



Lecture 1-A: Systems of Linear Equations - Matrices

CEDC102: Linear Algebra

Manara University

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- Introduction to Systems of Linear Equations
- Gaussian Elimination
- Operations with Matrices
- Properties of Matrix Operations
- Inverse matrices
- Elementary Matrices
- LU factorization

Introduction to Systems of Linear Equations

- A linear equation in n variables: $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$

a_1, a_2, \dots, a_n, b : real numbers

a_1 : leading coefficient

x_1 : leading variable

- Notes:

(1) Linear equations have no products or roots of variables and no variables involved in trigonometric, exponential, or logarithmic functions.

(2) Variables appear only to the first power.

- Ex : (Linear or Nonlinear)

Linear (a) $3x + 2y = 7$

Linear (c) $x_1 - 2x_2 + 10x_3 + x_4 = 0$

NonLinear (e) $xy + z = 2$

Products

NonLinear (g) $\sin x_1 + 2x_2 - 3x_3 = 0$

Trigonometric functions

(b) $\frac{1}{2}x + y - \pi z = \sqrt{2}$ Linear

(d) $(\sin \frac{\pi}{2})x_1 - 4x_2 = e^2$ Linear

Exponential
(f) $e^x - 2y = 4$

NonLinear

(h) $\frac{1}{x} + \frac{1}{y} = 4$

Not the first power

- A solution of a linear equation in n variables:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

$$x_1 = s_1, x_2 = s_2, \dots, x_n = s_n \quad \text{such that: } a_1s_1 + a_2s_2 + \cdots + a_ns_n = b$$

- Solution set: the set of all solutions of a linear equation
- Ex : (Parametric representation of a solution set)

$$x_1 + 2x_2 = 4 \quad (2, 1) \text{ is a solution, i.e. } x_1 = 2, x_2 = 1$$

If you solve for x_1 in terms of x_2 , you obtain $x_1 = 4 - 2x_2$

By letting $x_2 = t$ you can represent the solution set as $x_1 = 4 - 2t$

And the solutions are $\{(4 - 2t, t) | t \in R\}$ or $\{(s, 2 - \frac{1}{2}s) | s \in R\}$

In vector form: $(x_1, x_2) = (4, 0) + t(-2, 1) = (0, 2) + s(1, -\frac{1}{2})$

- A system of m linear equations in n variables:

$$a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n = b_1$$

$$a_{21}X_1 + a_{22}X_2 + \cdots + a_{2n}X_n = b_2$$

⋮

$$a_{m1}X_1 + a_{m2}X_2 + \cdots + a_{mn}X_n = b_m$$

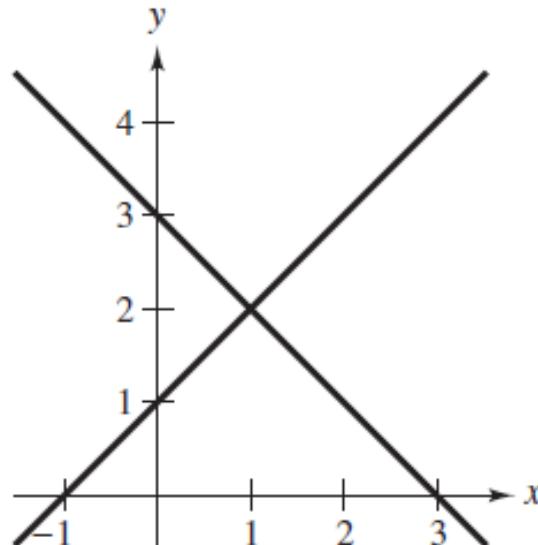
- **Consistent:** A system of linear equations has at least one solution.
- **Inconsistent:** A system of linear equations has no solution.
- **Notes:** Every system of linear equations has either
 - (1) exactly one solution,
 - (2) infinitely many solutions, or
 - (3) no solution.

- Ex : (Solution of a system of linear equations)

$$x + y = 3$$

$$x - y = -1$$

two intersecting lines

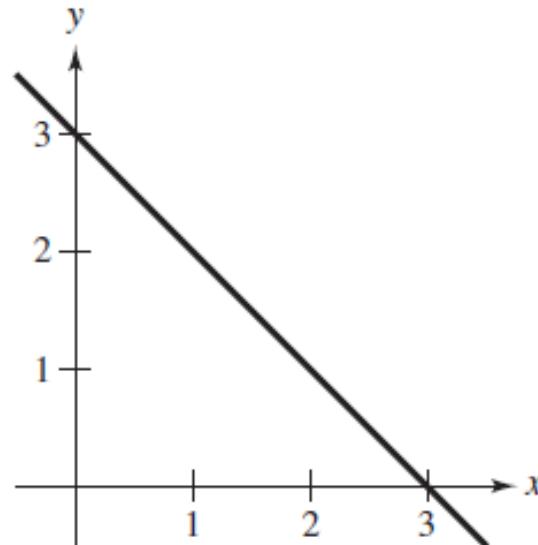


exactly one solution

$$x + y = 3$$

$$2x + 2y = 6$$

two coincident lines

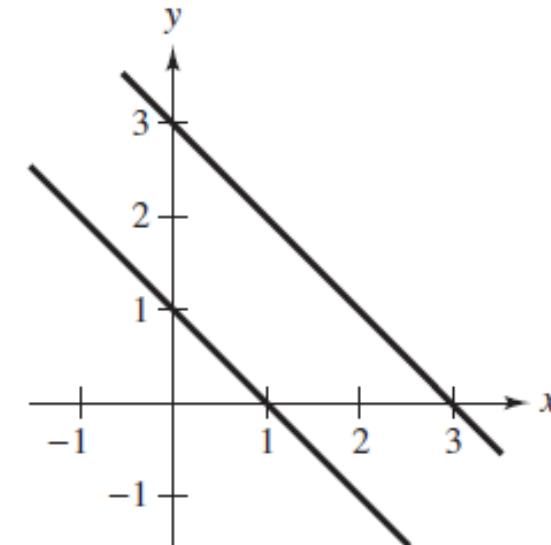


infinite number

$$x + y = 3$$

$$x + y = 1$$

two parallel lines



no solution

- Ex : (Using back substitution to solve a system in row echelon form)

$$x - 2y = 5 \quad (1)$$

$$y = -2 \quad (2)$$

Sol: By substituting $y = -2$ into (1), you obtain

$$x - 2(-2) = 5$$

$$x = 1$$

The system has exactly one solution: $x = 1, y = -2$

- Ex : (Using back substitution to solve a system in row echelon form)

$$x - 2y + 3z = 9 \quad (1)$$

$$y + 3z = 5 \quad (2)$$

$$z = 2 \quad (3)$$

Sol: Substitute $z=2$ into (2)

$$\begin{aligned}y + 3(2) &= 5 \\y &= -1\end{aligned}$$

and substitute $y=-1$ and $z=2$ into (1)

$$\begin{aligned}x - 2(-1) + 3(2) &= 9 \\x &= 1\end{aligned}$$

The system has exactly one solution: $x=1, y=-1, z=2$

- **Equivalent:**

Two systems of linear equations are called **equivalent** if they have precisely the same solution set

- **Notes:** Each of the following operations on a system of linear equations produces an equivalent system.

- (1) Interchange two equations.
- (2) Multiply an equation by a nonzero constant.
- (3) Add a multiple of an equation to another equation.

- **Ex 6: Solve a system of linear equations (consistent system)**

$$x - 2y + 3z = 9 \quad (1)$$

$$-x + 3y = -4 \quad (2)$$

$$2x - 5y + 5z = 17 \quad (3)$$

Sol:

$$(1) + (2) \rightarrow (2)$$

$$x - 2y + 3z = 9$$

$$y + 3z = 5$$

$$2x - 5y + 5z = 17$$

(4)

$$(1) \times (-2) + (3) \rightarrow (3)$$

$$x - 2y + 3z = 9$$

$$y + 3z = 5$$

$$-y - z = -1$$

(5)

$$(4) + (5) \rightarrow (5)$$

$$x - 2y + 3z = 9$$

$$y + 3z = 5$$

$$2z = 4$$

(6)

$$(6) \times \frac{1}{2} \rightarrow (6)$$

$$x - 2y + 3z = 9$$

$$y + 3z = 5$$

$$z = 2$$

So the solution is: $x=1, y=-1, z=2$

- Ex : Solve a system of linear equations (inconsistent system)

$$x_1 - 3x_2 + x_3 = 1 \quad (1)$$

$$2x_1 - x_2 - 2x_3 = 2 \quad (2)$$

$$x_1 + 2x_2 - 3x_3 = -1 \quad (3)$$

Sol: $(1) \times (-2) + (2) \rightarrow (2)$ $(1) \times (-1) + (3) \rightarrow (3)$

$$x_1 - 3x_2 + x_3 = 1$$

$$5x_2 - 4x_3 = 0 \quad (4)$$

$$5x_2 - 4x_3 = -2 \quad (5)$$

$$(4) \times (-1) + (5) \rightarrow (5)$$

$$x_1 - 3x_2 + x_3 = 1$$

$$5x_2 - 4x_3 = 0$$

$$0 = -2$$

(a false statement)

- Ex : Solve a system of linear equations (infinitely many solutions)

$$x_2 - x_3 = 0 \quad (1)$$

$$x_1 - 3x_3 = -1 \quad (2)$$

$$-x_1 + 3x_2 = 1 \quad (3)$$

Sol: $(1) \leftrightarrow (2)$

$$x_1 - 3x_3 = -1 \quad (1)$$

$$x_2 - x_3 = 0 \quad (2)$$

$$-x_1 + 3x_2 = 1 \quad (3)$$

$$(1) + (3) \rightarrow (3)$$

$$x_1 - 3x_3 = -1$$

$$x_2 - x_3 = 0$$

$$3x_2 - 3x_3 = 0$$

(4)

$$(2) \times (-3) + (4) \rightarrow (4)$$

$$x_1 - 3x_3 = -1$$

$$x_2 - x_3 = 0$$

0 = 0 **(a True statement)**

$$\Rightarrow x_2 = x_3, \quad x_1 = -1 + 3x_3$$

letting $x_3 = t$, then the solutions are: $\{(3t-1, t, t) | t \in R\}$

Gaussian Elimination and Gauss-Jordan Elimination

- ***m×n matrix:***

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

n columns

m rows

- **Notes:**

- (1) Every **entry** a_{ij} in a matrix is a number.
- (2) A matrix with *m rows* and *n columns* is said to be of **size** $m\times n$.
- (3) If $m = n$, then the matrix is called **square of order *n***.
- (4) For a square matrix, $a_{11}, a_{22}, \dots, a_{nn}$ are called **the main diagonal entries**.

- **Ex :** Matrix

$$[2]$$

Size

$$1 \times 1$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$2 \times 2$$

$$\begin{bmatrix} 1 & -3 & 0 & \frac{1}{2} \end{bmatrix}$$

$$1 \times 4$$

$$\begin{bmatrix} e & \pi \\ 2 & \sqrt{2} \\ -7 & 4 \end{bmatrix}$$

$$3 \times 2$$

- **Note:** One very common use of matrices is to represent a system of linear equations.

- A system of m equations in n variables:

$$\begin{aligned} a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n &= b_1 \\ a_{21}X_1 + a_{22}X_2 + \cdots + a_{2n}X_n &= b_2 \\ \vdots & \\ a_{m1}X_1 + a_{m2}X_2 + \cdots + a_{mn}X_n &= b_m \end{aligned}$$

Matrix form: $A\mathbf{x} = \mathbf{b}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

- Augmented matrix:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array} \right] = [A | \mathbf{b}]$$

- Coefficient matrix:

$$\left[\begin{array}{cccc} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{array} \right] = A$$

- Elementary row operation:

- (1) Interchange two rows.
- (2) Multiply a row by a nonzero constant.
- (3) Add a multiple of a row to another row.

$$r_{ij}: R_i \leftrightarrow R_j$$

$$r_i^{(k)}: (k)R_i \rightarrow R_i$$

$$r_{ij}^{(k)}: (k)R_i + R_j \rightarrow R_j$$

- Row equivalent: Two matrices are said to be **row equivalent** if one can be obtained from the other by a finite sequence of **elementary row operation**.

- Ex : (Elementary row operation)

$$\begin{bmatrix} 0 & 1 & 3 & 4 \\ -1 & 2 & 0 & 3 \\ 2 & -3 & 4 & 1 \end{bmatrix} \xrightarrow{r_{12}} \begin{bmatrix} -1 & 2 & 0 & 3 \\ 0 & 1 & 3 & 4 \\ 2 & -3 & 4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -4 & 6 & -2 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix} \xrightarrow{r_1^{(\frac{1}{2})}} \begin{bmatrix} 1 & -2 & 3 & -1 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 2 & 1 & 5 & -2 \end{bmatrix} \xrightarrow{r_{13}^{(-2)}} \begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 0 & -3 & 13 & -8 \end{bmatrix}$$

- Ex : Using elementary row operations to solve a system

Linear System

$$\begin{array}{l} x - 2y + 3z = 9 \\ -x + 3y = -4 \\ 2x - 5y + 5z = 17 \end{array}$$

$$\begin{array}{l} x - 2y + 3z = 9 \\ y + 3z = 5 \\ 2x - 5y + 5z = 17 \end{array}$$

$$\begin{array}{l} x - 2y + 3z = 9 \\ y + 3z = 5 \\ -y - z = -1 \end{array}$$

Augmented Matrix

$$\left[\begin{array}{cccc} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array} \right]$$

$$\left[\begin{array}{cccc} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 2 & -5 & 5 & 17 \end{array} \right]$$

$$\left[\begin{array}{cccc} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{array} \right]$$

Elementary Row Operation

$$r_{12}^{(1)}: (1)R_1 + R_2 \rightarrow R_2$$

$$r_{13}^{(-2)}: (-2)R_1 + R_3 \rightarrow R_3$$

Linear System

$$\begin{aligned}x - 2y + 3z &= 9 \\y + 3z &= 5 \\2z &= 4\end{aligned}$$

$$\begin{aligned}x - 2y + 3z &= 9 \\y + 3z &= 5 \\z &= 2\end{aligned}$$

$$\begin{array}{ccc|c}x & & & = 1 \\ \longrightarrow & y & & = -1 \\ & z & & = 2\end{array}$$

Associated Augmented Matrix

$$\left[\begin{array}{cccc} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{array} \right]$$

$$\left[\begin{array}{cccc} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

Elementary Row Operation

$$r_{23}^{(1)}: (1)R_2 + R_3 \rightarrow R_3$$

$$r_3^{(\frac{1}{2})}: (\frac{1}{2})R_3 \rightarrow R_3$$

Solve a system by Gauss-Jordan elimination method

The Idea of Elimination

- 1 For $m = n = 3$, there are three equations $Ax = b$ and three unknowns x_1, x_2, x_3 .
- 2 The first two equations are $a_{11}x_1 + \dots = b_1$ and $a_{21}x_1 + \dots = b_2$.
- 3 Multiply the first equation by a_{21}/a_{11} and subtract from the second : then x_1 is eliminated.
- 4 The corner entry a_{11} is the first “pivot” and the ratio a_{21}/a_{11} is the first “multiplier.”
- 5 Eliminate x_1 from every remaining equation i by subtracting a_{i1}/a_{11} times the first equation
- 6 Now the last $n - 1$ equations contain $n - 1$ unknowns x_2, \dots, x_n . Repeat to eliminate x_2 .
- 7 Elimination breaks down if zero appears in the pivot. Exchanging two equations may save it.

- Ex : Solve a system by Gauss-Jordan elimination method (one solution)

Sol:

augmented matrix

$$\left[\begin{array}{cccc} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array} \right] \xrightarrow{r_{12}^{(1)}, r_{13}^{(-2)}} \left[\begin{array}{cccc} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{array} \right] \xrightarrow{r_{23}^{(1)}} \left[\begin{array}{cccc} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{array} \right]$$

$$\xrightarrow{r_3^{(\frac{1}{2})}} \left[\begin{array}{cccc} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array} \right] \xrightarrow{r_{31}^{(-3)}, r_{32}^{(-3)}, r_{21}^{(2)}} \left[\begin{array}{cccc} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array} \right] \rightarrow \begin{aligned} x &= 1 \\ y &= -1 \\ z &= 2 \end{aligned}$$

- Ex : Solve a system by G.J. elimination method (infinitely many solutions)

$$\begin{array}{rcl} 2x_1 + 4x_2 - 2x_3 & = & 0 \\ 3x_1 + 5x_2 & = & 1 \end{array}$$

Sol:

augmented matrix

$$\left[\begin{array}{rrr|r} 2 & 4 & -2 & 0 \\ 3 & 5 & 0 & 1 \end{array} \right] \xrightarrow{r_1^{(\frac{1}{2})}, r_2^{(-3)}, r_2^{(-1)}, r_{21}^{(-2)}} \left[\begin{array}{rrr|r} 1 & 0 & 5 & 2 \\ 0 & 1 & -3 & -1 \end{array} \right]$$

$$\rightarrow \begin{array}{rcl} x_1 & + & 5x_3 = 2 \\ & x_2 & - 3x_3 = -1 \end{array}$$

leading variables: x_1, x_2

free variable: x_3

$$x_1 = 2 - 5x_3$$

$$x_2 = -1 + 3x_3$$

$x_3 = t$, then the solutions are: $\{(2 - 5t, -1 + 3t, t) | t \in R\}$

So the system has infinitely many solutions.

- Ex : Solve a system by Gauss-Jordan elimination method (no solution)

$$X_1 - X_2 + 2X_3 = 4$$

$$X_1 + X_3 = 6$$

$$2X_1 - 3X_2 + 5X_3 = 4$$

$$3X_1 + 2X_2 - X_3 = 1$$

Sol:

augmented matrix

$$\left[\begin{array}{cccc} 1 & -1 & 2 & 4 \\ 1 & 0 & 1 & 6 \\ 2 & -3 & 5 & 4 \\ 3 & 2 & -1 & 1 \end{array} \right] \xrightarrow{r_{12}^{(-1)}, r_{13}^{(-2)}, r_{14}^{(-3)}, r_{23}^{(1)}} \left[\begin{array}{cccc} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & -2 \\ 0 & 5 & -7 & -11 \end{array} \right]$$

$$\begin{array}{rcll} x_1 & - & x_2 & + 2x_3 = 4 \\ & & x_2 & - x_3 = 2 \\ \longrightarrow & & & 0 = -2 \\ & & 5x_2 & - 7x_3 = -11 \end{array}$$

Because the third equation is not possible, the system has no solution.

■ REVIEW OF THE KEY IDEAS ■

1. A linear system ($Ax = b$) becomes **upper triangular** ($Ux = c$) after elimination.
2. We **subtract** ℓ_{ij} times equation j from equation i , to make the (i, j) entry zero.
3. The **multiplier** is $\ell_{ij} = \frac{\text{entry to eliminate in row } i}{\text{pivot in row } j}$. **Pivots** can not be zero!
4. When zero is in the pivot position, **exchange rows** if there is a nonzero below it.
5. The upper triangular $Ux = c$ is solved by **back substitution** (starting at the bottom).
6. When **breakdown** is permanent, $Ax = b$ has no solution or infinitely many.

Operations with Matrices

- **Matrix:**

$$A = [a_{ij}]_{m \times n} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \in M_{m \times n}(K)$$

K : a field (R or C)

$M_{m \times n}(R)$: Real matrices

$M_{m \times n}(C)$: Complex matrices

(i, j)-th entry: a_{ij}

row: m

column: n

size: $m \times n$

- **i -th row vector**

$$r_i = [a_{i1} \ a_{i2} \ \dots \ a_{in}] \quad \text{row matrix}$$

- j -th column vector

$$c_j = \begin{bmatrix} c_{1j} \\ c_{2j} \\ \vdots \\ c_{mj} \end{bmatrix}$$

column matrix

- Square matrix: $m = n$

- Diagonal matrix:

$$A = \text{diag}(d_1, d_2, \dots, d_n) =$$

$$\begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \in M_{n \times n}(K)$$

- **Trace:**

If $A = [a_{ij}]_{n \times n}$ Then $Tr(A) = a_{11} + a_{22} + \cdots + a_{nn} = \sum_{i=1}^n a_{ii}$

- **Ex :**

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \Rightarrow r_1 = [1 \ 2 \ 3], \quad r_2 = [4 \ 5 \ 6]$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = [c_1 \ c_2 \ c_3] \Rightarrow c_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \quad c_3 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

- **Complex matrices**

$$\begin{bmatrix} 1 & -1 & i \\ 3 & 2i & 0 \end{bmatrix}, \quad [1 \ i \ -i \ 1], \quad \begin{bmatrix} 1 \\ \sqrt{2}i \end{bmatrix}, \quad \begin{bmatrix} 1+i & 1 \\ i & 1-i \end{bmatrix}$$

- **Equal matrix:**

If $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$ Then $A = B$ if and only if $a_{ij} = b_{ij} \forall 1 \leq i \leq m, 1 \leq j \leq n$

- **Ex : (Equal matrix)**

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If $A = B$ Then $a = 1, b = 2, c = 3, d = 4$

- **Matrix addition:**

If $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$ Then $A + B = [a_{ij}]_{m \times n} + [b_{ij}]_{m \times n} = [a_{ij} + b_{ij}]_{m \times n}$

- Ex : (Matrix addition)

$$\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1+1 & 2+3 \\ 0-1 & 1+2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1-1 \\ -3+3 \\ -2+2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Scalar multiplication:

If $A = [a_{ij}]_{m \times n}$, c : scalar ($\in K$) Then $cA = [ca_{ij}]_{m \times n}$

- Matrix subtraction:

$$A - B = A + (-1)B$$

- Ex : (Scalar multiplication and matrix subtraction)

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$$

Find (a) $3A$, (b) $-B$, (c) $3A - B$

Sol:

(a)

$$3A = 3 \begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3(1) & 3(2) & 3(4) \\ 3(-3) & 3(0) & 3(-1) \\ 3(2) & 3(1) & 3(2) \end{bmatrix} = \begin{bmatrix} 3 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix}$$

(b)

$$-B = (-1) \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ -1 & 4 & -3 \\ 1 & -3 & -2 \end{bmatrix}$$

(c)

$$3A - B = \begin{bmatrix} 3 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 12 \\ -10 & 4 & -6 \\ 7 & 0 & 4 \end{bmatrix}$$

- Matrix multiplication: First way

If $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{n \times p}$

Then $AB = [a_{ij}]_{m \times n} [b_{ij}]_{n \times p} = [c_{ij}]_{m \times p}$



Size of AB

where $c_{ij} = \sum_{k=1}^n a_{ik} b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$

$$\left[\begin{array}{cccc} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{array} \right] \left[\begin{array}{ccc} b_{11} & \dots & b_{1j} & \dots & b_{1p} \\ b_{21} & \dots & b_{2j} & \dots & b_{2p} \\ \vdots & & \vdots & & \vdots \\ b_{nj} & \dots & b_{nj} & \dots & b_{np} \end{array} \right] = \left[\begin{array}{cccc} c_{11} & \dots & c_{1j} & \dots & c_{1p} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \dots & c_{ij} & \text{(circled)} & c_{ip} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \dots & c_{mj} & \dots & c_{mp} \end{array} \right]$$

- **Notes:** (1) $A + B = B + A$, (2) $AB \neq BA$
 - **Ex :** (Find AB) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$A = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}$$

$$\text{Sol: } AB = \begin{bmatrix} (-1)(-3) + (3)(-4) & (-1)(2) + (3)(1) \\ (4)(-3) + (-2)(-4) & (4)(2) + (-2)(1) \\ (5)(-3) + (0)(-4) & (5)(2) + (0)(1) \end{bmatrix} = \begin{bmatrix} -9 & 1 \\ -4 & 6 \\ -15 & 10 \end{bmatrix}$$

- Matrix form of a system of linear equations

$$\left\{
 \begin{array}{l}
 a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n = b_1 \\
 a_{21}X_1 + a_{22}X_2 + \cdots + a_{2n}X_n = b_2 \\
 \vdots \\
 a_{m1}X_1 + a_{m2}X_2 + \cdots + a_{mn}X_n = b_m
 \end{array}
 \right.$$

↓

$$\begin{bmatrix}
 a_{11} & a_{12} & \dots & a_{1n} \\
 a_{21} & a_{22} & \dots & a_{2n} \\
 \vdots & \vdots & \vdots & \vdots \\
 a_{m1} & a_{m2} & \dots & a_{mn}
 \end{bmatrix}
 \begin{bmatrix}
 X_1 \\
 X_2 \\
 \vdots \\
 X_n
 \end{bmatrix}
 =
 \begin{bmatrix}
 b_1 \\
 b_2 \\
 \vdots \\
 b_m
 \end{bmatrix}$$

|| || ||
 A \mathbf{x} \mathbf{b}

m linear equation

↓

single matrix equation

$A\mathbf{x} = \mathbf{b}$

$m \times n$ $n \times 1$ $m \times 1$

$$A\mathbf{X} = \begin{bmatrix} a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n \\ a_{21}X_1 + a_{22}X_2 + \cdots + a_{2n}X_n \\ \vdots \\ a_{m1}X_1 + a_{m2}X_2 + \cdots + a_{mn}X_n \end{bmatrix} = \mathbf{b}$$

$$A\mathbf{X} = X_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + X_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + X_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = X_1 \mathbf{a}_1 + X_2 \mathbf{a}_2 + \cdots + X_n \mathbf{a}_n = \mathbf{b}$$

$$X_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + X_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + X_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Linear combination of the column matrices $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ with coefficients X_1, X_2, \dots, X_n

- Note:

The system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if \mathbf{b} can be expressed as a linear combination of the columns of A , where the coefficients of the linear combination are a solution of the system.

- Ex : (Solve a system of linear equations)

$$x_1 + 2x_2 + 3x_3 = 0$$

The linear system $4x_1 + 5x_2 + 6x_3 = 3$

$$7x_1 + 7x_2 + 8x_3 = 6$$

can be rewritten as a matrix equation $A\mathbf{x} = \mathbf{b}$

$$x_1 \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} = \mathbf{b}$$

Using Gaussian elimination, you can show that this system has infinitely many solutions, one of which is $x_1 = 1$, $x_2 = 1$, $x_3 = -1$

$$1 \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix} + (-1) \begin{bmatrix} 3 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$$

That is, \mathbf{b} can be expressed as a linear combination of the columns of A

■ Partitioned Matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & | & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & | & a_{24} & a_{25} \\ \hline a_{31} & a_{32} & a_{33} & | & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & | & a_{44} & a_{45} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

submatrix

$$A = \begin{bmatrix} a_{11} & a_{12} & | & a_{13} & a_{14} & | & a_{15} \\ a_{21} & a_{22} & | & a_{23} & a_{24} & | & a_{25} \\ \hline a_{31} & a_{32} & | & a_{33} & a_{34} & | & a_{35} \\ a_{41} & a_{42} & | & a_{43} & a_{44} & | & a_{45} \end{bmatrix} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & \hat{A}_{13} \\ \hat{A}_{21} & \hat{A}_{22} & \hat{A}_{23} \end{bmatrix}$$

Block multiplication

$$A = \begin{bmatrix} a_{11} & a_{12} & | & a_{13} & a_{14} \\ a_{21} & a_{22} & | & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & | & a_{33} & a_{34} \\ a_{41} & a_{42} & | & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$B = \begin{bmatrix} b_{11} & b_{12} & | & b_{13} \\ b_{21} & b_{22} & | & b_{23} \\ \hline b_{31} & b_{32} & | & b_{33} \\ b_{41} & b_{42} & | & b_{43} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & | & A_{11}B_{12} + A_{12}B_{22} \\ \hline A_{21}B_{11} + A_{22}B_{21} & | & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

- Ex :

$$A = \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 2 & 3 & -1 \\ \hline 2 & 0 & -4 & 0 \\ 0 & 1 & 0 & 3 \end{array} \right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$B = \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 & 2 & 2 \\ \hline 1 & 3 & 0 & 0 & 1 & 0 \\ -3 & -1 & 2 & 1 & 0 & -1 \end{array} \right] = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

Sol: $AB = \left[\begin{array}{c|c} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ \hline A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{array} \right] = C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$

$$C_{11} = A_{11}B_{11} + A_{12}B_{21} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ -3 & -1 & 2 \end{bmatrix}$$

$$C_{11} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 0 \\ 6 & 10 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 0 \\ 6 & 12 & 0 \end{bmatrix}$$

$$C_{12} = \begin{bmatrix} 1 & 2 & -1 \\ -3 & 7 & 5 \end{bmatrix}, \quad C_{21} = \begin{bmatrix} 0 & -12 & 0 \\ -9 & -2 & 7 \end{bmatrix}, \quad C_{22} = \begin{bmatrix} 2 & -2 & -2 \\ 2 & 2 & -1 \end{bmatrix}$$

$$AB = C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \left[\begin{array}{ccc|ccc} 3 & 3 & 0 & 1 & 2 & -1 \\ 6 & 12 & 0 & -3 & 7 & 5 \\ 0 & -12 & 0 & 2 & -2 & -2 \\ -9 & -2 & 7 & 2 & 2 & -1 \end{array} \right]$$

Properties of Matrix Operations

- Three basic matrix operators:

- (1) matrix addition
- (2) scalar multiplication
- (3) matrix multiplication

- Zero matrix: $O_{m \times n}$

- Identity matrix of order n : I_n

- Properties of matrix addition and scalar multiplication:

If $A, B, C \in M_{m \times n}(K)$ then

$$(1) A + B = B + A$$

$$(4) 1A = A$$

$$(2) A + (B + C) = (A + B) + C$$

$$(5) c(A + B) = cA + cB$$

$$(3) (cd)A = c(dA)$$

$$(6) (c + d)A = cA + dA$$

- Properties of zero matrices:

If $A \in M_{m \times n}(K)$, c : scalar then

$$(1) A + O_{m \times n} = A$$

$$(2) A + (-A) = O_{m \times n}$$

$$(3) cA = O_{m \times n} \Rightarrow c = 0 \text{ or } A = O_{m \times n}$$

- Notes:

(1) $O_{m \times n}$: **the additive identity** for the set of all $m \times n$ matrices

(2) $-A$: **the additive inverse** of A

- Properties of matrix multiplication:

$$(1) A(BC) = (AB)C$$

$$(2) A(B + C) = AB + AC$$

$$(3) (A + B)C = AC + BC$$

$$(4) c(AB) = (cA)B = A(cB)$$

- Properties of identity matrix:

If $A \in M_{m \times n}(K)$, then

$$(1) AI_n = A$$

$$(2) I_mA = A$$

- Transpose of a matrix:

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in M_{m \times n}(K)$$

$$\text{Then } A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix} \in M_{n \times m}(K)$$

- Ex : (Find the transpose of the following matrix)

$$(a) \ A = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$$

$$(b) \ A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$(c) \ A = \begin{bmatrix} 0 & 1 \\ 2 & 4 \\ 1 & -1 \end{bmatrix}$$

Sol: (a) $A = \begin{bmatrix} 2 \\ 8 \end{bmatrix} \Rightarrow A^T = [2 \ 8]$

$$(b) \ A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

$$(c) \ A = \begin{bmatrix} 0 & 1 \\ 2 & 4 \\ 1 & -1 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 4 & -1 \end{bmatrix}$$

- Properties of transposes:

$$(1) \ (A^T)^T = A$$

$$(2) \ (A + B)^T = A^T + B^T$$

$$(3) \ (cA)^T = c(A^T)$$

$$(4) \ (AB)^T = B^T A^T$$

- Symmetric matrix:

A square matrix A is **symmetric** if $A^T = A$

- Skew-symmetric matrix:

A square matrix A is **skew-symmetric** if $A^T = -A$

- Ex :

If $A = \begin{bmatrix} 1 & 2 & 3 \\ a & 4 & 5 \\ b & c & 6 \end{bmatrix}$ is symmetric, find a, b, c ?

Sol:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ a & 4 & 5 \\ b & c & 6 \end{bmatrix}, A^T = \begin{bmatrix} 1 & a & b \\ 2 & 4 & c \\ 3 & 5 & 6 \end{bmatrix} \quad A = A^T \Rightarrow a = 2, \ b = 3, \ c = 5$$

- Ex 3:

If $A = \begin{bmatrix} 0 & 1 & 2 \\ a & 0 & 3 \\ b & c & 0 \end{bmatrix}$ is a skew-symmetric, find a, b, c ?

Sol:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ a & 0 & 3 \\ b & c & 0 \end{bmatrix}, \quad -A^T = \begin{bmatrix} 0 & -a & -b \\ -1 & 0 & -c \\ -2 & -3 & 0 \end{bmatrix}$$

$$A = -A^T \Rightarrow a = -1, \quad b = -2, \quad c = -3$$

▪ Notes:

(1) AA^T is symmetric

(2) Every square matrix $A \in M_n(R)$ can be expressed as the sum of a symmetric matrix B and a skew-symmetric matrix C

$$B = \frac{1}{2}(A + A^T), \quad C = \frac{1}{2}(A - A^T)$$

- Real (Complex) number:

$ab = ba$ (Commutative law for multiplication)

- Matrix:

$$AB \neq BA$$

$m \times n$ $n \times p$

Three situations:

(1) If $m \neq p$, then AB is defined, BA is undefined

(2) If $m = p$, $m \neq n$, then

$$AB \in M_{m \times m}(K), BA \in M_{n \times n}(K)$$

(Sizes are not the same)

(3) If $m = p = n$, then

$$AB \in M_{m \times m}(K), BA \in M_{m \times m}(K)$$

(Sizes are the same, but matrices are not equal)

- **Ex :**

Show that AB and BA are not equal for the matrices.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$$

Sol:

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 4 & -4 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 7 \\ 4 & -2 \end{bmatrix}$$

- **Note:** $AB \neq BA$

- Real (Complex) number:

$$ac = bc, c \neq 0 \Rightarrow a = b \quad (\text{Cancellation law})$$

- Matrix:

$$AC = BC, \quad C \neq O$$

(1) If C is invertible, then $A = B$

(2) If C is not invertible, then $A \neq B$ (Cancellation is not valid)

- Ex : (An example in which cancellation is not valid)

Show that $AC = BC$

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$$

Sol:

$$AC = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$$

$$BC = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$$

So $AC = BC$ but $A \neq B$