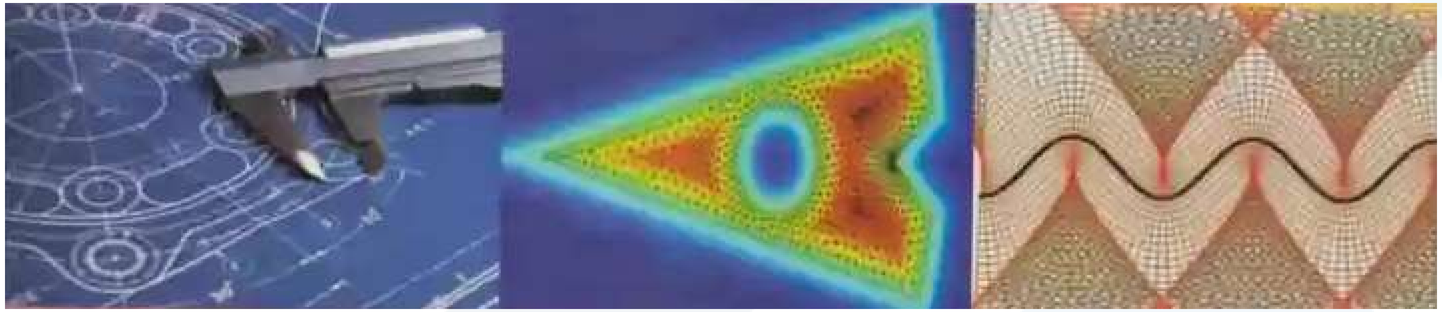
CEDC301: Engineering Mathematics

Lecture Notes 1: Functions of a Complex Variable: Part A



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Chapter 1

Functions of a Complex Variable

1. Complex Numbers
2. Powers and Roots
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4. Functions of a Complex Variable
5. Cauchy-Riemann Equations
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1. Complex Numbers

- **Definition:** A number of the form $z = x + iy$, where x and y are real numbers and $i = \sqrt{-1}$ (imaginary unit), is called a complex number.

  x is called the real part of z and is written as $Re(z)$ and y is called the imaginary part and is written as $Im(z)$.

  For example, if $z = 4 + 9i$, then $Re(z) = 4$ and $Im(z) = 9$

  A real constant multiple of the imaginary unit is called a pure imaginary number

- **Definition:** Complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are equal, $z_1 = z_2$, if $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$.

  A complex number $x + iy = 0$ if $x = 0$ and $y = 0$.

Arithmetic Operations

- If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

Addition: $z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$

Subtraction: $z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2)$

Multiplication: $z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1 x_2 - y_1 y_2 + i(y_1 x_2 + x_1 y_2)$

Division: $\dfrac{z_1}{z_2} = \dfrac{x_1 + iy_1}{x_2 + iy_2} = \dfrac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\,\dfrac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2}$

Commutative laws: $\begin{cases} z_1 + z_2 = z_2 + z_1 \\ z_1 z_2 = z_2 z_1 \end{cases}$

Associative laws: $\begin{cases} z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3 \\ z_1 (z_2 z_3) = (z_1 z_2) z_3 \end{cases}$

Distributive law: $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

- If $z = x + iy$ is a complex number, then the complex number $\bar{z} = x - iy$ is called the complex conjugate or, simply, the conjugate of z.

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}, \qquad \overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$$

$$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}, \qquad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$$

For example, if $z = 4 + 9i$, then $\bar{z} = 4 - 9i$

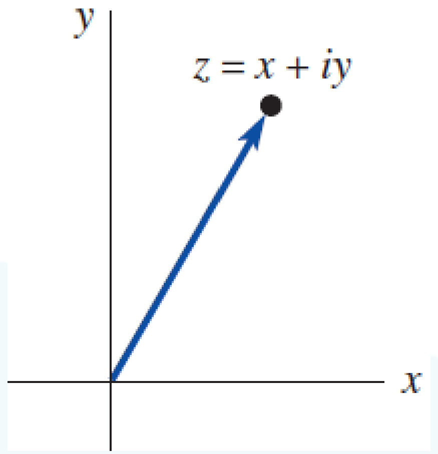
$$z + \bar{z} = (x + iy) + (x - iy) = 2x = 2Re(z)$$
$$z - \bar{z} = (x + iy) - (x - iy) = 2iy = 2Im(z)$$

$$\Rightarrow Re(z) = \frac{z + \bar{z}}{2}, \quad Im(z) = \frac{z - \bar{z}}{2i}$$

$$z\bar{z} = (x + iy)(x - iy) = x^2 + y^2$$

Geometric Interpretation

A complex number $z = x + iy$ can be viewed as a vector whose initial point is the origin and whose terminal point is (x, y). The coordinate plane is called the complex plane or simply the z-plane. The horizontal or x-axis is called the real axis and the vertical or y-axis is called the imaginary axis.



- Definition: The modulus or absolute value of $z = x + iy$, denoted by $|z|$, is the real number

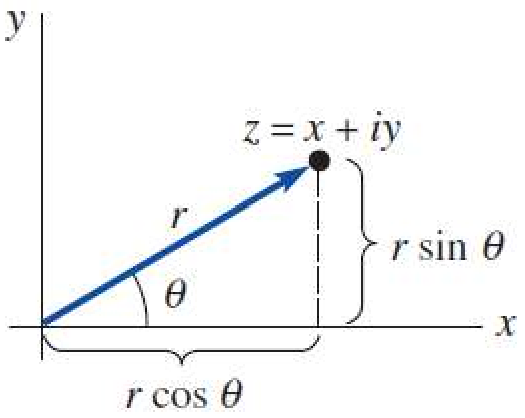
$$|z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$$

For example, if $z = 2 - 3i$, then $|z| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$

$|z_1 + z_2| \le |z_1| + |z_2|$ the triangle inequality $|z_1 + z_2| \ge |z_1| - |z_2|$

2. Powers and Roots

Polar Form



- A nonzero complex number $z = x + iy$ can be written as
  $z = (r\cos\theta) + i(r\sin\theta)$ or $z = r(\cos\theta + i\sin\theta)$ polar form
  $r = |z|$ $\theta = \arg z = \tan^{-1}(y/x)$
  θ measured in radians is called an argument of z $(\arg z)$.
- If θ_0 is an argument of z, then the angles $\theta_0 \pm 2\pi k$, are also arguments.
- The argument of a complex number in the interval $-\pi < \theta \leq \pi$ is called the principal argument of z and is denoted by $\operatorname{Arg} z$.

For example, if $z = 1 - \sqrt{3}i$, then $z = 2\left[\cos\left(-\dfrac{\pi}{3}\right) + i\sin\left(-\dfrac{\pi}{3}\right)\right]$

- If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

$$|z_1 z_2| = |z_1||z_2|$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$$

- Note: It is not true, in general, that $\text{Arg}(z_1 z_2) = \text{Arg } z_1 + \text{Arg } z_2$ and $\text{Arg}(z_1/z_2) = \text{Arg } z_1 - \text{Arg } z_2$ (although it may be true for some complex numbers).

  For example, if $z_1 = -1$ and $z_2 = 5i$, then

  $$\text{Arg}(z_1) = \pi,\ \text{Arg}(z_2) = \pi/2,\ \text{Arg}(z_1 z_2) = -\pi/2,\ \text{Arg } z_1 + \text{Arg } z_2 = 3\pi/2 \neq \text{Arg}(z_1 z_2)$$

  If $z_1 = -1$ and $z_2 = -5i$, then

  $$\text{Arg}(z_1) = \pi,\ \text{Arg}(z_2) = -\pi/2,\ \text{Arg}(z_1/z_2) = -\pi/2,\ \text{Arg } z_1 - \text{Arg } z_2 = 3\pi/2 \neq \text{Arg}(z_1/z_2)$$

Integer Powers of z

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

For example, if $z = 1 - \sqrt{3}i$, then $z^3 = 2^3\left[\cos\left(-\pi\right) + i\sin\left(-\pi\right)\right] = -8$

DeMoivre's Formula

$$(\cos\ \theta + i\sin\ \theta)^n = \cos\ n\theta + i\sin\ n\theta$$

Roots

A number w is said to be an n^{th} root of a nonzero complex number z if $w^n = z$.

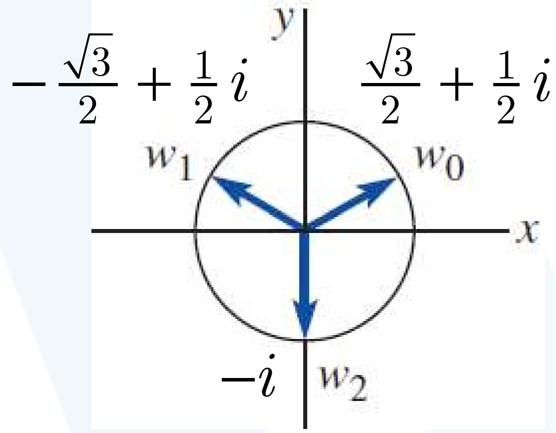
$z = r(\cos\theta + i\sin\theta) \Rightarrow$

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$$

where $k = 0, 1, 2, \ldots, n - 1$

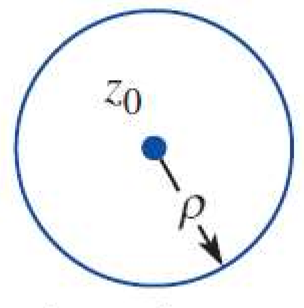
For example, the three cube roots of $z = i$ are:

$$w_k = 1^{1/3}\left[\cos\left(\frac{\pi/2 + 2\pi k}{3}\right) + i\sin\left(\frac{\pi/2 + 2\pi k}{3}\right)\right],\ k = 0, 1, 2$$



3. Sets in the Complex Plane

- Suppose $z_0 = x_0 + iy_0$. $\left|z - z_0\right| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$ is the distance between the points $z = x + iy$ and $z_0 = x_0 + iy_0$, the points $z = x + iy$ that satisfy the equation $\left|z - z_0\right| = \rho$, $\rho > 0$, lie on a circle of radius ρ centered at the point z_0.

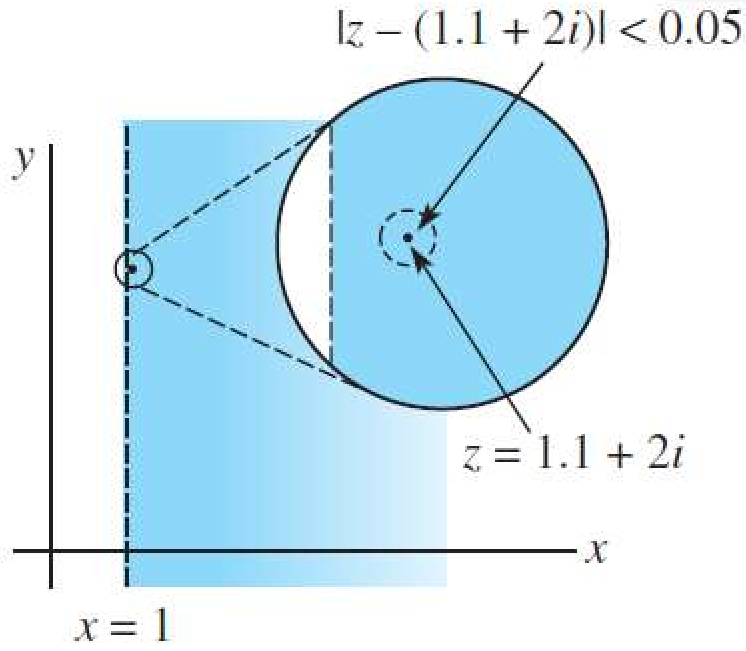


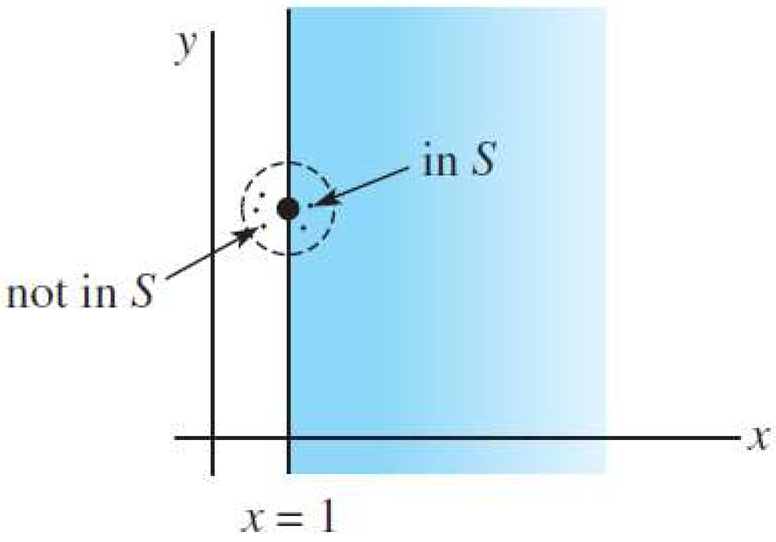
- The points z satisfying the inequality $\left|z - z_0\right| < \rho$, $\rho > 0$, lie within, but not on, a circle of radius ρ centered at the point z_0. This set is called a neighborhood of z_0 or an open disk.

- A point z_0 is said to be an interior point of a set S of the complex plane if there exists some neighborhood of z_0 that lies entirely within S. If every point z of a set S is an interior point, then S is said to be an open set.

For example, the inequality $Re(z) > 1$ is an open set.

- The set S of points in the complex plane defined by $Re(z) \geq 1$ is not an open set.

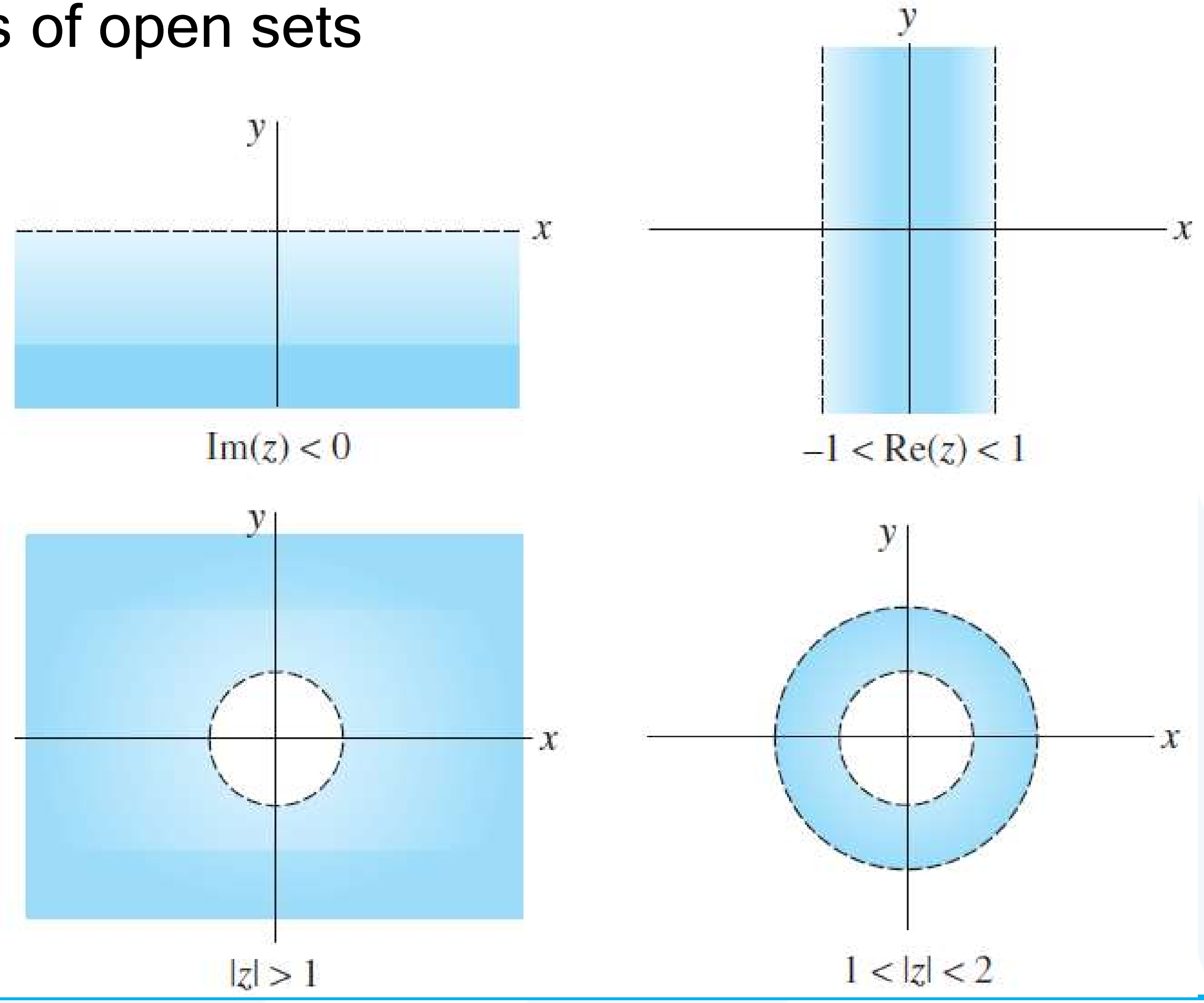




Open set magnified view of a point near $x = 1$

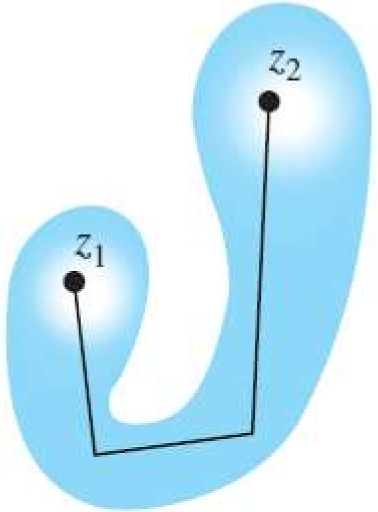
Set S is not open

Four examples of open sets



- The set of numbers satisfying $\rho_1 < |z - z_0| < \rho_2$ is called an open annulus.
- If every neighborhood of a point z_0 contain ns at least one point that is in a set S and at least one point that is not in S, then z_0 is said to be a boundary point of S.
- The boundary of a set S is the set of all boundary points of S.
- For the set of points defined by $Re(z) \geq 1$, the points on the line $x = 1$ are boundary points.
- The points on the circle $|z - i| = 2$ are boundary points for the disk $|z - i| \leq 2$.
- If any pair of points z_1 and z_2 in an open set S can be connected by a polygonal line that lies entirely in the set, then the open set S is said to be connected.
- An open connected set is called a domain.

- The set of numbers satisfying $Re(z) \neq 4$ is an open set but is not connected.
- A region is a domain in the complex plane with all, some, or none of its boundary points.
- Since an open connected set does not contain any boundary points, it is automatically a region.
- A region containing all its boundary points is said to be closed. The disk defined by $|z - i| \leq 2$ is an example of a closed region and is referred to as a closed disk.

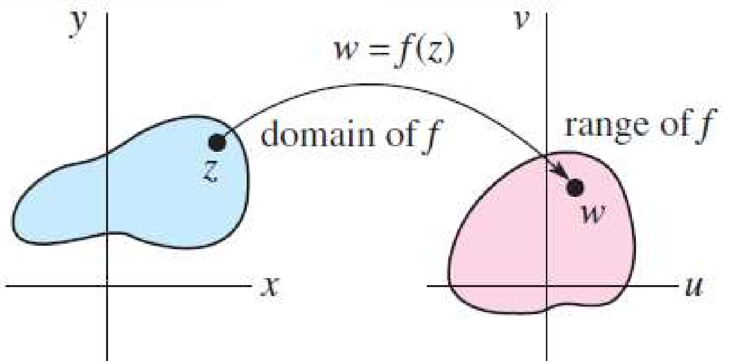


- A region may be neither open nor closed; the annular region defined by $1 \leq |z - 5| < 3$ contains only some of its boundary points and so is neither open nor closed.

- Note: Do not confuse the concept of "domain" defined here as open connected set with the concept of the "domain of a function."

4. Functions of a Complex Variable

- Definition: A complex function is a function f whose domain and range are subsets of the set C of complex numbers.
- The image w of a complex number $z = x + iy$ will be some complex number $w = u + iv$; that is, $w = u(x, y) + iv(x, y) = f(z)$, where u, v are real functions of x and y.
- If to each value of z, there corresponds one and only one value of w, then w is said to be a single-valued function of z otherwise a multi-valued function.



For example, $w = 1/z$ is a single-valued function and $w = \sqrt{z}$ is a multi-valued function of z. The former is defined at all points of the z-plane except at $z = 0$ and the latter assumes two values for each value of z except at $z = 0$.

Some examples of functions of a complex variable are:

$$f(z) = z^2 - 4z = (x^2 - y^2 - 4x) + i(2xy - 4y), \quad z \in C$$

$$f(z) = \frac{z}{z^2 + 1}, \quad z \in C \backslash \{i, -i\} \qquad\qquad f(z) = z + Re(z), \quad z \in C$$

- Note: we cannot draw a graph of a complex function $w = f(z)$. We, say that a curve C in the z-plane is mapped into the corresponding curve C' in the w-plane by the function $w = f(z)$ which defines a mapping or transformation of the z-plane into the w-plane.

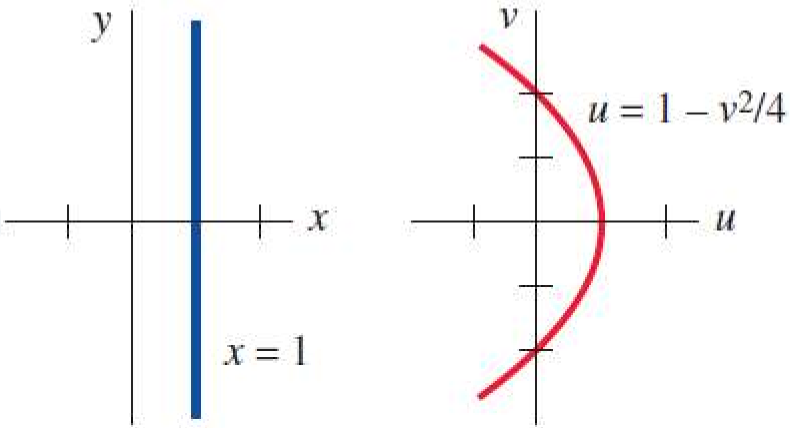
- Example 1: Image of a Vertical Line

  Find the image of the line $Re(z) = 1$ under the mapping $f(z) = z^2$

  $$f(z) = z^2 \Rightarrow u(x, y) = x^2 - y^2 \text{ and } v(x, y) = 2xy$$

  $$Re(z) = x = 1 \Rightarrow u(x, y) = 1 - y^2 \text{ and } v(x, y) = 2y$$

  $$\Rightarrow u = 1 - v^2/4$$



(a) z-plane (b) w-plane

Principal Square Root Function $z^{1/2}$

The square root of a nonzero complex number $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ is given by:

$$\sqrt{r}\left[\cos\left(\frac{\theta + 2\pi k}{2}\right) + i\sin\left(\frac{\theta + 2\pi k}{2}\right)\right] = \sqrt{r}e^{i(\theta+2k\pi)/2},\ k = 0, 1$$

By setting $\theta = \text{Arg}(z)$ and $k = 0$ $z^{1/2} = \sqrt{|z|}e^{i\text{Arg}(z)/2}$ principal square root function

- Example 2: Values of $z^{1/2}$ for $z = -2i$

$$(-2i)^{1/2} = \sqrt{2}e^{i(-\pi/2+2k\pi)/2},\ k = 0, 1$$

$$(-2i)^{1/2} = \begin{cases} \sqrt{2}e^{i(-\pi/4)} = 1 - i \\ \sqrt{2}e^{i(3\pi/4)} = -1 + i \end{cases}$$

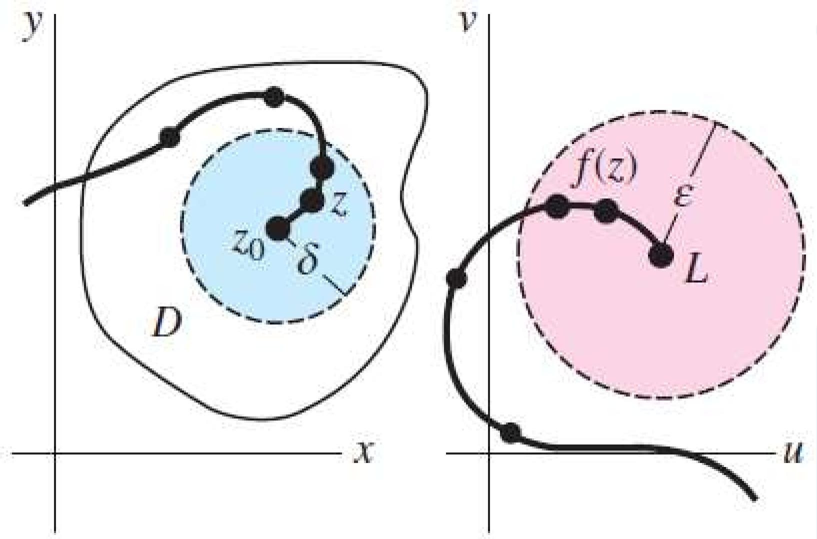
principal square root

Limits and Continuity

- Definition: Suppose the function f is defined in some neighborhood of z_0, except possibly at z_0 itself. Then f is said to possess a limit at z_0, written

$$\lim_{z \to z_0} f(z) = L$$

if, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(z) - L| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.

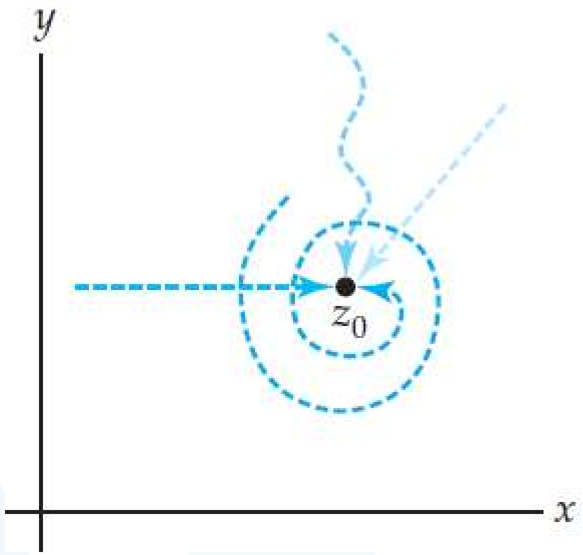


(a) δ-neighborhood (b) ε-neighborhood

- Complex and real limits have many common properties, but there is at least one very important difference. For real functions, $\lim_{x \to x_0} f(x) = L$ if and only if:

$$\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = L$$ two directions

- For limits of complex functions, z is allowed to approach z_0 from any direction in the complex plane, that is, along any path through z_0.



- In order that $\lim_{z \to z_0} f(z)$ exists and equals L, we require that $f(z)$ approach the same complex number L along every possible path through z_0.

Criterion for the Nonexistence of a Limit

- If f approaches two complex numbers $L_1 \neq L_2$ for two different paths or paths through z_0, then $\lim_{z \to z_0} f(z)$ does not exist.

- **Example 3:** A Limit That Does Not Exist

  Show that $\lim_{z\to 0} \frac{z}{\overline{z}}$ does not exist

  z approach 0 along the real axis $\lim_{z\to 0} \frac{z}{\overline{z}} = \lim_{x\to 0} \frac{x+0i}{x-0i} = 1$

  z approach 0 along the imaginary axis $\lim_{z\to 0} \frac{z}{\overline{z}} = \lim_{y\to 0} \frac{0+iy}{0-iy} = -1$

- **Theorem 1:** Suppose that $f(z) = u(x, y) + iv(x, y)$, $z_0 = x_0 + iy_0$, and $L = u_0 + iv_0$. Then $\lim_{z\to z_0} f(z) = L$ if and only if

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$$

- Example 4: Using Theorem 1 to Compute a Limit

Use Theorem 1 to compute $\lim_{z \to 1+i} (z^2 + i)$

$$f(z) = z^2 + i = x^2 - y^2 + (2xy + 1)i$$

$$u_0 = \lim_{(x,y)\to(1,1)} (x^2 - y^2) = 1^2 - 1^2 = 0 \quad \text{and}$$

$$v_0 = \lim_{(x,y)\to(x_0,y_0)} (2xy + 1) = 3$$

$$\lim_{z \to 1+i} (z^2 + i) = L = u_0 + iv_0 = 3i$$

- Theorem 2: Suppose $\lim_{z \to z_0} f(z) = L_1$ and $\lim_{z \to z_0} g(z) = L_2$. Then

$$\lim_{z \to z_0} [f(z) + g(z)] = L_1 + L_2 \qquad \lim_{z \to z_0} [f(z)g(z)] = L_1 L_2 \qquad \lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{L_1}{L_2},\ L_2 \neq 0$$

- Definition: A function f is continuous at a point z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

- As a consequence, if two functions f and g are continuous at a point z_0, then their sum and product are continuous at z_0. The quotient of the two functions is continuous at z_0 provided $g(z_0) \neq 0$.

  A polynomial of degree n

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 z,\ a_n \neq 0,\ a_i \in C,\ i = 0, 1, \cdots, n$$

  is continuous everywhere.

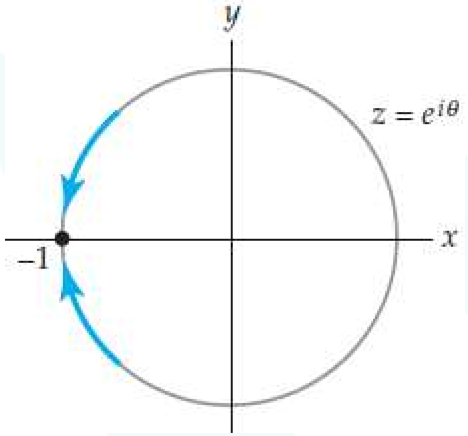
  A rational function $f(z) = \dfrac{g(z)}{h(z)}$, where g and h are polynomial functions, is continuous except at those points at which $h(z)$ is zero.

- Example 5: Discontinuity of Principal Square Root Function

Show that the principal square root function $f(z) = z^{1/2}$ is discontinuous at $z_0 = -1$

z approaching -1 along the second quadrant. That is, $z = e^{i\theta}$, $\pi/2 < \theta < \pi$, with θ approaching π

$$\lim_{z \to -1} z^{1/2} = \lim_{z \to -1} \sqrt{|z|} e^{i\text{Arg}(z)/2} = \lim_{\theta \to \pi} e^{i\theta/2} = \lim_{\theta \to \pi} (\cos\frac{\theta}{2} + \sin\frac{\theta}{2}) = i$$



z approaching -1 along the third quadrant. That is, $z = e^{i\theta}$, $-\pi < \theta < -\pi/2$, with θ approaching $-\pi$

$$\lim_{z \to -1} z^{1/2} = \lim_{z \to -1} \sqrt{|z|} e^{i\text{Arg}(z)/2} = \lim_{\theta \to -\pi} e^{i\theta/2} = \lim_{\theta \to -\pi} (\cos\frac{\theta}{2} + \sin\frac{\theta}{2}) = -i$$

$\lim_{z \to -1} z^{1/2}$ does not exist. Therefore, the principal square root function $f(z) = z^{1/2}$ is discontinuous at $z_0 = -1$

Derivative

- Definition: Suppose the complex function f is defined in a neighborhood of a point z_0. The derivative of f at z_0 is

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

provided this limit exists.

- If the limit exists, the function f is said to be differentiable at z_0.
- As in real variables, If f is differentiable at z_0, then f is continuous at z_0.

  Moreover, the rules of differentiation are the same as in the calculus of real variables.
- If f and g are differentiable at a point z, and c is a complex constant, then:

Constant Rules: $\frac{d}{dz}c = 0, \quad \frac{d}{dz}cf(z) = cf'(z)$

Sum Rule: $\frac{d}{dz}[f(z)+g(z)] = f'(z)+g'(z)$

Product Rule: $\frac{d}{dz}[f(z)g(z)] = f'(z)g(z)+f(z)g'(z)$

Quotient Rule: $\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{f'(z)g(z)-f(z)g'(z)}{[g(z)]^2}$

Chain Rule: $\frac{d}{dz}f(g(z)) = f'(g(z))g'(z)$

Power Rule: $\frac{d}{dz}z^n = nz^{n-1}$, n an integer

- Note: In order for a complex function f to be differentiable at a point z_0,

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

must approach the same complex number from any direction.

- Example 6: A Function That Is Nowhere Differentiable.

Show that the function $f(z) = x + 4iy$ is nowhere differentiable

$$\Delta z = \Delta x + i\Delta y \;\Rightarrow\; f(z + \Delta z) - f(z) = (x + \Delta x) + 4i(y + \Delta y) - x - 4iy = \Delta x + 4i\Delta y$$

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta x + 4i\Delta y}{\Delta x + i\Delta y}$$

$\Delta z \to 0$ along a line parallel to the x-axis, then $\Delta y = 0$ and the limit is 1.

$\Delta z \to 0$ along a line parallel to the y-axis, then $\Delta x = 0$ and the limit is 4.

Analytic Functions

- Definition: A complex function $w = f(z)$ is said to be analytic (holomorphic) at a point z_0 if f is differentiable at z_0 and at every point in some neighborhood of z_0.

  A function f is analytic in a domain D if it is analytic at every point in D.

  $f(z) = |z|^2$ is differentiable at $z = 0$ but is differentiable nowhere else. Hence, $f(z) = |z|^2$ is nowhere analytic.

  In contrast, the simple polynomial $f(z) = z^2$ is differentiable at every point z in the complex plane. Hence, $f(z) = z^2$ is analytic everywhere.

- A function that is analytic at every point z is said to be an entire function.

  Polynomial functions are differentiable at every point z and so are entire functions.

5. Cauchy-Riemann Equations

A Necessary Condition for Analyticity

- **Theorem 3 (Cauchy-Riemann Equations):** Suppose $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point $z = x + iy$. Then at z the first-order partial derivatives of u and v exist and satisfy the Cauchy Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

This result is a necessary condition for analyticity

For example he polynomial $f(z) = z^2 + z$ is analytic for all z

$$f(z) = x^2 - y^2 + x + i(2xy + y)$$

$$\frac{\partial u}{\partial x} = 2x + 1 = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$$

Cauchy-Riemann equations are satisfied

- Example 7: Using the Cauchy-Riemann Equations

Show that the function $f(z) = 2x^2 + y + i(y^2 - x)$ is not analytic at any point.

$$\frac{\partial u}{\partial x} = 4x \quad \text{and} \quad \frac{\partial v}{\partial y} = 2y$$

$$\frac{\partial u}{\partial y} = 1 \quad \text{and} \quad \frac{\partial v}{\partial x} = -1$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ is satisfied only on the line $y = 2x$

However, for any point z on the line, there is no neighborhood or open disk about z in which f is differentiable. We conclude that f is nowhere analytic.

Criterion for Analyticity

- Theorem 4: (Criterion for Analyticity) Suppose the real-valued functions $u(x, y)$ and $v(x, y)$ are continuous and have continuous first-order partial derivatives in a domain D. If u and v satisfy the Cauchy-Riemann equations at all points of D, then the complex function $f(z) = u(x, y) + iv(x, y)$ is analytic in D.

  The function $f(z) = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}$

  is analytic in any domain not containing the point $z = 0$.

- Note: If the real-valued functions $u(x, y)$ and $v(x, y)$ are continuous and have continuous first order partial derivatives in a neighborhood of z, and if u and v satisfy the Cauchy-Riemann equations at the point z, then the complex function $f(z) = u(x, y) + iv(x, y)$ is differentiable at z and $f'(z)$ is given by:

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Polar Coordinates

$$f(z) = u(r, \theta) + iv(r, \theta)$$

- In polar coordinates the Cauchy-Riemann equations become

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

The polar version of $f'(z)$ at a point z is

$$f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = \frac{1}{r}e^{-i\theta}\left(\frac{\partial v}{\partial \theta} - i\frac{\partial u}{\partial \theta}\right)$$

Harmonic Functions

- Definition: A real-valued function $\phi(x, y)$ that has continuous second-order partial derivatives in a domain D and satisfies Laplace's equation ($\partial^2\phi/\partial^2 x + \partial^2\phi/\partial^2 y = 0$) is said to be harmonic in D.

- Theorem 5 (Harmonic Functions): Suppose $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D. then the functions $u(x, y)$ and $v(x, y)$ are harmonic functions.

Harmonic Conjugate Functions If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then u and v are harmonic in D. Now suppose $u(x, y)$ is a given function that is harmonic in D. It is then sometimes possible to find another function $v(x, y)$ that is harmonic in D so that $u(x, y) + iv(x, y)$ is an analytic function in D. The function v is called a harmonic conjugate function of u.

- Example 8: Harmonic Function/Harmonic Conjugate Function

Verify that the function $u(x, y) = x^3 - 3xy^2 - 5y$ is harmonic in the entire complex plane. Find the harmonic conjugate function of u.

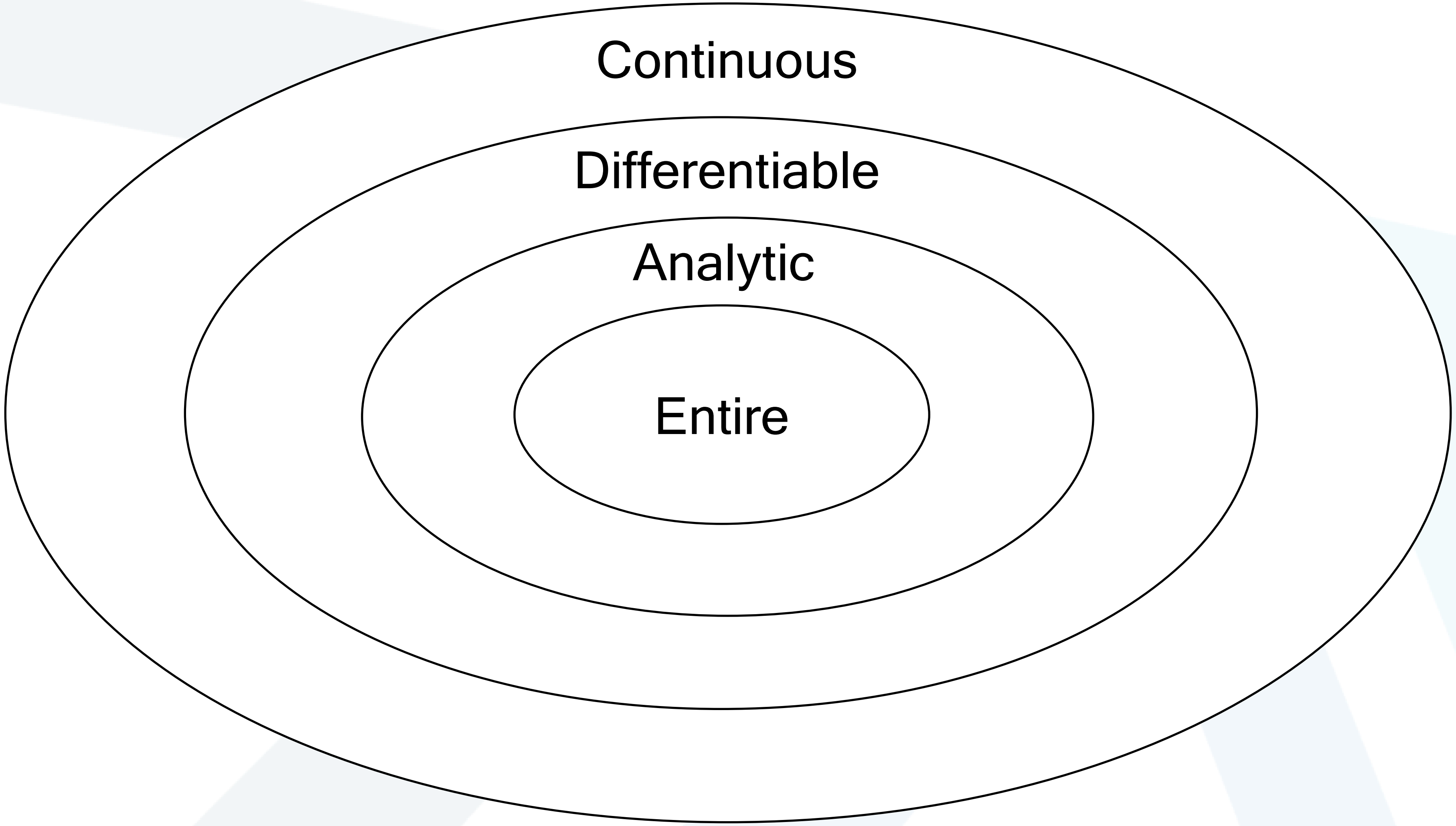
$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2, \quad \frac{\partial^2 u}{\partial x^2} = 6x, \quad \frac{\partial u}{\partial y} = -6xy - 5, \quad \frac{\partial^2 u}{\partial y^2} = -6x$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 3x^2 - 3y^2, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 6xy + 5$$

$$v(x, y) = 3x^2y - y^3 + h(x) \Rightarrow \frac{\partial v}{\partial x} = 6xy + h'(x) \Rightarrow h'(x) = 5 \Rightarrow h(x) = 5x + C$$

$$f(z) = x^3 - 3xy^2 - 5y + i(3x^2y - y^3 + 5x + C)$$


Continuous
Differentiable
Analytic
Entire