Calculus 2

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Calculus 2

Lecture 3

Exercices 3

Improper Integrals

Improper Integrals

The integral $\int_{-1}^{1} \frac{dx}{x^3}$ diverges.

Trying to compute that integral by the Fundamental Theorem of Calculus, one gets

$$\int_{-1}^{1} \frac{dx}{x^3} = \cancel{\frac{1}{-2} x^{-2} \Big|_{-1}^{1} = -\frac{1}{2} - \left(-\frac{1}{2}\right)} = 0.$$

This is an incorrect computation.

Evaluating an integral on $(-\infty,\ \infty)$

$$\int_{-\infty}^{\infty} e^{-|x|}\ dx$$

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Solution

$$\int_{-\infty}^{\infty} e^{-|x|}\, dx = \int_{-\infty}^{0} e^{x}\, dx + \int_{0}^{\infty} e^{-x}\, dx$$

$$= \lim_{a \to -\infty} \int_{a}^{0} e^{x}\, dx + \lim_{a \to \infty} \int_{0}^{a} e^{-x}\, dx$$

$$= \lim_{a \to -\infty} \left(1 - e^{a}\right) + \lim_{a \to \infty} \left(-e^{-a} + 1\right)$$

$$= 1 + 1 = 2$$

Evaluating an integral on $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$

Solution

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{\eta \to 1} \int_0^{\eta} \frac{dx}{\sqrt{1-x^2}} = \lim_{\eta \to 1} \arcsin(\eta) = \frac{\pi}{2}$$

Evaluate $$\lim_{b\to 1^-}\int_0^b \sqrt{\frac{1+x}{1-x}}\,dx$$

Solution $$\int \sqrt{\frac{1+x}{1-x}}\,\frac{\sqrt{1+x}}{\sqrt{1+x}}\,dx$$

Rationalize the numerator.

$$\int \frac{1+x}{\sqrt{1-x^2}}\,dx$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx+\int \frac{x}{\sqrt{1-x^2}}\,dx$$

$$\sin^{-1}x-\frac{1}{2}\int u^{-\frac{1}{2}}du$$

$$u=1-x^2$$

$$du=-2x\,dx$$

$$-\frac{1}{2}du=x\,dx$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx + \int \frac{x}{\sqrt{1-x^2}}\,dx$$

$$u = 1 - x^2$$

$$du = -2x\,dx$$

$$-\frac{1}{2}\,du = x\,dx$$

$$\sin^{-1} x - \frac{1}{2}\int u^{-\frac{1}{2}}\,du$$

$$\sin^{-1} x - u^{\frac{1}{2}}$$

This integral converges because it approaches a solution.

$$\lim_{b\to 1^-} \sin^{-1} x - \sqrt{1-x^2}\,\Big|_0^b$$

$$\lim_{b\to 1^-} \left(\cancelto{\frac{\pi}{2}}{\sin^{-1} b} - \cancelto{0}{\sqrt{1-b^2}}\right) - \left(\cancelto{0}{\sin^{-1} 0} - \sqrt{1}\right) = \boxed{\frac{\pi}{2} + 1}$$

For what values of p is the integral $\int_1^{+\infty} \frac{dx}{x^p}$ convergent?

Solution

$$\int_1^{\infty} \frac{dx}{x^P} \qquad P > 0$$

(P is a constant.)

$$\int_1^{\infty} x^{-P} \ dx$$

$$\lim_{b\to\infty} \int_1^{b} x^{-P} \ dx$$

$$\lim_{b\to\infty} \ \frac{1}{-P+1} x^{-P+1}\Big|_1^b$$

$$\lim_{b\to\infty} \ \frac{b^{-P+1}}{-P+1} - \frac{1^{-P+1}}{-P+1}$$

What happens here?

If $P<1$ then b^{-P+1} gets bigger and bigger as , $b \to \infty$ therefore the integral <u>diverges</u>.

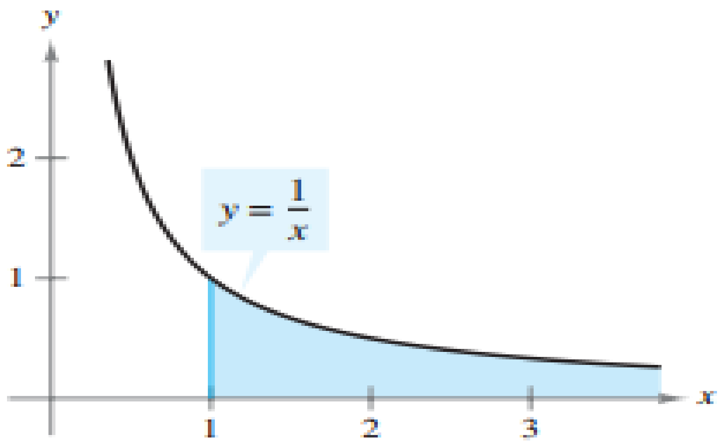
If $P>1$ then b has a negative exponent and , $b^{-P+1} \to 0$ therefore the integral <u>converges</u>.

When $p \neq 1,$

$$\int_1^{+\infty} \frac{dx}{x^p} = \lim_{b \to \infty} \frac{b^{-P+1}}{-P+1} - \frac{1^{-P+1}}{-P+1} = \begin{cases} +\infty & p < 1 \\ \dfrac{1}{p-1} & p > 1 \end{cases}.$$

When $p = 1,$

$$\int_1^{+\infty} \frac{dx}{x} = \lim_{b \to \infty} \ln x \Big|_1^b = \lim_{b \to \infty} (\ln b - \ln 1) = +\infty$$



Diverges (infinite area)

Thus, the integral diverges.

evaluate the integral or state that it diverges. $\int_{0}^{+\infty} \frac{dx}{e^{-x}+e^{x}}$

Solution

$$\frac{1}{e^{-x}+e^{x}}=\frac{e^{x}}{1+e^{2x}}$$

وبفرض

$$e^{x}=t$$

يكون

$$\int_{0}^{+\infty} \frac{dx}{e^{-x}+e^{x}}=\int_{1}^{+\infty} \frac{dt}{t\left[\frac{1}{t}+t\right]}=\int_{1}^{+\infty} \frac{dt}{1+t^{2}}=$$

$$=\lim_{B\to+\infty} \int_{1}^{B} \frac{dt}{1+t^{2}}=\lim_{B\to+\infty} [arctgt]_{1}^{B}=$$

$$=\lim_{B\to+\infty} [arctgB-arctg\,1]=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$$

هذا يعني أن التكامل يتقارب ويكون: $\int_{0}^{+\infty} \frac{dx}{e^{-x}+e^{x}}=\frac{\pi}{4}$

evaluate the integral or state that it diverges.

$$\int_{0}^{2} \frac{dx}{1-x^{2}}$$

Solution

$$\int_0^2 \frac{dx}{1-x^2} \text{ has an infinite discontinuity at } x = 1$$

$$= \int_0^1 \frac{dx}{1-x^2} + \int_1^2 \frac{dx}{1-x^2}$$

$$= \lim_{c \to 1-} \int_0^c \frac{dx}{1-x^2} + \lim_{c \to 1+} \int_c^2 \frac{dx}{1-x^2}$$

$$= \lim_{c \to 1-} \left(-\frac{1}{2} \ln\left(\frac{x-1}{x+1}\right)\right)_0^c + \lim_{c \to 1+} \left(-\frac{1}{2} \ln\left(\frac{x-1}{x+1}\right)\right)_c^2$$

$$= \infty \text{ , the integral diverges}$$

Use the comparison test to determine if the following integral are convergent or divergent

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{x}} dx \, .$$

(c) $\int_0^{\pi/2} \frac{\cos x}{\sqrt{x}}\,dx$ converges because

$$0 \le \frac{\cos x}{\sqrt{x}} \le \frac{1}{\sqrt{x}} \quad \text{on} \quad \left[0, \frac{\pi}{2}\right],$$

$0 \le \cos x \le 1$ on $\left[0, \frac{\pi}{2}\right]$

and

$$\int_0^{\pi/2} \frac{dx}{\sqrt{x}} = \lim_{a \to 0^+} \int_a^{\pi/2} \frac{dx}{\sqrt{x}}$$

$$= \lim_{a \to 0^+} \sqrt{4x}\,\Big]_a^{\pi/2}$$

$2\sqrt{x} = \sqrt{4x}$

$$= \lim_{a \to 0^+} \left(\sqrt{2\pi} - \sqrt{4a}\right) = \sqrt{2\pi}$$

converges.

Evaluate the improper integral or state that it diverges.

$$\int_1^{\infty} \frac{dx}{x^{\frac{3}{2}}}$$

$$\int_0^1 (\ln x)^2\, dx$$

$$\int_1^{\infty} \frac{dx}{e^{x} + e^{-x}}$$

$$\int_{-1}^{1} \frac{dx}{x^4}$$

Evaluate the improper integral $\int_0^1 (\ln x)^2\, dx$

Solution

$v = x \quad , du = \frac{2}{x} \ln x . dx \qquad dv = dx \quad , u = (\ln x)^2$ بفـــرض أن

$$\int (\ln x)^2 dx = (\ln x)^2 . x - \int x . 2 \ln x . \frac{1}{x} dx = x(\ln x)^2 - 2\int \ln x dx$$

نفرض $v = dx, u = \ln x$ ونكامل مرة ثانية بالتجزئة فيكون:

$$I = (\ln x)^2 x - 2\left[\ln x\, x - \int x . \frac{1}{x} dx\right] =$$

$$= x(\ln x)^2 - 2x . \ln x + 2x \qquad (*)$$

$$\lim_{x\to 0^+} x(\ln x)^2 = \lim_{x\to 0}\frac{(\ln x)^2}{\frac{1}{x}} = \lim_{x\to 0}\frac{2\ln x}{\frac{1}{x^2}} = \lim_{x\to 0}\frac{2.\frac{1}{x}}{\frac{1}{x^2}} = \lim_{x\to 0} 2x = 0$$

بالطريقة نفسها نجد أنّ $\lim_{x\to 0} \int_0^1 (lnx)^2\, dx$

ومن العلاقة (11) واستناداً إلى العلاقة (*) يكون:

$$I_2 = \int_0^1 (\ln x)^2 dx = \lim_{\eta'\to 0^+}\Big[x(\ln x)^2 - 2x\ln x + 2x\Big]_{\eta'}^{1} = 2$$

هذا يعني أن التكامل المعطى يتقارب و يكون : $I_2 = \int_0^1 (\ln x)^2 dx = 2$

Evaluate the improper integral or state that it diverges.

$$\int_1^{\infty} \frac{1}{x^3}\,dx$$

$$\int_1^{\infty} \frac{3}{\sqrt[3]{x}}\,dx$$

$$\int_{-\infty}^{0} xe^{-4x}\,dx$$

$$\int_0^{\infty} x^2 e^{-x}\,dx$$

$$\int_4^{\infty} \frac{1}{x(\ln x)^3}\,dx$$

$$\int_0^{\infty} \frac{e^x}{1+e^x}\,dx$$

$$\int_0^{\infty} \sin\frac{x}{2}\,dx$$

$$\int_0^{\infty} e^{-x}\cos x\,dx$$

TABLE 8.1 Basic integration formulas

1. $\int k\,dx = kx + C \quad$ (any number k)
2. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
3. $\int \dfrac{dx}{x} = \ln|x| + C$
4. $\int e^x\,dx = e^x + C$
5. $\int a^x\,dx = \dfrac{a^x}{\ln a} + C \quad (a > 0,\ a \neq 1)$
6. $\int \sin x\,dx = -\cos x + C$
7. $\int \cos x\,dx = \sin x + C$
8. $\int \sec^2 x\,dx = \tan x + C$
8. $\int \csc^2 x\,dx = -\cot x + C$
10. $\int \sec x \tan x\,dx = \sec x + C$
11. $\int \csc x \cot x\,dx = -\csc x + C$
12. $\int \tan x\,dx = \ln|\sec x| + C$
13. $\int \cot x\,dx = \ln|\sin x| + C$
14. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$
15. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$
16. $\int \sinh x\,dx = \cosh x + C$
17. $\int \cosh x\,dx = \sinh x + C$
18. $\int \dfrac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\dfrac{x}{a}\right) + C$
19. $\int \dfrac{dx}{a^2 + x^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right) + C$
20. $\int \dfrac{dx}{x\sqrt{x^2 - a^2}} = \dfrac{1}{a}\sec^{-1}\left|\dfrac{x}{a}\right| + C$
21. $\int \dfrac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\dfrac{x}{a}\right) + C \quad (a > 0)$
22. $\int \dfrac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\dfrac{x}{a}\right) + C \quad (x > a > 0)$

Thank you for your attention