Problem sets 1,2 : Systems of Linear Equations-Matrices

CEDC102: Linear Algebra and Matrix Theory

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2023-2024

Problem 1. Suppose A is a 3×3 matrix, B is a 2×3 matrix, x is a 3-component column vector, and y is a 2-component column vector.
What is the **shape** of the output, say **nonsense** if the operation doesn't make sense

$$AB, \quad BA, \quad ABy, \quad BAx, \quad y^T Ax \quad x^T y \quad yx^T \quad yx \quad A^2 \quad B^2 \quad \frac{x}{A} \quad \frac{x}{x} \quad \frac{x}{3} \quad \frac{B}{3}$$

Problem 2. Consider Gaussian elimination on the following system of equations

$$\begin{aligned} 2x + 5y + z &= 0 \\ 4x + dy + z &= 2 \\ y - z &= 3 \end{aligned}$$

(Write your solution in matrix form.)

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- What number d forces you to do a row exchange during elimination, and what (non-singular) triangular system do you obtain for that d?
- What value of d would make this system singular (no third pivot, i.e. no way to get a triangular system with 3 nonzero values on the diagonal)?

Problem 3 A system of linear equations $Ax = b$ cannot have **exactly two** solution , why?

Problem 4 Suppose we want to solve $Ax = b$ for **more than one right-hand side** b. For example, suppose

$$A = \begin{bmatrix} 1 & 6 & -3 \\ -2 & 3 & 4 \\ 1 & 0 & -2 \end{bmatrix}, \qquad b_1 = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix}, \qquad b_2 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

Explain why solving both $Ax_1 = b_1$and $Ax_2 = b_2$ is equivalent to solving $AX = B$ where X is an unknown matrix (of what shape?) and B is a given matrix on the right-hand-side. Give B explicitly, and relate X to your desired solutions x_1 and x_2.

Solve your $AX = B$ equation by forming the augmented matrix (AB), reducing it to upper-triangular form (once), and doing backsubstitution (twice) to obtain X and hence x_1 and x_2

Problem 5 Consider the three 4×4 matrices

$$A = \begin{pmatrix} 1 & -3 & 0 & 3 \\ & 1 & 1 & 2 \\ & & 1 & -1 \\ & & & 1 \end{pmatrix}, \; B = \begin{pmatrix} -3 & -1 & 1 & -2 \\ -3 & 0 & 0 & -3 \\ 1 & 3 & 3 & -3 \\ 2 & 3 & 2 & 0 \end{pmatrix}, \; C = A(AB)^{-1}$$

Compute the third column of C^{-1} without computing the whole inverse of any matrix.

Problem 6 Consider the following 5×5 matrix, which is almost upper triangular:

$$A = \begin{pmatrix} 2 & -2 & 0 & 1 & 1 \\ 2 & -1 & 1 & -2 & 1 \\ & 3 & 4 & -3 & 1 \\ & & -2 & -5 & -3 \\ & & & 7 & 3 \end{pmatrix}$$

Compute (by hand) its LU factorization $A = LU$ via Gaussian elimination (i.e. give both L and U). Notice any special pattern to the nonzero entries in L and/or U?

Problem 7 consider

$$A = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$$

for some numbers a, b, c.

- When you perform the usual Gaussian elimination steps to L, what matrix will you obtain?
- If you apply the *same* row operations to I, what matrix will you get?
- If you apply the *same* row operations to LB for some $3 \times n$ matrix B, what will you get?

Problem 8 Suppose that A is a 4×4 matrix:

$$A = \begin{pmatrix} 1 & 3 & 2 & 5 \\ -2 & 1 & 1 & \\ 4 & 2 & & \\ -1 & & & \end{pmatrix}$$

a. Convert A into a standard lower triangular matrix, of the form

as a product of A with one or more permutation matrices on the left and/or right.

b. Suppose that we want to solve $Ax = b$ for $b = [26, -1, 10, -1]$. Using your answer from (a), write $x = A^{-1}b$ in the following form:

- $y =$ something easy with b?
- forward-substitution solve of $Lz = y$
- $x =$ something easy with z?

Problem9 For which three numbers k does elimination break down? Which is fixed by a row exchange? In each case, is the number of solutions 0 or 1 or ∞?

$$kx + 3y = 6$$
$$3x + ky = -6$$

Problem 10: Consider the matrix

$$\begin{bmatrix} 2 & 1 & 0 \\ 5 & 4 & 3 \\ 4 & 7 & 6 \end{bmatrix}$$

1. write it as the sum of a symmetric matrix and an skew-symmetric matrix
2. For a general matrix X, suppose you want to write it as X = S + A, where S is symmetric and A is skew-symmetric. Can you find formulas for S and A in terms of X only?