# MATHEMATICAL ANALAYSIS 1

# Lecture 2

Prepared by

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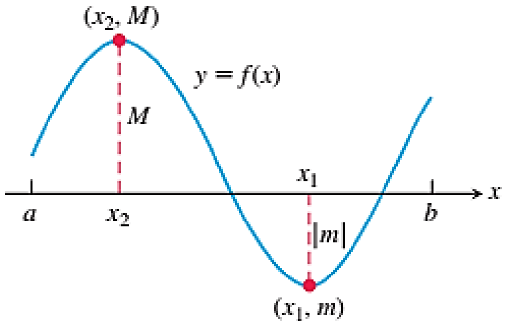
# Extreme Values of Functions on Closed Intervals

**DEFINITIONS** Let $f$ be a function with domain $D$. Then $f$ has an **absolute maximum** value on $D$ at a point $c$ if

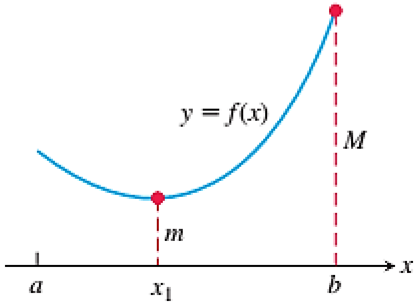
$$f(x) \leq f(c) \qquad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on $D$ at $c$ if

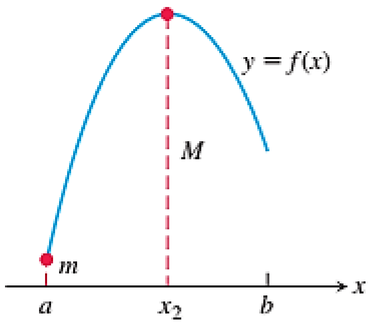
$$f(x) \geq f(c) \qquad \text{for all } x \text{ in } D.$$



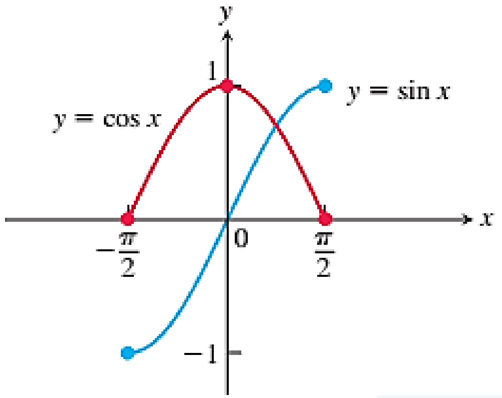
Maximum and minimum at interior points



Minimum at interior point, maximum at endpoint



Maximum at interior point, minimum at endpoint



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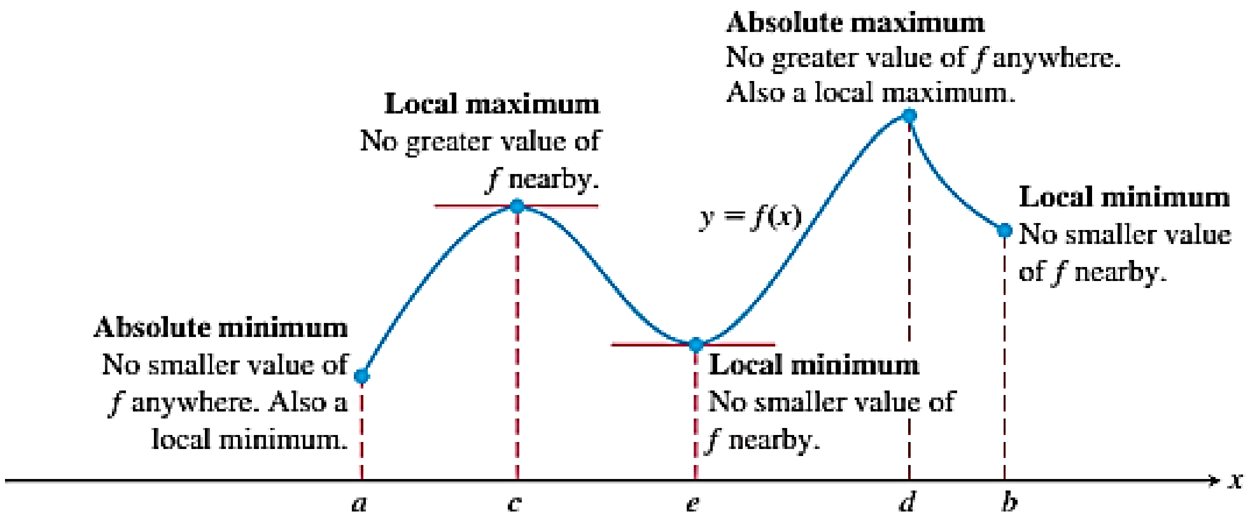
**DEFINITIONS** A function $f$ has a **local maximum** value at a point $c$ within its domain $D$ if $f(x) \leq f(c)$ for all $x \in D$ lying in some open interval containing $c$.

A function $f$ has a **local minimum** value at a point $c$ within its domain $D$ if $f(x) \geq f(c)$ for all $x \in D$ lying in some open interval containing $c$.

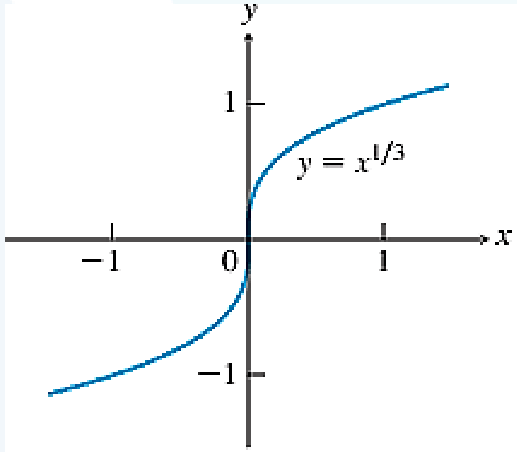


## Extreme Values of Functions on Closed Intervals

**DEFINITION** An interior point of the domain of a function $f$ where $f'$ is zero or undefined is a **critical point** of $f$.

**Finding the Absolute Extrema of a Continuous Function $f$ on a Finite Closed Interval**

1. Find all critical points of $f$ on the interval.
2. Evaluate $f$ at all critical points and endpoints.
3. Take the largest and smallest of these values.



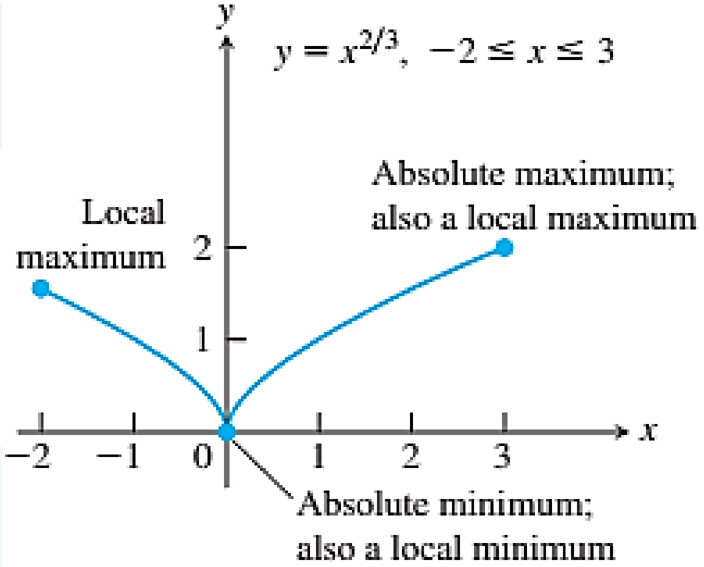
(b)

**EXAMPLE 2** Find the absolute maximum and minimum values of $f(x) = x^2$ on $[-2, 1]$.

absolute maximum value of 4 at x = -2 absolute minimum value of 0 at x = 0.

**EXAMPLE 4** Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-2, 3]$.

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$




Critical point value: $f(0) = 0$

Endpoint values: $f(-2) = (-2)^{2/3} = \sqrt[3]{4}$

$f(3) = (3)^{2/3} = \sqrt[3]{9}.$

# Exercises

- find the absolute maximum and minimum values of each function on the given interval.

$f(x) = 4 - x^3, \quad -2 \le x \le 1$

$F(x) = -\frac{1}{x^2}, \quad 0.5 \le x \le 2$

$f(\theta) = \tan\theta, \quad -\frac{\pi}{3} \le \theta \le \frac{\pi}{4}$

$f\left(\frac{-\pi}{3}\right) = -\sqrt{3}$ and $f\left(\frac{\pi}{4}\right) = 1$ no critical

$f(t) = |t - 5|, \quad 4 \le t \le 7$

$f(7) = 2 \quad f(5) = 0.$

- find the critical points and domain endpoints for each function. Then find the value of the function at each of these points and identify extreme values (absolute and local).

$y = x^{2/3}(x^2 - 4)$

| crit.pt. | derivative | extremum | value |
|---|---|---|---|
| $x = -1$ | 0 | minimum | $-3$ |
| $x = 0$ | undefined | local max | 0 |
| $x = 1$ | 0 | minimum | 3 |

$$y = \begin{cases} 4 - 2x, & x \le 1 \\ x + 1, & x > 1 \end{cases}$$

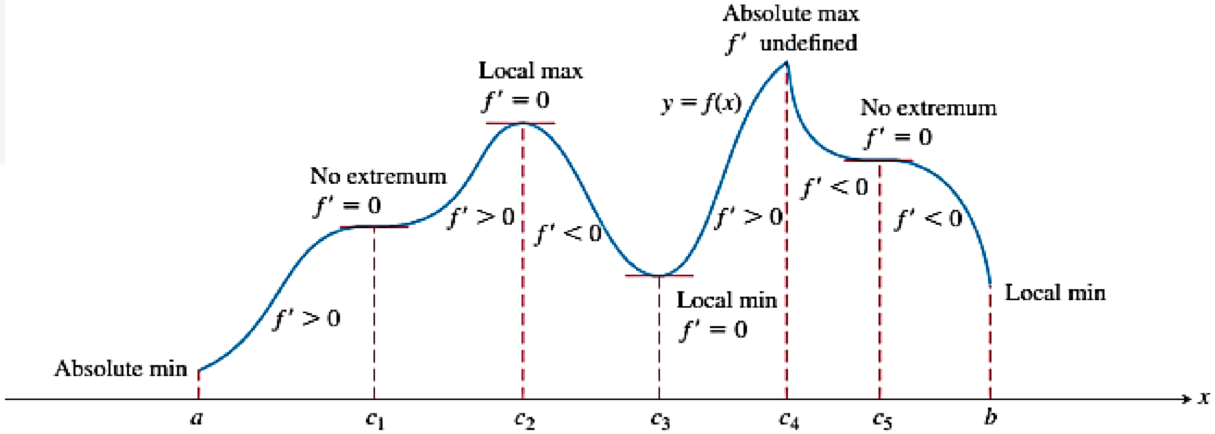
| crit.pt. | derivative | extremum | value |
|---|---|---|---|
| $x = 1$ | undefined | minimum | 2 |

# Exercises

- Suppose that at any given time $t$ (in seconds) the current $i$ (in amperes) in an alternating current circuit is $i = 2\cos t + 2\sin t$. What is the peak current for this circuit (largest magnitude)?

the peak current is $2\sqrt{2}$

# First Derivative Test for Local Extrema




## First Derivative Test for Local Extrema

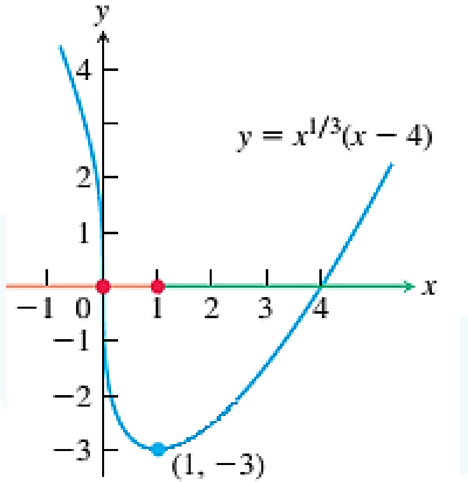
Suppose that $c$ is a critical point of a continuous function $f$, and that $f$ is differentiable at every point in some interval containing $c$ except possibly at $c$ itself. Moving across this interval from left to right,

1. if $f'$ changes from negative to positive at $c$, then $f$ has a local minimum at $c$;
2. if $f'$ changes from positive to negative at $c$, then $f$ has a local maximum at $c$;
3. if $f'$ does not change sign at $c$ (that is, $f'$ is positive on both sides of $c$ or negative on both sides), then $f$ has no local extremum at $c$.

**EXAMPLE 2** Find the critical points of

$$f(x) = x^{1/3}(x - 4) = x^{4/3} - 4x^{1/3}.$$

Identify the open intervals on which $f$ is increasing and decreasing. Find the function's local and absolute extreme values.




$$f'(x) = \frac{d}{dx}\left(x^{4/3} - 4x^{1/3}\right) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3} = \frac{4}{3}x^{-2/3}(x - 1) = \frac{4(x - 1)}{3x^{2/3}}$$

The critical points x = 0 and x = 1

| **Interval** | $x < 0$ | $0 < x < 1$ | $x > 1$ |
|---|---|---|---|
| **Sign of $f'$** | $-$ | $-$ | $+$ |
| **Behavior of $f$** | decreasing | decreasing | increasing |

# Exercises

- **a.** Find the open intervals on which the function is increasing and decreasing.

  **b.** Identify the function's local and absolute extreme values, if any, saying where they occur.

$$f(x) = \frac{x^2 - 3}{x - 2}, \quad x \neq 2$$

$$f' = +++\underset{1}{|}---\underset{2}{)(}---\underset{3}{|}+++$$

$$f(x) = \frac{x^3}{3x^2 + 1}$$

$$f' = +++\underset{0}{|}+++$$

- Determine the values of constants $a$ and $b$ so that $f(x) = ax^2 + bx$ has an absolute maximum at the point $(1, 2)$.

$a = -2, b = 4$

**DEFINITION** The graph of a differentiable function $y = f(x)$ is

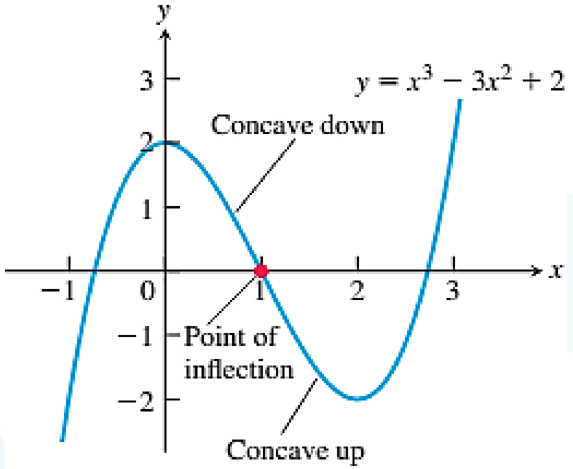
(a) **concave up** on an open interval $I$ if $f'$ is increasing on $I$;

(b) **concave down** on an open interval $I$ if $f'$ is decreasing on $I$.

**The Second Derivative Test for Concavity**

Let $y = f(x)$ be twice-differentiable on an interval $I$.

1. If $f'' > 0$ on $I$, the graph of $f$ over $I$ is concave up.
2. If $f'' < 0$ on $I$, the graph of $f$ over $I$ is concave down.

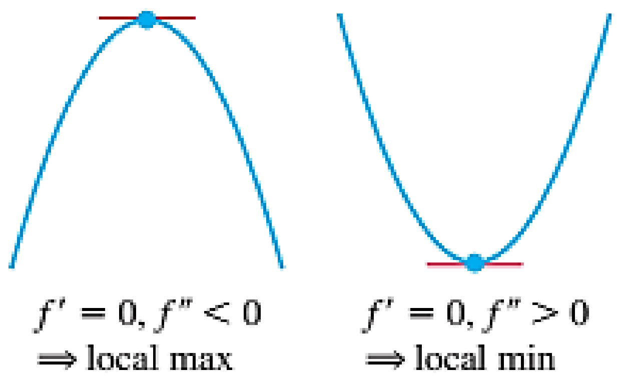



**DEFINITION** A point $(c, f(c))$ where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

## Second Derivative Test for Local Extrema

**THEOREM 5—Second Derivative Test for Local Extrema**
Suppose $f''$ is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then $f$ has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then $f$ has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function $f$ may have a local maximum, a local minimum, or neither.



# Exercises

- identify the coordinates of any local and absolute extreme points and inflection points.

$y = x + \sin x, \quad 0 \le x \le 2\pi$

$x = 0$ minimum

$x = 2\pi$ maximum

$x = \pi$ inflection

$y = (2 - x^2)^{3/2}$

$x = 0$ maximum

$x = \pm\sqrt{2}$ minima

$x = \pm 1$ inflection

$y = |x^2 - 2x|$

$x = 1$ maximum

$x = 0$ and $x = 2$. minima

no points of inflection

$$y = |x^2 - 2x| = \begin{cases} x^2 - 2x, & x < 0 \\ 2x - x^2, & 0 \le x \le 2, \\ x^2 - 2x, & x > 2 \end{cases}$$

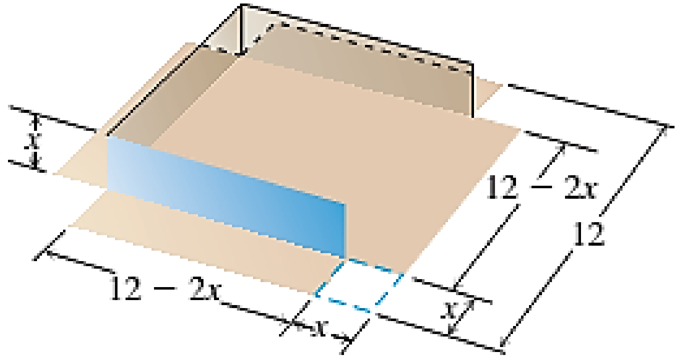
- Find the values of constants $a$, $b$, and $c$ so that the graph of $y = ax^3 + bx^2 + cx$ has a local maximum at $x = 3$, local minimum at $x = -1$, and inflection point at $(1, 11)$.

$a = -1, b = 3, \text{ and } c = 9$

## Solving Applied Optimization Problems

1. *Read the problem*. Read the problem until you understand it. What is given? What is the unknown quantity to be optimized?
2. *Draw a picture*. Label any part that may be important to the problem.
3. *Introduce variables*. List every relation in the picture and in the problem as an equation or algebraic expression, and identify the unknown variable.
4. *Write an equation for the unknown quantity*. If you can, express the unknown as a function of a single variable or in two equations in two unknowns. This may require considerable manipulation.
5. *Test the critical points and endpoints in the domain of the unknown*. Use what you know about the shape of the function's graph. Use the first and second derivatives to identify and classify the function's critical points.

**EXAMPLE 1** An open-top box is to be made by cutting small congruent squares from the corners of a 12-in.-by-12-in. sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?



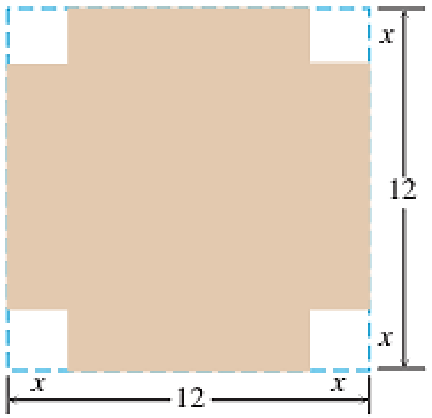

$$V = h.l.w$$

$$V = x(12-2x)^2 \qquad 0 \le x \le 6$$

$$\frac{dV}{dx} = 12(2-x)(6-x)$$

Critical point value: $V(2) = 128$

Endpoint values: $V(0) = 0, \quad V(6) = 0.$




The maximum volume is 128 in$^3$. The cutout squares should be 2 in. on a side.

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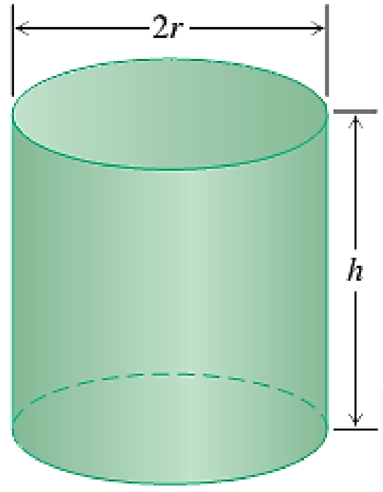
**EXAMPLE 2** You have been asked to design a one-liter can shaped like a right circular cylinder (Figure 4.38). What dimensions will use the least material?



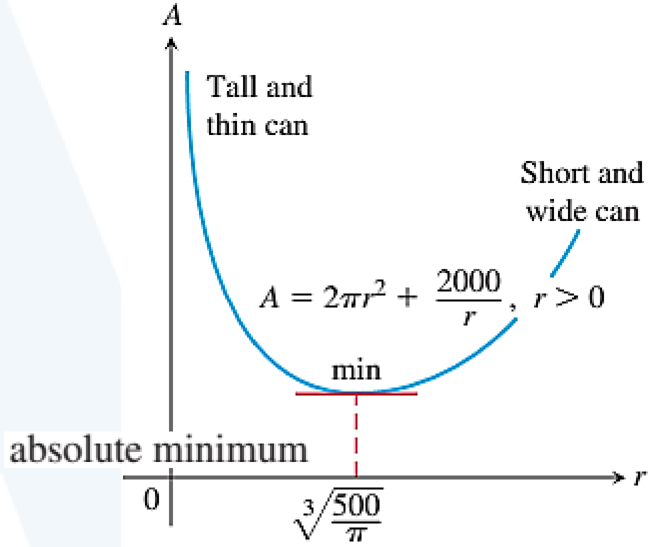

$$A = \underbrace{2\pi r^2}_{\text{circular ends}} + \underbrace{2\pi rh}_{\text{cylindrical wall}}$$

$$\pi r^2 h = 1000. \qquad 1 \text{ liter} = 1000 \text{ cm}^3 \quad \Rightarrow \quad h = \frac{1000}{\pi r^2}.$$

$$A = 2\pi r^2 + \frac{2000}{r} \quad \Rightarrow \quad \frac{dA}{dr} = 4\pi r - \frac{2000}{r^2}$$

$$r = \sqrt[3]{\frac{500}{\pi}} \simeq 5.42 \qquad \frac{d^2A}{dr^2} = 4\pi + \frac{4000}{r^3} > 0$$

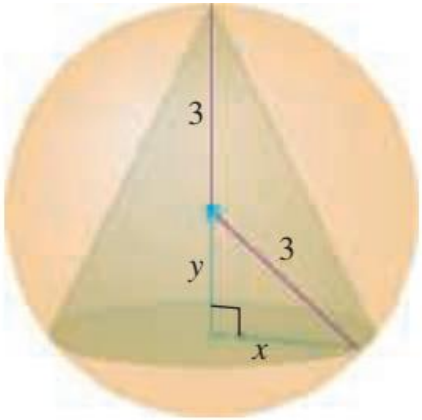
$$h = 2\sqrt[3]{\frac{500}{\pi}} \qquad r \approx 5.42 \text{ cm and } h \approx 10.84 \text{ cm}.$$



# Exercises

- Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3.

$V = \frac{1}{3}\pi r^2 h$  $r = x = \sqrt{9 - y^2}$  $h = y + 3$  $\frac{32\pi}{3}$




- A piece of cardboard measures 10 in. by 15 in. Two equal squares are removed from the corners of a 10-in. side as shown in the figure. Two equal rectangles are removed from the other corners so that the tabs can be folded to form a rectangular box with lid.

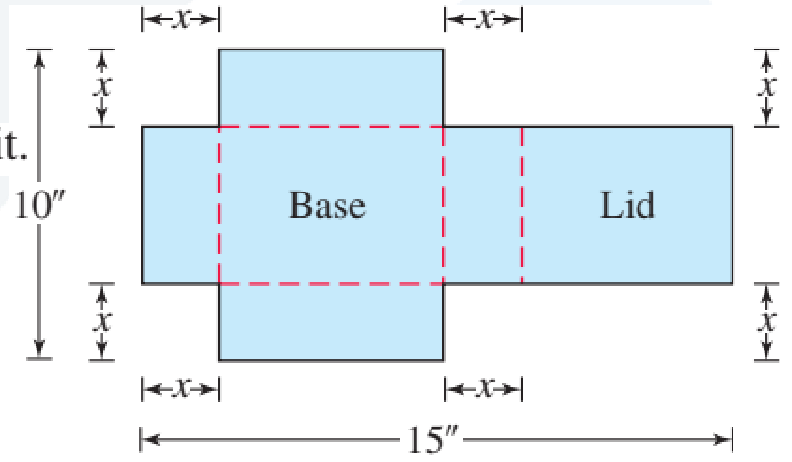
**a.** Write a formula $V(x)$ for the volume of the box.

find the maximum volume and the value of $x$ that gives it.

$$V(x) = \frac{x(10-2x)(15-2x)}{2} \quad x > 0,\ 2x < 10,\ \text{and } 2x < 15$$

interval (0, 5)

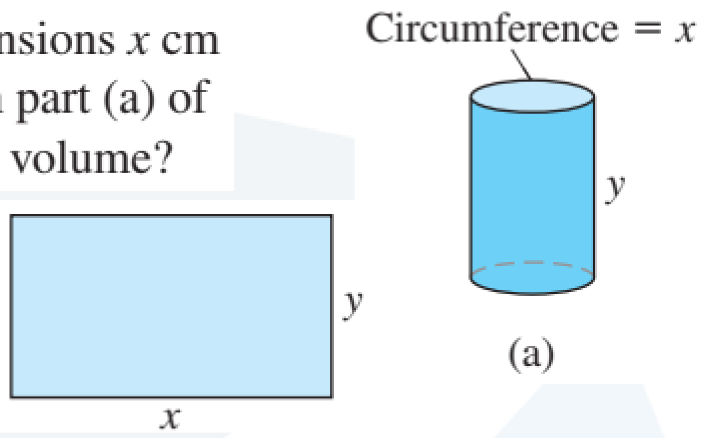
$x \approx 1.96$ or $x \approx 6.37$



# Exercises

- **a.** A rectangular sheet of perimeter 36 cm and dimensions $x$ cm by $y$ cm is to be rolled into a cylinder as shown in part (a) of the figure. What values of $x$ and $y$ give the largest volume?

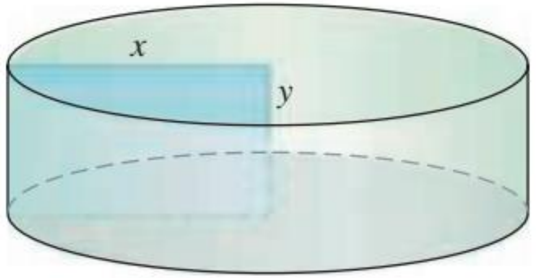
$x = 12.$ $\quad y = 6 \quad V(x) = \frac{18x^2 - x^3}{4\pi}$



(a)

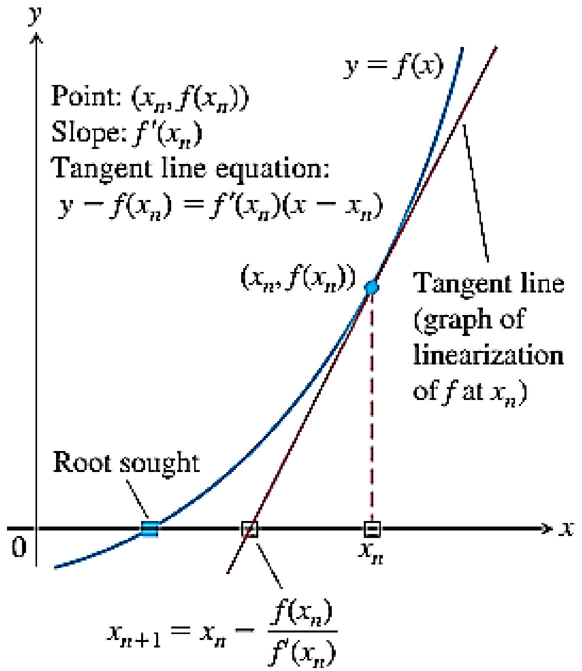
**b.** The same sheet is to be revolved about one of the sides of length $y$ to sweep out the cylinder as shown in part (b) of the figure. What values of $x$ and $y$ give the largest volume?

$V(x) = \pi x^2 (18 - x) \quad x = 12. \quad y = 6$



(b)

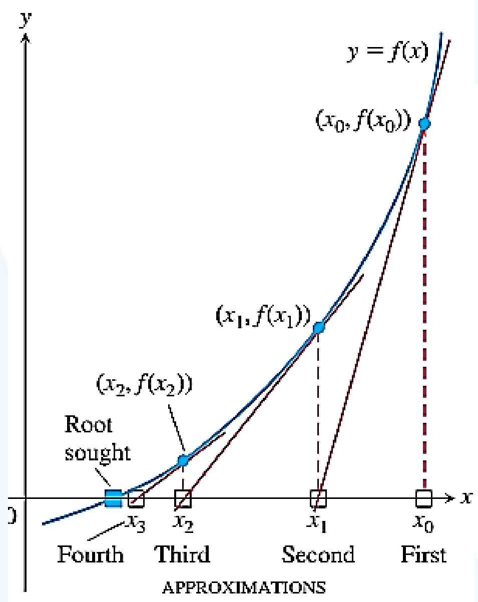
# Newton's Method




## Newton's Method

1. Guess a first approximation to a solution of the equation $f(x) = 0$. A graph of $y = f(x)$ may help.
2. Use the first approximation to get a second, the second to get a third, and so on, using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad \text{if } f'(x_n) \neq 0. \qquad (1)$$



## Newton's Method

**EXAMPLE 1** Approximate the positive root of the equation

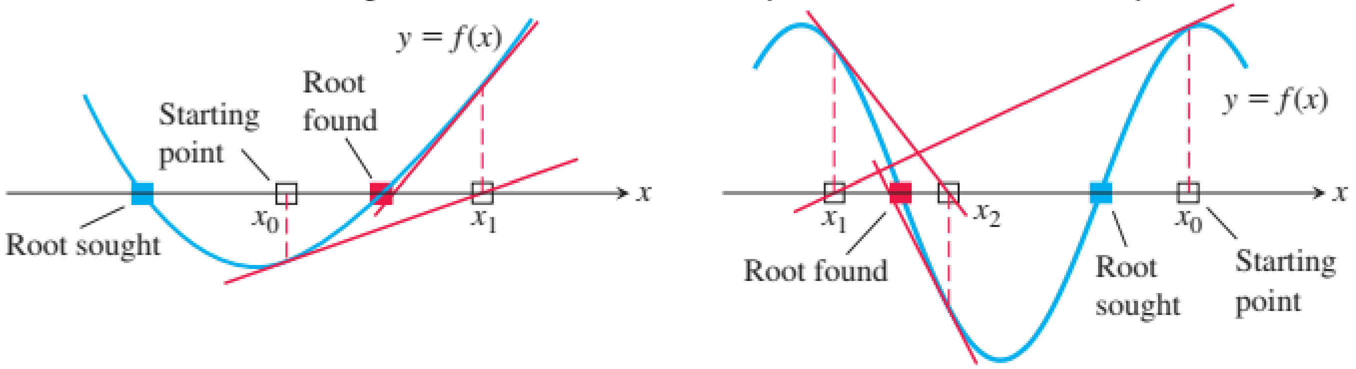
$$f(x) = x^2 - 2 = 0.$$

$$x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$$

$$\sqrt{2} = 1.41421$$

| | Error | Number of correct digits |
|---|---|---|
| $x_0 = 1$ | $-0.41421$ | 1 |
| $x_1 = 1.5$ | 0.08579 | 1 |
| $x_2 = 1.41667$ | 0.00246 | 3 |
| $x_3 = 1.41422$ | 0.00001 | 5 |

*When Newton's method converges to a root, it may not be the root you have in mind.*



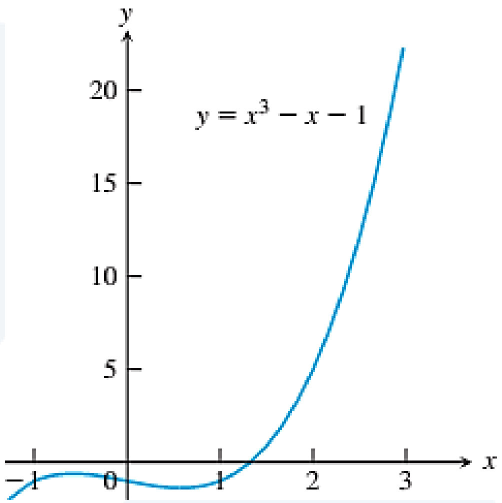
## Newton's Method

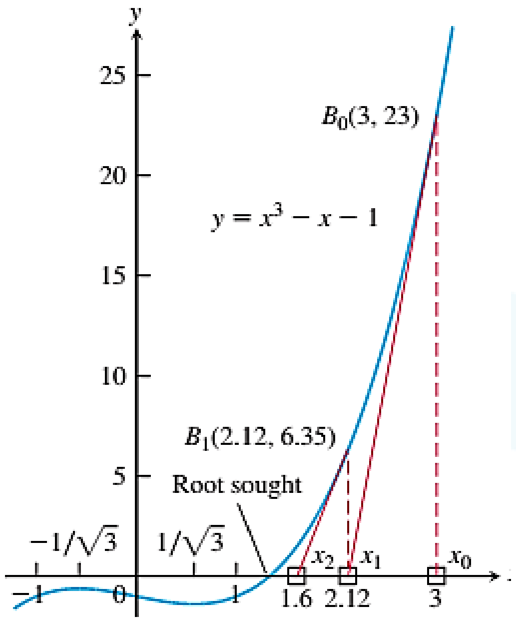
**EXAMPLE 2** Find the $x$-coordinate of the point where the curve $y = x^3 - x$ crosses the horizontal line $y = 1$.

$$x^3 - x - 1 = 0$$

$$x_{n+1} = x_n - \frac{x_n^3 - x_n - 1}{3x_n^2 - 1}$$

| $n$ | $x_n$ | $f(x_n)$ | $f'(x_n)$ | $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|---|---|---|
| 0 | 1 | −1 | 2 | 1.5 |
| 1 | 1.5 | 0.875 | 5.75 | 1.3478 26087 |
| 2 | 1.3478 26087 | 0.1006 82173 | 4.4499 05482 | 1.3252 00399 |
| 3 | 1.3252 00399 | 0.0020 58362 | 4.2684 68292 | 1.3247 18174 |
| 4 | 1.3247 18174 | 0.0000 00924 | 4.2646 34722 | 1.3247 17957 |
| 5 | 1.3247 17957 | −1.8672E-13 | 4.2646 32999 | 1.3247 17957 |



# Exercises

- The curve $y = \tan x$ crosses the line $y = 2x$ between $x = 0$ and $x = \pi/2$. Use Newton's method to find where.

$$x_{n+1} = x_n - \frac{\tan(x_n) - 2x_n}{\sec^2(x_n)}; \quad x_0 = 1 \Rightarrow x_1 = 1.2920445 \qquad \Rightarrow x_{16} = x_{17} = 1.165561185$$

**DEFINITION** The collection of all antiderivatives of $f$ is called the **indefinite integral** of $f$ with respect to $x$, and is denoted by

$$\int f(x)\,dx.$$

The symbol $\int$ is an **integral sign**. The function $f$ is the **integrand** of the integral, and $x$ is the **variable of integration**.

$$\int f(x)dx = F(x) \quad \Leftrightarrow \quad F'(x) = f(x) \qquad \int f(x)dx = F(x) + C$$

1) $\int c\,f(x)dx = c\int f(x)dx \quad ; \quad c = \text{constant}$

2) $\int [f_1(x) + f_2(x) - f_3(x)]dx = \int f_1(x)dx + \int f_2(x)dx - \int f_3(x)dx$

3) $\left[\int f(x)dx\right]' = f(x)$

**TABLE 8.1** Basic integration formulas

| | |
|---|---|
| 1. $\int k\, dx = kx + C \quad$ (any number $k$) | 12. $\int \tan x\, dx = \ln \lvert \sec x \rvert + C$ |
| 2. $\int x^n\, dx = \dfrac{x^{n+1}}{n+1} + C \quad (n \neq -1)$ | 13. $\int \cot x\, dx = \ln \lvert \sin x \rvert + C$ |
| 3. $\int \dfrac{dx}{x} = \ln \lvert x \rvert + C$ | 14. $\int \sec x\, dx = \ln \lvert \sec x + \tan x \rvert + C$ |
| 4. $\int e^x\, dx = e^x + C$ | 15. $\int \csc x\, dx = -\ln \lvert \csc x + \cot x \rvert + C$ |
| 5. $\int a^x\, dx = \dfrac{a^x}{\ln a} + C \quad (a > 0,\ a \neq 1)$ | 16. $\int \sinh x\, dx = \cosh x + C$ |

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6. $\int \sin x\, dx = -\cos x + C$

7. $\int \cos x\, dx = \sin x + C$

8. $\int \sec^2 x\, dx = \tan x + C$

8. $\int \csc^2 x\, dx = -\cot x + C$

10. $\int \sec x \tan x\, dx = \sec x + C$

11. $\int \csc x \cot x\, dx = -\csc x + C$

17. $\int \cosh x\, dx = \sinh x + C$

18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$

19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$

20. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1}\left|\frac{x}{a}\right| + C$

21. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C \qquad (a > 0)$

22. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C \qquad (x > a > 0)$

# Definite Integral

**THEOREM 3—The Mean Value Theorem for Definite Integrals**
If $f$ is continuous on $[a, b]$, then at some point $c$ in $[a, b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

**THEOREM 4—The Fundamental Theorem of Calculus, Part 1**
If $f$ is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on $(a, b)$ and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x). \qquad (2)$$

**THEOREM 4 (Continued)—The Fundamental Theorem of Calculus, Part 2**
If $f$ is continuous over $[a, b]$ and $F$ is any antiderivative of $f$ on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

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**EXAMPLE 4** Find $\displaystyle\int_0^{\pi/4} \frac{dx}{1 - \sin x}$.

$$\int_0^{\pi/4} \frac{dx}{1 - \sin x} = \int_0^{\pi/4} \frac{1}{1 - \sin x} \cdot \frac{1 + \sin x}{1 + \sin x}\, dx = \int_0^{\pi/4} \frac{1 + \sin x}{1 - \sin^2 x}\, dx = \int_0^{\pi/4} \frac{1 + \sin x}{\cos^2 x}\, dx$$

$$= \int_0^{\pi/4} (\sec^2 x + \sec x \tan x)\, dx = \Big[\tan x + \sec x\Big]_0^{\pi/4} = \sqrt{2}.$$

**EXAMPLE 5** Evaluate

$$\int \frac{3x^2 - 7x}{3x + 2}\, dx.$$

$$\int \frac{3x^2 - 7x}{3x + 2}\, dx = \int \left( x - 3 + \frac{6}{3x + 2} \right) dx = \frac{x^2}{2} - 3x + 2 \ln |3x + 2| + C.$$

**THEOREM 6—The Substitution Rule**
If $u = g(x)$ is a differentiable function whose range is an interval $I$, and $f$ is continuous on $I$, then

$$\int f(g(x)) \cdot g'(x)\, dx = \int f(u)\, du.$$

**The Substitution Method to evaluate $\int f(g(x))g'(x)\, dx$**

1. Substitute $u = g(x)$ and $du = (du/dx)\, dx = g'(x)\, dx$ to obtain $\int f(u)\, du$.
2. Integrate with respect to $u$.
3. Replace $u$ by $g(x)$.

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**EXAMPLE 6** Evaluate

$$\int \frac{3x + 2}{\sqrt{1 - x^2}}\, dx.$$

$$\int \frac{3x + 2}{\sqrt{1 - x^2}}\, dx = -3\sqrt{1 - x^2} + 2 \sin^{-1} x + C.$$

**EXAMPLE 7** Evaluate

$$\int \frac{dx}{(1 + \sqrt{x})^3}.$$

$$= C - \frac{1 + 2\sqrt{x}}{(1 + \sqrt{x})^2}.$$

## Integration by Parts

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

Find $\int \ln x \, dx$

**Solution**

$$\left.\begin{array}{r} f(x) = \ln x \\ g'(x) = 1 \end{array}\right\} \Rightarrow \left.\begin{array}{r} f'(x) = \dfrac{1}{x} \\ g(x) = x \end{array}\right\}$$

$$\int \ln x \, dx = f(x)g(x) - \int f'(x)g(x)dx = x \ln x - \int x \frac{1}{x} dx$$

$$= x \ln x - \int dx = x \ln x - x + C$$

1- $I = \int P(x)e^{\alpha x}dx \Rightarrow \begin{cases} f(x) = P(x) \\ g'(x) = e^{\alpha x} \end{cases}$

Find $\int (x^2 - 4x)e^{2x}dx$

**Solution**

$$\left. \begin{array}{l} f(x) = x^2 - 4x \\ g'(x) = e^{2x} \end{array} \right\} \Rightarrow \left. \begin{array}{l} f'(x) = 2x - 4 \\ g(x) = \frac{1}{2}e^{x} \end{array} \right\}$$

$$\int (x^2 - 4x)e^{2x}dx = f(x)g(x) - \int f'(x)g(x)dx = \frac{1}{2}(x^2 - 4x)e^{2x} - \frac{1}{2}\int (2x-4)e^{2x}dx$$

$$\left.\begin{array}{l} f(x) = 2x - 4 \\ g'(x) = e^{2x} \end{array}\right\} \Rightarrow \left.\begin{array}{l} f'(x) = 2 \\ g(x) = \frac{1}{2}e^{x} \end{array}\right\}$$

$$\int (2x-4)e^{2x}dx = f(x)g(x) - \int f'(x)g(x)dx$$

$$= \frac{1}{2}(2x-4)e^{2x} - \int e^{2x}dx = \frac{1}{2}(2x-4)e^{2x} - \frac{1}{2}e^{2x} + C$$

$$\int (x^2-4x)e^{2x}dx = \frac{1}{2}(x^2-4x)e^{2x} - \frac{1}{4}(2x-4)e^{2x} + \frac{1}{4}e^{2x} + c$$

2- $I = \int P(x)\cos\alpha x dx$ or $I = \int P(x)\sin\alpha x dx \Rightarrow \begin{cases} f(x) = P(x) \\ g'(x) = \cos\alpha x \\ \quad or \\ g'(x) = \sin\alpha x \end{cases}$

Find $\int x \sin 2x \, dx$

**Solution**

$$\left.\begin{array}{l} f(x) = x \\ g'(x) = \sin 2x \end{array}\right\} \Rightarrow \left.\begin{array}{l} f'(x) = 1 \\ g(x) = \dfrac{-1}{2}\cos 2x \end{array}\right\}$$

$$\int x \sin 2x \, dx = f(x)g(x) - \int f'(x)g(x)dx$$

$$= \frac{-1}{2} x \cos 2x + \frac{1}{2}\int \cos 2x dx = -\frac{1}{2} x \cos 2x + \frac{1}{4}\sin 2x + C$$

3- $I=\int e^{\alpha x}\cos\beta x\,dx$ or $I=\int e^{\alpha x}\sin\beta x\,dx$

Find $I=\int e^{x}\sin x\,dx$

**Solution**

$$I=\int e^{x}\sin x\,dx \underset{f(x)=\sin x\ ,\ g'(x)=e^{x}}{=} e^{x}\sin x-\int e^{x}\cos x\,dx \underset{f(x)=\cos x\ ,\ g'(x)=e^{x}}{=}$$

$$e^{x}\sin x-\Big(e^{x}\cos x+\underbrace{\int e^{x}\sin x\,dx}_{I}\Big)=e^{x}\sin x-e^{x}\cos x-I$$

$$\Rightarrow\ I=\frac{1}{2}\left(e^{x}\sin x-e^{x}\cos x\right)+C$$

$$\int_a^b f(x)g'(x)dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)dx$$

Calculate $\int_0^1 \tan^{-1} x dx$

**Solution**

$$\left.\begin{aligned} f(x) &= \tan^{-1} x \\ g'(x) &= 1 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} f'(x) &= \frac{1}{1+x^2} \\ g(x) &= x \end{aligned}\right\}$$

$$\int_0^1 \tan^{-1} x dx = [f(x)g(x)]_0^1 - \int_0^1 f'(x)g(x)dx$$

$$= [x \tan^{-1} x]_0^1 - \int_0^1 \frac{x}{x^2+1}dx = \frac{\pi}{4} - \int_0^1 \frac{x}{x^2+1}dx$$

$$\int_0^1 \frac{x}{x^2+1}dx$$

$$t = x^2 + 1 \qquad x = 0 \Rightarrow t = 1$$

$$dt = 2x\,dx \qquad x = 1 \Rightarrow t = 2$$

$$\int_0^1 \frac{x}{x^2+1}dx = \frac{1}{2}\int_1^2 \frac{1}{t}dt = \frac{1}{2}[\ln t]_1^2 = \frac{1}{2}[\ln 2 - \ln 1] = \frac{1}{2}\ln 2$$