



Calculus 2

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Calculus 2

Lecture 3

Improper Integrals

Improper Integrals

- **Evaluate an improper integral that has an infinite limit of integration.**
- **Evaluate an improper integral that has an infinite discontinuity.**
- **Tests for Convergence and Divergence**

Quick Review

If f is continuous on the interval $[a,b]$ and F is any function that satisfies $F'(x) = f(x)$ throughout this interval then

$$\int_a^b f(x) dx = F(b) - F(a)$$

REMEMBER: $[a,b]$ is a closed interval



introduction

Up to now we have focused on **definite integrals** with **continuous integrands** and **finite intervals of integration**.

we extend the concept of a definite integral to the cases where:

- The interval is infinite
- f has an infinite discontinuity in $[a, b]$

These are called **improper integrals**

Recall in the definition of $\int_a^b f(x)dx$,

the interval $[a, b]$ was finite. If a or b (or both) are ∞ or $-\infty$, we call the integral an improper integral of type 1 with an infinite interval. **For example:**

- $\int_0^{\infty} \frac{dx}{1+x^2}$
- $\int_{-\infty}^{-1} xe^{-x^2} dx$
- $\int_{-\infty}^{\infty} x^2 e^{-x^2} dx$



Improper integral: type I:

- 1) If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx$$

- 2) If $f(x)$ is continuous on $[-\infty, b)$, then

$$\int_{-\infty}^b f(x) dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) dx$$

- 3) If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x) dx = \underbrace{\int_{-\infty}^c f(x) dx + \int_c^{\infty} f(x) dx}$$

If the Limit is finite, then the Improper Integral converges.

If the Limit fails, then it diverges

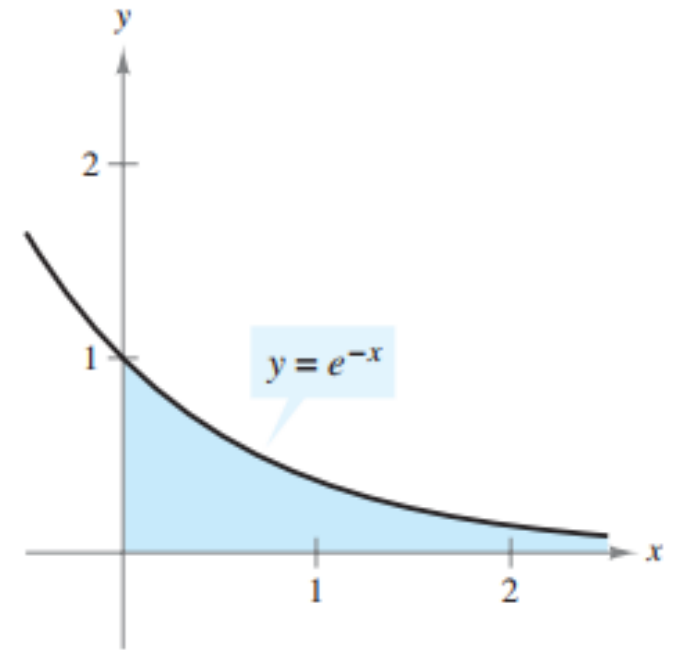
If both Improper Integrals converge, then so does



Evaluating an Improper Integral on $[1, \infty)$

Does the improper integral $\int_0^{\infty} e^{-x} dx$ converge or diverge?

$$\begin{aligned}\int_0^{\infty} e^{-x} dx &= \lim_{b \rightarrow \infty} \int_0^b e^{-x} dx \\ &= \lim_{b \rightarrow \infty} \left(-e^{-x} \Big|_0^b \right) \\ &= \lim_{b \rightarrow \infty} \left(-e^{-b} + e^{-0} \right) \\ &= \lim_{b \rightarrow \infty} \left(1 - e^{-b} \right) \\ &= 1\end{aligned}$$





Example Using L'Hôpital's Rule with Improper Integrals

Evaluate $\int_1^{\infty} x e^{-x} dx$.

$$\int_1^{\infty} x e^{-x} dx = \lim_{b \rightarrow \infty} \int_1^b x e^{-x} dx$$

$$\begin{array}{l} u = x \\ du = 1 dx \end{array} \quad \begin{array}{l} dv = e^{-x} dx \\ v = -e^{-x} \end{array}$$

$$\begin{aligned} &= \lim_{b \rightarrow \infty} \left(-x e^{-x} \Big|_1^b - \int_1^b -e^{-x} dx \right) \\ &= \lim_{b \rightarrow \infty} \left(\left(-x e^{-x} - e^{-x} \right) \Big|_1^b \right) \\ &= \lim_{b \rightarrow \infty} \left(\left(-b e^{-b} - e^{-b} \right) - \left(-1 \cdot e^{-1} - e^{-1} \right) \right) \end{aligned}$$

$$= \lim_{b \rightarrow \infty} \left(-e^{-b} (b+1) + 2e^{-1} \right)$$

$$= \lim_{b \rightarrow \infty} \left(-\frac{(b+1)}{e^b} \right) + \frac{2}{e}$$

Use L'Hôpital's Rule

$$= \lim_{b \rightarrow \infty} \left(-\frac{1}{e^b} \right) + \frac{2}{e}$$

$$\boxed{= \frac{2}{e}}$$



Express the Example Evaluating an Integral on $(-\infty, \infty)$

Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$.

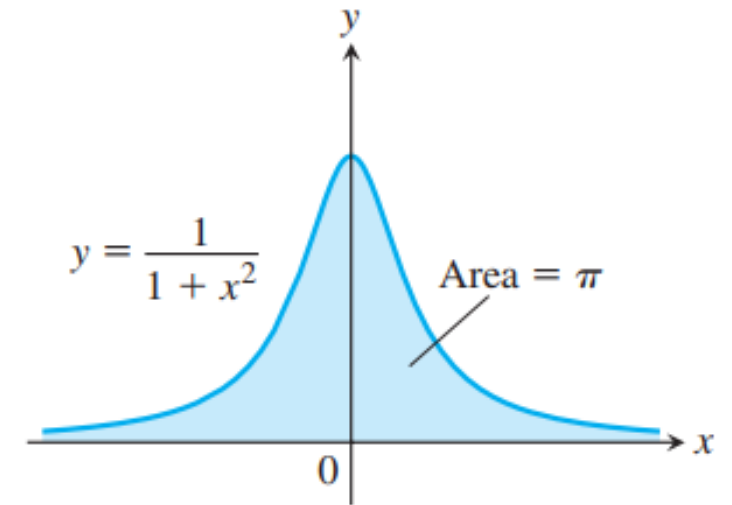
$$= \left(0 - -\frac{\pi}{2} \right) + \left(\frac{\pi}{2} - 0 \right) = \pi$$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \int_{-\infty}^0 \frac{1}{1+x^2} dx + \int_0^{\infty} \frac{1}{1+x^2} dx$$

$$= \lim_{a \rightarrow -\infty} \int_a^0 \frac{1}{1+x^2} dx + \lim_{b \rightarrow \infty} \int_0^b \frac{1}{1+x^2} dx$$

$$= \lim_{a \rightarrow -\infty} \left(\arctan x \Big|_a^0 \right) + \lim_{b \rightarrow \infty} \left(\arctan x \Big|_0^b \right)$$

$$= \lim_{a \rightarrow -\infty} \left(\arctan 0 - \arctan a \right) + \lim_{b \rightarrow \infty} \left(\arctan b - \arctan 0 \right)$$



IMPROPER INTEGRALS OF TYPE 2: INFINITE INTEGRANDS

Recall in the definition of $\int_a^b f(x)dx$,

the functions f was bounded on $[a, b]$. If f is not bounded on $[a, b]$ (that is, has an x -value, $a \leq x \leq b$, where the limit is ∞ or $-\infty$), we call the integral an improper integral of type 2 with infinite integrand. For example

$$\int_0^1 \frac{dx}{x^2}, \quad \int_{-2}^0 \frac{dx}{x^2}, \quad \int_{-2}^1 \frac{dx}{x^2}$$



Improper integral: type 2:

1) If $f(x)$ is continuous on $(a, b]$, then

$$\int_a^b f(x) dx = \lim_{c \rightarrow a^+} \int_c^b f(x) dx$$

2) If $f(x)$ is continuous on $[a, b)$, then

$$\int_a^b f(x) dx = \lim_{c \rightarrow b^-} \int_a^c f(x) dx$$

3) If $f(x)$ is continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

If the Limit is finite, then the Improper Integral converges.

If the Limit fails, then it diverges

If both Improper Integrals converge, then so does $\int_a^b f(x) dx$.



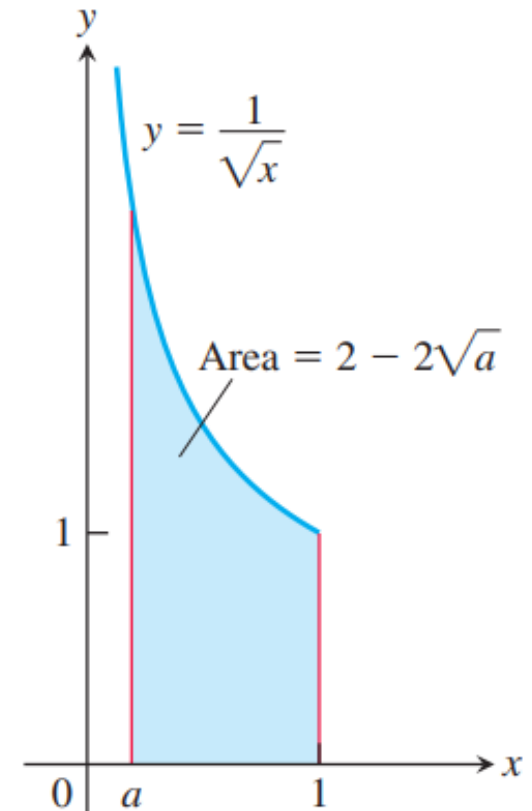
Example Infinite Discontinuity at an Interior Point

Evaluate $\int_0^1 \frac{1}{\sqrt{x}} dx$.

$$\int_0^1 \frac{1}{\sqrt{x}} dx = \lim_{c \rightarrow 0^+} \int_c^1 \frac{1}{\sqrt{x}} dx$$

$$= \lim_{c \rightarrow 0^+} \left(2\sqrt{x} \Big|_c^1 \right)$$

$$= \lim_{c \rightarrow 0^+} \left(2\sqrt{1} - 2\sqrt{c} \right) = 2$$





Example Infinite Discontinuity at an Interior Point

Evaluate $\int_0^3 \frac{1}{(x-1)^{2/3}} dx$.

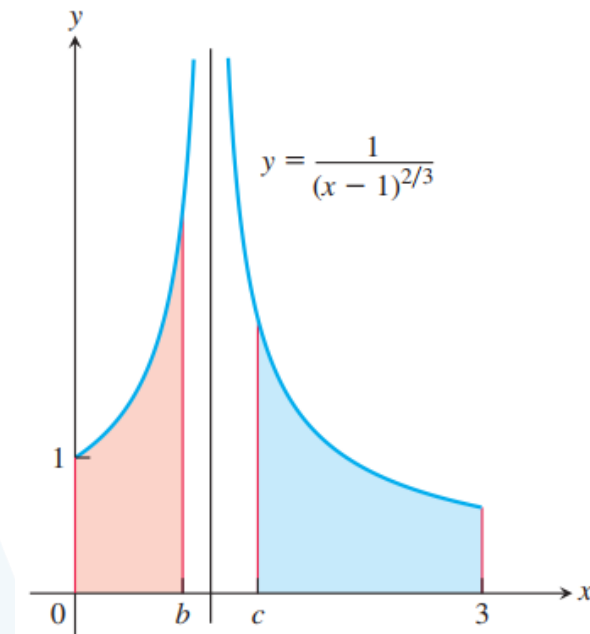
$$\int_0^3 \frac{1}{(x-1)^{2/3}} dx = \int_0^1 \frac{1}{(x-1)^{2/3}} dx + \int_1^3 \frac{1}{(x-1)^{2/3}} dx$$

$$= \lim_{c \rightarrow 1^-} \int_0^c \frac{1}{(x-1)^{2/3}} dx + \lim_{c \rightarrow 1^+} \int_c^3 \frac{1}{(x-1)^{2/3}} dx$$

$$= \lim_{c \rightarrow 1^-} \left(3(x-1)^{1/3} \Big|_0^c \right) + \lim_{c \rightarrow 1^+} \left(3(x-1)^{1/3} \Big|_c^3 \right)$$

$$= \lim_{c \rightarrow 1^-} \left(3(c-1)^{1/3} - 3(0-1)^{1/3} \right) + \lim_{c \rightarrow 1^+} \left(3(3-1)^{1/3} - 3(c-1)^{1/3} \right)$$

$$= (0 + 3) + (3\sqrt[3]{2} - 0) = 3 + 3\sqrt[3]{2}$$





Example : Improper integral

$$\int_1^{\infty} e^{-x} dx \quad \leftarrow \text{Converges}$$

$$\lim_{b \rightarrow \infty} \int_1^b e^{-x} dx$$

$$\lim_{b \rightarrow \infty} -e^{-x} \Big|_1^b$$

$$\lim_{b \rightarrow \infty} -e^{-b} - (-e^{-1})$$

$$\lim_{b \rightarrow \infty} -\frac{1}{e^b} + \frac{1}{e} = \frac{1}{e}$$



Example : Improper integral

Does $\int_1^{\infty} e^{-x^2} dx$ converge?

Compare:

$\frac{1}{e^{x^2}}$ to $\frac{1}{e^x}$ for positive values of x .

For $x > 1$, $e^{x^2} > e^x \therefore \frac{1}{e^{x^2}} < \frac{1}{e^x}$

Since $\frac{1}{e^x}$ converges to a finite number, $\frac{1}{e^{x^2}}$ must also converge!

Comparison Test (Using estimation to show convergence or divergence)

Let f and g be continuous on $[a, \infty)$ with $0 \leq f(x) \leq g(x)$ for all $x \geq a$, then:

① $\int_a^{\infty} f(x) dx$ converges if $\int_a^{\infty} g(x) dx$ converges.

② $\int_a^{\infty} g(x) dx$ diverges if $\int_a^{\infty} f(x) dx$ diverges.



Using estimation to show convergence or divergence

Example

$$\int_1^{\infty} \frac{\sin^2 x}{x^2} dx$$

Solution:

$$0 \leq \frac{\sin^2 x}{x^2} \leq \frac{1}{x^2} \quad \text{on } [1, \infty)$$

Since $\frac{1}{x^2}$ converges, $\frac{\sin^2 x}{x^2}$ converges.



Using estimation to show convergence or divergence

Example

$$\int_1^{\infty} \frac{1}{\sqrt{x^2 - 0.1}} dx$$

Solution:

$$\sqrt{x^2 - 0.1} < x \quad \text{for positive values of } x, \text{ so:}$$

$$\frac{1}{\sqrt{x^2 - 0.1}} \geq \frac{1}{x} \quad \text{on} \quad [1, \infty)$$

Since $\frac{1}{x}$ diverges, $\frac{1}{\sqrt{x^2 - 0.1}}$ diverges.



Thank you for your attention