^2}dx=\int_0^\infty \frac{1}{2}t^{-\frac{1}{2}}e^{-t}\,dt=\frac{1}{2}\Gamma\left(-\frac{1}{2}+1\right)=\frac{1}{2}\Gamma\left(\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2}$$

The Eulerian Integrals

The Gamma Function

Example Use Gamma function to compute the integral $I = \int_0^{\infty} 3^{-x^2} dx$

$$x^2 = t \Rightarrow x = 0 \Leftrightarrow t = 0 \quad , \quad x \to \infty \Leftrightarrow t \to \infty \qquad x = t^{\frac{1}{2}} \qquad dx = \frac{1}{2} t^{-\frac{1}{2}} dt$$

$$\int_0^{\infty} 3^{-x^2} dx = \frac{1}{2}\int_0^{\infty} t^{-\frac{1}{2}} 3^{-t}\, dt = \frac{1}{2}\int_0^{\infty} t^{-\frac{1}{2}} e^{-t\ln 3}\, dt$$

$$u = t\ln 3 \Rightarrow dt = \frac{du}{\ln 3}$$

$$\int_0^{\infty} 3^{-x^2} dx = \frac{1}{2}\int_0^{\infty} t^{-\frac{1}{2}} e^{-t\ln 3}\, dt = \frac{1}{2\sqrt{\ln 3}}\int_0^{\infty} u^{-\frac{1}{2}} e^{-u} du = \frac{1}{2\sqrt{\ln 3}}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\frac{\pi}{\ln 3}}$$

The Eulerian Integrals

The Beta Function

$$\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx$$

- If $m\geq 1$ and $n\geq 1$ this is a proper integral.
- If $m>0,\ n>0$ and either or both $m<1,\ n<1$ the integral is improper but convergent.

Properties of The Beta Function

1. $\beta(m,n)=\beta(n,m)$ Symmetric
2. $\beta(m,n)=\dfrac{m-1}{m+n-1}\beta(m-1,n)\ ,\ m>1\ ,\ n>0$ **The Reduction Formula** Integration by parts
3. $\beta(m,n)=\dfrac{(n-1)!}{(m+n-1)\dots m}\ ,\ n\in\mathbb{N}$

The Eulerian Integrals

The Beta Function

4 $$\beta(m,n)=\frac{(n-1)!(m-1)!}{(m+n-1)!}\ ,\ n,m\in\mathbb{N}$$

5 $x=\sin^2\theta$ $\Rightarrow$ $$\beta(m,n)=2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta d\theta$$ Trigonometric formula

6 $$\beta(m,n)=\frac{\Gamma(m).\Gamma(n)}{\Gamma(m+n)}\ ,\ n,m\in\mathbb{R}^{+,*}$$

7 $$\beta(p,1-p)=\frac{\pi}{\sin p\pi}\ ,\ 0<p<1$$

The Eulerian Integrals

Example Use Gamma and Beta functions to compute the integral $\int_0^1 \sqrt{x}\,(1-x)^3 dx$

$$m-1=\frac{1}{2}\,,\; n-1=3 \quad \Longrightarrow \quad m=\frac{3}{2}\,,\; n=4$$

$$\int_0^1 \sqrt{x}\,(1-x)^3 dx = \beta\left(\frac{3}{2},4\right)=\frac{\Gamma\left(\frac{3}{2}\right)\Gamma(4)}{\Gamma\left(\frac{3}{2}+4\right)}=\frac{\left(\frac{1}{2}\right)\left(\sqrt{\pi}\right)(3!)}{\left(\frac{9}{2}\right)\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(\sqrt{\pi}\right)}=\frac{32}{315}$$

The Eulerian Integrals

Example Use Gamma and Beta functions to compute the integral $\int_0^{\pi/2} \sin^4\theta \cos^3\theta \, d\theta$

$$2m-1=4 \;,\; 2n-1=3 \quad \Longrightarrow \quad m=\frac{5}{2} \,,\, n=2$$

$$\int_0^{\pi/2} \sin^4\theta \cos^3\theta = \frac{1}{2}\beta\left(\frac{5}{2},2\right) = \frac{1}{2}\frac{\Gamma\left(\frac{5}{2}\right)\Gamma(2)}{\Gamma\left(\frac{9}{2}\right)} = \frac{1}{2}\frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(\sqrt{\pi}\right)(1)}{\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(\sqrt{\pi}\right)} = \frac{2}{35}$$

The Eulerian Integrals

Example Use Gamma and Beta functions to compute the integral $\int_0^{\pi} \cos^4\theta\, d\theta$

$$\int_0^{\pi} \cos^4\theta\, d\theta = \int_0^{\pi/2} \cos^4\theta\, d\theta + \int_{\pi/2}^{\pi} \cos^4\theta\, d\theta$$

setting $\theta = \theta_1 + \frac{\pi}{2}$ $\Rightarrow$ $\int_{\pi/2}^{\pi} \cos^4\theta\, d\theta = \int_0^{\frac{\pi}{2}} (-\sin\theta_1)^4\, d\theta_1 = \int_0^{\frac{\pi}{2}} \sin^4\theta_1\, d\theta_1$

$$\int_0^{\pi} \cos^4\theta\, d\theta = \int_0^{\pi/2} \cos^4\theta\, d\theta + \int_0^{\pi/2} \sin^4\theta\, d\theta$$

The Eulerian Integrals

$$\int_0^{\pi/2} \cos^4\theta \, d\theta$$

$$2m-1=0 \, , \, 2n-1=4$$

$$\Downarrow$$

$$m=\frac{1}{2} \, , \, n=\frac{5}{2}$$

$$\Downarrow$$

$$\int_0^{\pi/2} \cos^4\theta \, d\theta = \frac{1}{2}\beta\left(\frac{1}{2},\frac{5}{2}\right)$$

$$\int_0^{\pi/2} \sin^4\theta \, d\theta$$

$$2m-1=4 \, , \, 2n-1=0$$

$$\Downarrow$$

$$m=\frac{5}{2} \, , \, n=\frac{1}{2}$$

$$\Downarrow$$

$$\int_0^{\pi/2} \sin^4\theta \, d\theta = \frac{1}{2}\beta\left(\frac{5}{2},\frac{1}{2}\right)$$

The Eulerian Integrals

$$\beta(m,n)=\beta(n,m)$$

$$\int_0^{\pi}\cos^4\theta\,d\theta=\frac{1}{2}\left[\beta\left(\frac{5}{2},\frac{1}{2}\right)+\beta\left(\frac{1}{2},\frac{5}{2}\right)\right]=\beta\left(\frac{5}{2},\frac{1}{2}\right)$$

$$=\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(3)}=\frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(\sqrt{\pi}\right)\left(\sqrt{\pi}\right)}{2!}=\frac{3\pi}{8}$$

Example Use Gamma and Beta functions to compute the integral $\int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta$

$$\int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta=\int_0^{\pi/2}\sin^{1/2}\theta\cos^{-1/2}\theta\,d\theta$$

$$2m-1=\frac{1}{2}\ ,\ 2n-1=\frac{-1}{2}\ \Rightarrow\ m=\frac{3}{4}\ ,\ n=\frac{1}{4}$$

$$\Rightarrow\ \int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta=\frac{1}{2}\beta\left(\frac{3}{4},\frac{1}{4}\right)=\frac{1}{2}\,\frac{\pi}{\sin\frac{\pi}{4}}=\frac{\pi}{\sqrt{2}}$$